Applied Mathematics and Statistics

Foundation Qualifying Examination Part B

in Computational Applied Mathematics

Spring 2019 (January)

(Closed Book Exam)

Please solve 3 out of 4 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part B: 1 2 3 4

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: January 28, 2019

Time: 11:15 AM – 13:15 PM

B1. (10 points)

(a) Show that two Lagrangian functions L_1 and L_2 which differ by the total time derivative $d\Lambda/dt$ of some function $\Lambda(q, t)$,

$$L_1 = L_2 + d\Lambda/dt,$$

are equivalent, leading to the same Lagrange's equation of motion.

(b) What is the relation between the generalized momenta p_1 and p_2 that these two Langangians yield?

(c) What is the relation between the Hamiltonian functions H_1 and H_2 that these two Langangians yield?

(d) Show explicitly that Hamilton's equations written using two sets of quantities (with indices 1 and 2) are equivalent.

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B2. (10 points)

Consider the following Riccati equation

$$y' + xy^2 + \frac{1}{x}y - 1 = 0.$$

(a) Transform this equation to a second order linear equation.

(b) Perform analysis of singular points of the second order linear equation on the interval $x \in [0, +\infty)$.

(c) Find the leading behavior of two solutions to the second order linear equation as $x \to +\infty$.

(d) Using previous results, obtain the leading behavior of the general solution to the Riccati equation as $x \to +\infty$.

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B3. (10 points)

Let $A \in \mathbb{R}^{m \times n}$, where $m \geq n$, and A has full rank. Let $B = \begin{bmatrix} \alpha I & A \\ A^T & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R} \backslash \{0\}$.

a) (2 points) Show that B is nonsingular.

b) (2 points) Is B positive definite, positive semidefinite, or indefinite? Justify your answer.

c) (3 points) Show that the largest singular value of B is greater than that of A.

d) (3 points) Show that $B \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ has a solution with $x = A^+ b$.

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B4. (10 points)
Consider the minimal residual (MINRES) method for solving $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is symmetric.

a) (3 points) What is the smallest value for k such that $\mathcal{K}_k(A, b) = \mathcal{K}_{k+1}(A, b)$? Justify your answer. (Hint: You need to take into account both A and b).

b) (2 points) Assuming exact arithmetic, how many iterations does it take for MINRES to arrive at the exact solution?

c) (2 points) If A is nonsymmetric, what Krylov subspace method can be applied to solve $Ax = b$?

d) (3 points) Do the answers in parts (a) and (b) apply to this method for nonsymmetric A in (c)? Justify your answer.

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