AMS Foundation Exam (January 2019): Probability Questions

Solve any three of the following four problems.

All problems are weighted equally. On this cover page write which three problems you want graded.

problems to be graded:

Name (PRINT CLEARLY), ID number

1. Suppose X and Y are two independent random variables with p.m.f.'s

$$P(X = x) = (e-1)e^{-x} \text{ and } P(Y = y) = \frac{1}{(e-1)y!}$$

for $x, y = 1, 2, \ldots$. Let $U_1, U_2, \ldots$ be a sequence of i.i.d. uniform random variables on $[0, 1]$ that is independent of X and Y. Define $M = \max\{U_1, \ldots, U_Y\}$. Find the distribution of $Z = X - M$.

2. Let (X, Y) be the coordinates of a point uniformly selected from the unit square $[0, 1]^2$. Compute the conditional expectation $E[X|XY]$.

3. Suppose X and Y are two continuous random variables with joint p.d.f.

$$f(x,y) = \begin{cases} 2x^2y + \sqrt{y}, & \text{if } x \in (0,1) \text{ and } y \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Find the joint p.d.f. of U and V.

4. Let X be a non-negative random variable with p.d.f. $f(x)$. Prove that

$$E[X^k] = \int_0^\infty kx^{k-1} P(X > x)dx$$

for any integer $k \geq 1$ (assume that $E[X^k]$ is finite).