Mathematical Statistics Qualifer Examination
Part I of the STAT AREA EXAM
May 22, 2024; 9:00 AM - 11:00 AM

There are 4 problems. You are required to solve them all. Show detailed work for full credit.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name: ______________________________ **ID**: ______________________________

Signature: ______________________________

1. Let $X_1, \ldots, X_3$ and $Y_1, \ldots, Y_3$ be two independent random samples from uniform distributions on the interval $(0, 1)$. Define $V = \max(X_1, \ldots, X_3) - \max(Y_1, \ldots, Y_3)$. Find the probability density function of V, and calculate the expected value $\mathbb{E}[V]$ and the variance $\mathrm{Var}(V)$.

2. Prove that log of a moment generating function of random variable Z is convex. That is,

$$\log M_Z(\lambda t_1 + (1-\lambda)t_2) \leq \lambda \log M_Z(t_1) + (1-\lambda)\log M_Z(t_2).$$

   for any t_1, t_2, and for any λ such that $0 \leq \lambda \leq 1$.

3. Suppose that Y follows a Binomial distribution Binomial(n, θ) and θ follow a beta distribution with positive parameters α, β, whose density is

$$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad 0 < \theta < 1.$$

   The mean and variance of θ are $\mathrm{E}(\theta) = \xi = \alpha/(\alpha+\beta)$ and $\mathrm{Var}(\theta) = \xi(1-\xi)/(\gamma+1)$, where $\gamma = \alpha + \beta$.

   (a) Show that the (marginal) mean and variance of Y are $\mathrm{E}(Y) = n\xi$ and $\mathrm{Var}(Y) = n\tau^{-1}\xi(1-\xi)$, where $\tau = (1+\gamma)/(n+\gamma)$.

   (b) Show that the Bayes estimator of θ unter the squared error loss is

$$\xi^* = \frac{y + \gamma\xi}{n+\gamma}.$$

   (c) Show that the conditional distribution of θ given $Y = y$ has variance

$$\frac{\xi^*(1-\xi^*)}{n+\gamma+1}.$$

4. Let $X \sim N(\theta_1, \theta_3)$ and $Y \sim N(\theta_2, \theta_3)$ are independent random variables. Then the parameter space is $\Omega = \{(\theta_1, \theta_2, \theta_3) : -\infty < \theta_1 < \infty, \ -\infty < \theta_2 < \infty, \ \theta_3 > 0\}$. Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ denote independent random samples from these distributions. The hypothesis H_0: $\theta_1 = \theta_2$ is to be tested against all alternatives. Then the parameter space under H_0 is $\omega = \{(\theta_1, \theta_2, \theta_3) : -\infty < \theta_1 = \theta_2 < \infty, \ \theta_3 > 0\}$. Find the maximum likelihood estimators of θ_1, θ_2 and θ_3.