AMS Foundation Exam - Part A, January 18, 2024

Name: ____________________________ ID Num. ____________________

LA: ______ / 30

AC: ______ / 30 Total: ______ / 60

This component of the Foundation Exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with three problems in each. Each question is worth 10 points; answer all **THREE** questions from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Show all work and/or justify your responses.

Your solutions must be submitted within 5 minutes of the end of the exam. Submission instructions:

1. Scan your pages, ordered and oriented appropriately, into a single PDF file. Make sure that each problem's solution is clearly labeled.

2. Email the PDF file to Professor Li (`xiaolin.li@stonybrook.edu`), CCing Professor Green (`david.green@stonybrook.edu`). Your email should be titled "AMS Foundation Exam Part A" and does not need to contain any text in its body.

3. Late submissions – those received after 11:05 am EST on January 18, 2024 (as timestamped by the SBU email service) – will not be scored.

Good Luck!

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Signature

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Section 1: Linear Algebra

1. Let I be an $n \times n$ identity sub-matrix, A be an $m \times m$ square sub-matrix, prove

$$\begin{vmatrix} I & B \\ \mathbf{0} & A \end{vmatrix} = |A|$$

where $\mathbf{0}$ and B are $m \times n$ and $n \times m$ rectangular zero and non-zero sub-matrcies.

2. Given the linear transformation $F : R^4 \to R^3$ defined by:

$$F(x, y, z, w) = (x + 2y + w, 2x - y + 2z - w, x - 3y + 2z - 2w)$$

(a). Find the matrix representation of $F = A[x, y, z, w]^T$.

(b). Find the basis and dimension of the image of A.

(c). Find the basis and dimension of the kernel of A.

(d). Find a vector that is NOT in the kernel of A.

(e). Find a vector that is NOT in the image of A.

3. Given the matrix A as follows

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

(a). Find all eigenvalues and eigenvectors of A.

(b). Find P and P^{-1} such that $P^{-1}AP = \Lambda$, where Λ is diagonal.

(c). Find A^{10}.

Section 2: Advanced Calculus

1. Give the best estimate of lower bound for the following double integral

$$I = \int_0^\infty \int_0^\infty \frac{\sqrt{1+x^2}}{\sqrt{1+y^2}} e^{-(x^2+y^2)} dx dy.$$

   Hint: apply Cauchy-Schwarz.

2. A rectangular box without a lid (top side) is to be made from $27m^2$ of cardboard. Find the maximum volume of such box.

3. Find the Taylor expansion at x_0 for the following functions, calculate the radius of convergence.

(a).

$$f(x) = \frac{1}{(1+2x^2)^2} \qquad x_0 = 0.$$

(b).

$$f(x) = \frac{1}{x^2} \qquad x_0 = 1.$$