Applied Mathematics and Statistics

Foundation Qualifying Examination Part B

in Computational Applied Mathematics,

Spring 2024 (January)

(Closed Book Exam)

Instructions: There are 3 problems, and you are required to solve all of them. All problems are weighted equally. Please show detailed work for full credit. Start each answer on a new page. Print your name, and the appropriate question number at the top of every page used to answer any question. Hand in all answer pages.

NAME ____

Student ID ____

Date of Exam: January 18, 2024

Time: 11:15 AM – 13:15 PM

B1. Consider the following Riccati equation

$$y' + xy^2 + \frac{1}{x}y - 1 = 0.$$

(a) Transform this equation to a second order linear equation by using the substitution $y(x) = v'(x)/(x\,v(x))$.

(b) Classify singular points of the second order linear equation on the interval $x \in [0, +\infty)$.

(c) Find the leading behavior (the controlling factor and a correction term) of two solutions to the second order linear equation as $x \to +\infty$.

(d) Using the larger-valued solution in part (c), approximate the general solution to the Riccati equation as $x \to +\infty$.

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B2. Consider a singular matrix $A \in \mathbb{R}^{n \times n}$ with rank $n-1$. Suppose its left and right singular vectors corresponding to the zero singular value are u_n and v_n, respectively.

a) (5 points) Suppose $s \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$, where $s^T v_n \neq 0$ and $t^T u_n \neq 0$. Show that

$$\begin{bmatrix} A & t \\ s^T & 0 \end{bmatrix}$$

is nonsingular.

b) (5 points) Show that the solution to

$$\begin{bmatrix} A & u_n \\ v_n^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

is a least squares solution to $Ax = b$. In addition $\|x\|_2$ is minimized. (In other words, x is the pseudoinverse solution to the least squares problem $Ax = b$).

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B3. Given $A \in \mathbb{R}^{m \times m}$ and $b \in \mathbb{R}^m \backslash \{0\}$, consider the Arnoldi iteration

$$AQ_k = Q_{k+1}\tilde{H}_k,$$

where $\tilde{H}_k \in \mathbb{R}^{(k+1)\times k}$ is upper Hessenberg, and Q_k is composed of orthonormal columns with $q_1 = b/\|b\|$.

a) (5 points) Show that if A is skew-symmetric, then $\tilde{H}_k$ is tridiagonal.

b) (5 points) Show that span$\{q_1, q_2, \ldots q_k\}$ is equal to the Krylov subspace $\mathcal{K}_k(A, b) = \text{span}\{b, Ab, \ldots, A^{k-1}b\}$, given that the dimension of $\mathcal{K}_k(A, b)$ is equal to k.

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