Outline

1. General solution to Hill's eqn } ← emergence of Hill eqn in conventional & plasma accelerators
2. Emergence of Twiss Parameters
3. emergence of constant of motion.
4. Dispersion & evolution of ϵ

Relation between plasma physics & Hill eqn will be discussed throughout.

Emittance and Phase Space

In the study of conventional accelerators, emittance emerges as an invariant in the study of a particle's transverse motion. In the simplest case of a particle in a linear focusing force (quadruples in conventional accelerators or blowout regime in plasma accelerators) the particle's transverse trajectory from the axis is described by the Hill's equation

spatially varying focusing force

$$\underbrace{x''}_{\frac{d^2x}{ds^2}} + \overbrace{K(s)}x = 0 \quad \cdots (1)$$

coordinate along the accelerator

In a conventional accelerator, this describes for instance the transverse trajectory of a particle going off axis in a quadrupole:

Consider a particle moving through a magnet w/ gradient

$B' = \frac{\partial B_y}{\partial x}$ over a distance of Δs,

$$\Delta x' = \frac{dx}{ds} = -\frac{B' \Delta s}{(B\rho)} x$$

$$\frac{\Delta x'}{\Delta s} = -\frac{B'(s)}{(B\rho)} x$$

$$\Rightarrow \boxed{x'' + \frac{B'(s)}{(B\rho)} x = 0}$$

→ The ratio of momentum to charge, $\frac{P}{e}$ is often called the magnetic rigidity, and written as $B\rho$ (single symbol)

Like a quadrupole magnet, the blowout regime also has a linear force:

$$\frac{dP_x}{ds} = -(\partial_x \psi)\, x = -\frac{\omega_p^2}{2c^2} x$$

* The two F_x, F_y are coupled through F_r

$$P_x = P x'$$

total momentum

$$x' = \frac{x_\perp}{x_\parallel}$$

normalized units

$$\bar{P}_x = \frac{P_x}{mc} = \frac{P}{mc} x' \approx \gamma x' \quad (v \approx c)$$

First, consider the case with no acceleration (constant γ)

$$x'' + \frac{\omega_p^2}{2\gamma c^2} x = 0$$

$$k_p \equiv \frac{\omega_p}{c} \Rightarrow \frac{\omega_p^2}{2\gamma c^2} = \frac{k_p^2}{2\gamma} \equiv k_\beta^2$$ called the betatron frequency

$$\therefore \; x'' + k_\beta^2 x = 0$$

Compared to equation 1, $K = k_\beta^2$.

There are two methods of solution for Hill's equation:

- Consider where K is constant. Within each component, we use harmonic oscillator solution and piece them together at interfaces.
- We can get a closed form solution using the properties of Hill's equation.

Piecewise method of solution

We can describe the motion of a particle by a 2x2 matrix. There are two cases to consider:

K=0: this case is the same as having a drift space L. In matrix form,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

This corresponds to a drift space, e.g. after the plasma accelerator. Incidentally, the other important linear transport matrix is that of a thin lens (both in accelerators as well as optics):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

K>0: over distance ℓ, the equation of motion is just a simple harmonic oscillator, and in matrix term,

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{out} = \begin{bmatrix} \cos(\sqrt{K}\,L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,L) \\ -\sqrt{K}\sin(\sqrt{K}\,L) & \cos(\sqrt{K}\,L) \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{in}$$

Here,

$$\sqrt{K} = k_\beta$$

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} \cos k_\beta L & \frac{1}{k_\beta}\sin(k_\beta L) \\ -k_\beta \sin(k_\beta L) & \cos(k_\beta L) \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{in}$$

in thin lens approximation, $\delta \equiv k_\beta L \to 0$, $\cos\delta \to 1 + \frac{\delta^2}{2}$ & $\sin\delta \to \delta - \frac{\delta^3}{3}$
this matrix approaches that of a lens with $f = \frac{1}{k_\beta^2 L}$

Now, consider the case where the betatron frequency is changing (for instance due to plasma density variation), but energy is still constant, $\gamma' = 0$
In this case, we need a solution for Hill's equation. This was a differential equation extensively studied in the nineteenth century and the result can be expressed in the form

Constants of integration

$$x = A\, w(s) \cos\left[\psi(s) + \delta\right]$$

spatially varying amplitude & phase

Note: w(s), and the phase will no longer be linear in 's'.

To find $w(s)$ and $\psi(s)$, substitute the general solution into the eqn:

$$x'' + Kx = 0 \Rightarrow -A\left(w\psi'' + 2w'\psi'\right)\sin[\psi + \delta] + A\left(w'' - w\psi' + Kw\right)\cos[\psi + \delta] = 0$$

These two coefficients have to go to zero.

$w \times \sin$ term : $2ww'\psi' + w^2\psi'' = (w^2\psi')' = 0$

$$\therefore \psi' = \frac{c}{w(s)^2}$$ ← c, arbitrary constant, Let's look at a case where it is set to 1.

$$\boxed{\psi' = \frac{1}{w(s)^2}} \quad \cdots (2)$$

the cosine term turns into:

$$w^3(w'' + Kw) = 1 \quad \cdots (3)$$

Now, we can represent the transport matrix components in terms of parameters introduced here:

i.e.

$$x = A\, w(s) \cos\left[\psi(s) + \delta\right]$$

Rewrite as:

$$x = w(s)\left(A_1 \cos\psi + A_2 \sin\psi\right)$$

then,

$$x' = \left(A_1 w' + \frac{A_2}{w}\right)\cos\psi + \left(A_2 w' - \frac{A_1}{w}\right)\sin\psi$$

then given initial conditions x_0, x_0' at $s = s_0$, the constants are

$$A_1 = \frac{x_0}{w},$$

$$A_2 = x_0' w - x_0 w'$$

The matrix propagation from $s_0 \longrightarrow s_0 + d$

small value so w is roughly the same or for synchrotron where w is fixed at each pt around the ring

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{s_0+d} = \begin{bmatrix} \cos\Delta\psi_c - ww' \sin\Delta\psi_c & w^2 \sin\Delta\psi_c \\ -\frac{1+(ww')^2}{w^2} \sin\Delta\psi_c & \cos\Delta\psi_c + ww' \sin(\Delta\psi_c) \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{s_0} \quad (4)$$

the phase of particle oscillation advances by

$$\psi(s_0 \rightarrow s_0 + d) \equiv \Delta\psi_c = \int_{s_0}^{s_0+d} \frac{ds}{w^2(s)}$$

Courant-Snyder Parameters

The constant in equation 2 was set to one arbitrarily. To remove this ambiguity, it is customary to define more fundamental variables as

$$\begin{cases} \beta(s) = \dfrac{w^2(s)}{c} \\ \alpha(s) = -\dfrac{1}{2}\dfrac{d\beta(s)}{ds} = -\dfrac{1}{2}\dfrac{d}{ds}\left(\dfrac{w^2(s)}{c}\right) \\ \gamma_{cs} \equiv \dfrac{1+\alpha^2}{\beta} \end{cases}$$

Note: Eqn 4 will have c sprinkled through it if not set to 1, and will be different for different c

not related to Lorentz factor

and rewrite Eqn 4 as

$$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \begin{bmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{1/2}(\cos\Delta\psi + \alpha_1 \sin\Delta\psi) & (\beta_1\beta_2)^{1/2}\sin\Delta\psi \\ -\frac{1-\alpha_1\alpha_2}{(\beta_1\beta_2)^{1/2}}\sin\Delta\psi + \frac{\alpha_1-\alpha_2}{(\beta_1\beta_2)^{1/2}}\cos\Delta\psi & \left(\frac{\beta_1}{\beta_2}\right)^{1/2}(\cos\Delta\psi - \alpha_2\sin\Delta\psi) \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$$

* see Appendix for $\alpha_2, \beta_2, \gamma_2$

Here, the phase advance is $\Delta\psi(s_1 \rightarrow s_2) = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}$

$\beta(s)$ may be interpretted as the local wavelength of the oscillation divided by 2π, particularly based on above phase relation.

You can see this by comparing $\Delta\psi = \int_{s_1}^{s_2} \frac{ds}{\beta(s)} = \frac{s}{\beta}$ for constant β with the customary $\Delta\psi_c = \frac{2\pi}{\lambda} s$ (or ks) to see that $\beta \sim \lambda$

$\alpha, \beta, \gamma_{cs}$: Courant-Snyder parameters

β : amplitude function

General Solution of motion: $\boxed{x(s) = A\sqrt{\beta(s)}\cos[\psi(s) + \delta]}$ ... (6)

From eqn 3, $w^3(w'' + Kw) = 1$,
the amplitude function must satisfy

$$\boxed{2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4} \quad \cdots (7)$$

or

$$\boxed{K\beta = \gamma_{cs} + \alpha'} \quad \cdots (8)$$

Equation 7 shows that in the limit where β is constant,

$$\beta' \sim \beta'' \sim 0 \Rightarrow \beta^2 K = (\beta K_\beta)^2 = 1 \Rightarrow \boxed{\beta = \frac{1}{K_\beta}} \quad \cdots (9)$$

The transport matrix in equation 5 can be written as

$$M = I \cos\Delta\psi_c + J \sin\Delta\psi_c$$

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma_{cs} & -\alpha \end{pmatrix}$$

Note: $J^2 = -I$, $\quad I$: Identity matrix

$$\therefore \ M = e^{J\Delta\psi_c}$$

$\rightsquigarrow$ Complex matrix analogous to a complex number: $\begin{cases} \cos\Delta\psi + j\sin\Delta\psi \\ j^2 = -1 \end{cases}$

Emittance

It turns out that the constant "A" in equation 6 plays an important role in accelerator physics

$$(6) \cdots x = A\sqrt{\beta(s)}\cos(\psi(s) + \delta)$$

$$x' = \frac{A}{2}\beta^{-\frac{1}{2}}\beta'\cos(\psi(s) + \delta) + A\beta^{\frac{1}{2}}\sin(\psi(s) + \delta)\,\psi'$$

$$\Rightarrow \alpha(s)x = -\frac{1}{2}\beta' A\sqrt{\beta}\cos(\psi(s) + \delta)$$

$$\beta(s)\,x' = \frac{A}{2}\sqrt{\beta}\,\beta'\cos(\psi(s)+\delta) - A\,\beta\sqrt{\beta}\,\sin(\psi(s)+\delta)\,\psi'$$

recall $\psi' = \frac{1}{W(s)^2} = \frac{1}{\beta}$

$$\alpha(s)\,x + \beta(s)\,x' = -A\sqrt{\beta}\,\sin(\psi(s)+\delta) \quad \cdots (10)$$

square and add equations (1) & (2) :

$$x^2(s) + [\alpha(s)\,x(s) + \beta(s)\,x'(s)]^2$$

$$= A^2\beta(s)\left[\sin^2(\psi(s)+\delta) + \cos(\psi(s)+\delta)\right]^2$$

$$= A^2\beta(s)$$

$$\therefore A^2 = \frac{1}{\beta}\left[x^2(s) + \alpha^2(s)\,x^2(s) + \beta^2 x'^2(s) + 2\alpha(s)\beta(s)\,x\,x'\right]$$

$$= x^2(s)\underbrace{\left[\frac{1+\alpha^2}{\beta}\right]}_{\gamma_{cs}} + \beta(s)\,x'^2 + 2\alpha(s)\,\alpha(s)\,x'(s)\,]$$

$$\therefore A^2 = \gamma_{cs}(s)\,x^2(s) + 2\,\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'^2(s) \quad \cdots (11)$$

Comparing this equation to the general equation of an ellipse:

$$a x^2 + 2 b x y + c y^2 = d$$

One can observe that equation 1 describes an ellipse in x-x' space. According to the analytic geometry, the area of an ellipse is given by

$$\text{area} = \frac{\pi d}{\sqrt{ac - b^2}}$$

Substituting the constants from equation 1,

$$\text{area} = \frac{\pi A^2}{\sqrt{\beta\,\gamma_{cs} - \alpha^2}} = \pi A^2$$

Therefore, for different location throughout an accelerator lattice, the ellipse will have a different orientation depending on the value of the amplitude function (β) and its derivative (α), but they will all have the same value "A", corresponding to the area of the phase space. The phase space area occupied by the beam is called emittance and is denoted by $\pi\epsilon$.

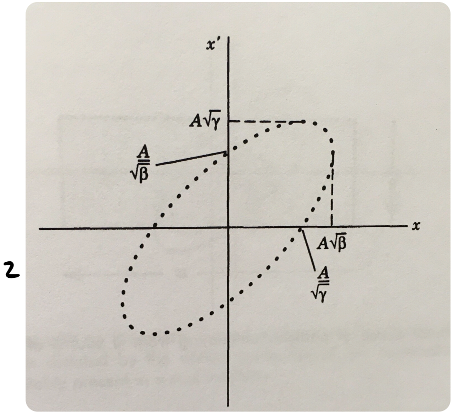


Figure 1, adiabatic invariant

Note : some sources define ϵ = area = πA^2 rather than $\epsilon = A^2$ as is done here.

An equivalent description of an ellipse may be given by matrix representation:

$$u^T \sigma^{-1} u = 1 \quad \ldots (12)$$

where $u = \begin{bmatrix} x \\ x' \end{bmatrix}$ is the coordinate matrix & σ is an unknow symmetric matrix

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \qquad \sigma_{12} = \sigma_{21}$$

Substituting these variables in equation 12 gives

$$\sigma_{11} x^2 + 2\sigma_{12} x x' + \sigma_{22} x'^2 = 1 \quad \ldots (13)$$

Comparing equations 13 and 11, and using $A^2 = \epsilon$,

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma_{cs} \end{pmatrix} \quad \ldots (14)$$

The area of this matrix is given by

$$\pi\sqrt{\det \sigma} = \pi\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2} = \pi\epsilon \quad \ldots (15)$$

Emittance of Realistic Beams

In the ideal scenario, the beam would have zero cross sectional area, and all the particles would be headed in the same direction. In that case all the particles would occupy the same point in the phase space of this degree of freedom.

For a distribution of particles, the motion of the individual particles will lie on the elliptical invariant curves. In the case of linear forces with no energy gain, the orientation of the ellipse will change as the Courant-Snyder parameters evolve (see figure and Appendix 2), but the area stays the same.

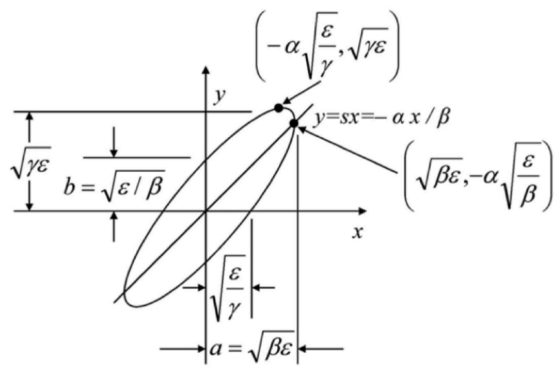


Figure: C-S parameters and shape of phase space

<u>In Summary,</u> if the beam in an accelerator has emittance epsilon, the phase space area is bounded by a curve

$$\epsilon = \gamma_{cs}x^2 + 2\alpha x x' + \beta x'^2$$

Often, we want to speak about the Emittance in terms of the RMS beam size:

→ Gaussian Distribution:

$$n(x)dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma_x^2} dx \longleftarrow$$

normalized to one particle

i.e. $\int_{-\infty}^{\infty} n(x)dx = 1$

This is a probability distribution. For a beam with N particles, the particle distribution in x, x' space is

$$N(x,x',t)\,dx\,dx' = \frac{N_0}{2\pi\sigma\sigma'} e^{-x^2/2\sigma_x^2} e^{-x'^2/2\sigma_{x'}^2} = N_0\, n(x,x',t)$$

Here, both x and x' space are assumed to have a Gaussian distributions. Using the trajectory of the rms spot size as the suitable trajectory to define the emittance, we have

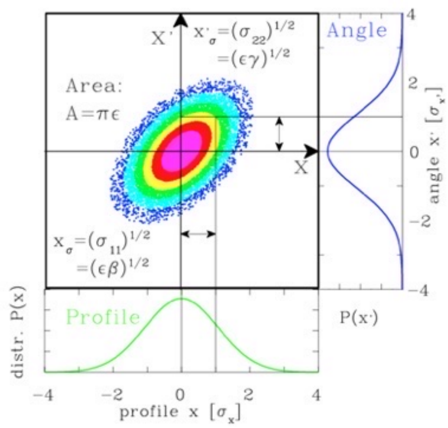
$$\sigma^2 = \langle x^2 \rangle = \int dx\,dx'\, n(x,x',t)\,x^2 = \beta\epsilon$$

$$\sigma'^2 = \langle x'^2 \rangle = \int dx\,dx'\, n(x,x',t)\,x'^2 = \gamma_{cs}\,\epsilon$$

& the cross term

$$\langle x x' \rangle = \qquad\qquad = -\epsilon\alpha$$

The first two expressions can be confirmed from the ellipse profile shown in Figure 1. Comparing these expressions with equations 14 and 15 results in the following expression for emittance:



$$\epsilon = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2} \quad \cdots (16)$$

Also frequently used are the expressions for the maximum displacement and angle

$$x_{max} = \sqrt{\epsilon\,\beta_{max}}$$

$$x'_{max} = \sqrt{\epsilon\,\gamma_{cs}^{max}}$$

Adiabatic damping of betatron oscillations

In the previous section, we considered motion with constant total momentum. We now want to show how the amplitude of motion changes as a function of the particle momentum, since the particle will continually gain energy as it is accelerated in the blowout regime. Here, we assume that the momentum is a slowly changing function of time, or equivalently, all g the length of the accelerator.

$$\frac{d}{ds}(px') + \frac{Kp^2}{2}x = 0$$

$$px'' + p'x' + \frac{Kp^2}{2}x = 0$$

$$x'' + \frac{p'}{p}x' + K_\beta^2 x = 0$$

Hill's equation, but with a damping term

To solve this equation, we use a method known as the method of integrated phase

assume a solution of form $x = uv$

$$x' = u'v + v'u$$

$$x'' = u''v + 2u'v' + v''u$$

$$\therefore\ u''v + \underline{2v'u'} + v''u + \underline{\frac{p'}{p}u'v} + \frac{p'}{p}v'u + K_\beta^2 uv = 0$$

Collect the u terms

$$vu'' + \underbrace{\left(2v' + \frac{p'}{p}v\right)} u' + \left(v'' + \frac{p'}{p}v' + K_\beta^2 v\right)u = 0$$

Choose v such that this term goes to zero. This will convert the u equation into another Hill's equation

$$2\frac{v'}{v} = -\frac{p'}{p} \Rightarrow \boxed{v = v_0\left(\frac{p_0}{p}\right)^{1/2}}$$

Since momentum is changing slowly, we can ignore second order terms v'' & $p'v'$.

$$\therefore u'' + K_\beta^2 u = 0$$ ← the Hill equation

Solution: $x = uv = A_0 \left(\frac{P_0}{P}\right)^{1/2} \beta^{1/2}(s) \cos(\psi(s) + \delta)$ ... (17)

The amplitude of the betatron oscillations is therefore reduced by a factor of $\gamma^{-1/2}$, where γ is the relativistic Lorentz factor. Since the beam Emittance is the phase space area bounded by the Courant Snyder invariant curve, and since this area is proportional to the square of the betatron amplitude, we see that the beam emittance varies inversely with the beam momentum. The use of normalized emittance,

$$\epsilon_N = \epsilon \left(\gamma \frac{v}{c}\right) \approx \gamma \epsilon$$

Permits comparisons of the phase space areas independent of the kinematic factors. This factor should stay constant in an ideal world, but it does not in practice due to nonlinear forces as well as energy spread.

To preserve a beam normalized emittance under these circumstances, we use <u>matched</u> beams. A matched beam has <u>little to no variation</u> in its spot size.

Recall a beam spot size is given by

$$\sigma^2 = \langle x^2 \rangle = \epsilon\beta$$

Constant spot size: $\beta' = \beta'' = 0 \Rightarrow$ from Eqn 7, $4\beta^2 K = 4$

multiply both sides by ϵ^2

$$\epsilon^2 - \epsilon^2\beta^2 K_\beta^2 = 0$$ $\sigma = \sqrt{\epsilon\beta}$

$$\epsilon^2 - \sigma^4 K_\beta^2 = 0$$

$$\boxed{1 - \frac{\sigma_0^4 K_\beta^2}{\epsilon^2} = 0} \quad \cdots (18)$$

Equation 18 allows us to calculate a matched spot size for any value of K_β.

$$\sigma_0^4 = \frac{\epsilon^2}{K_\beta^2} \Rightarrow \sigma_0 = \left(\frac{\epsilon}{K_\beta}\right)^{1/2} = \left(\frac{\epsilon_N}{\gamma K_\beta}\right)^{1/2}$$

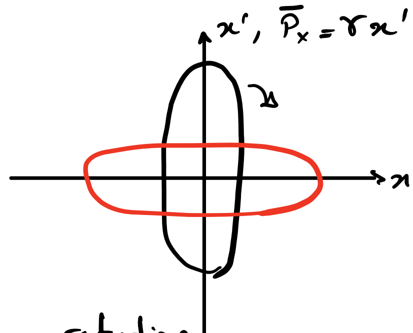
Also, note that from equation 9, this is synonymous with

$$\beta = \frac{1}{K_\beta}$$

<u>Evolution of emittance for an unmatched case, physical picture</u>

Consider a beam with a phase space represented by a vertical ellipse.

i.e. $\alpha = 0$, $\beta = \beta_{min}$, beam at waist



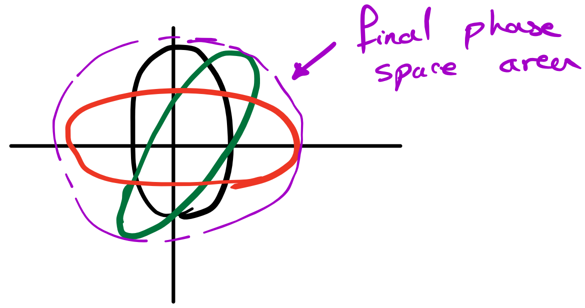
Area $\propto \langle x^2 \rangle \langle \bar{P}_x^2 \rangle$ (eqn 15)

As beam evolves, values of α & β & γ change. This translates to the phase space rotation of the ellipse.

If $\gamma' = 0$ & $\Delta\gamma = 0$ & the focusing force is linear, then all particles rotate together. After a $1/4$ rotation (to red) $\langle x^2 \rangle$ is Larger and $\langle \bar{p}^2 \rangle$ is smaller, but $\langle x^2 \rangle \langle \bar{p}^2 \rangle$ is the same

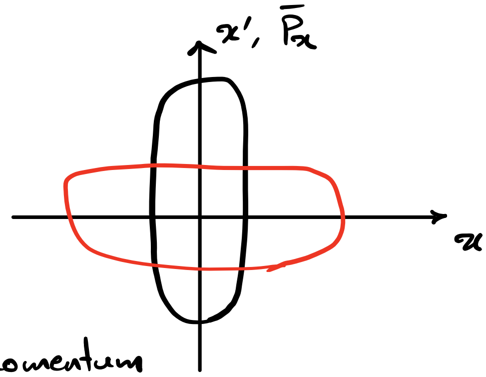
If particles do not rotate together, then ψ' varies and after many "rotations" the phase space gets filled.

However, the amplitudes X_{max} & P_{max} are very similar, so the phase space area fills up



This physical picture provides a quick way to estimate the emittance growth of unmatched beams.

Assume a beam that is nearly matched & the ellipse is initially vertical:



$\alpha \approx 0$

$\beta' \approx \beta'' \approx 0 \Rightarrow$ Equation 9: $\beta \simeq \frac{1}{k_\beta}$

if initial beam size is σ_0, initial momentum spread is

$$\bar{P}_0 = \gamma \sigma_{x'} = \gamma(\epsilon \gamma_{cs})^{1/2} = \gamma\sqrt{\epsilon/\beta} = \frac{\gamma\sqrt{\epsilon\beta}}{\beta} = \boxed{\gamma \sigma_0 k_\beta}$$

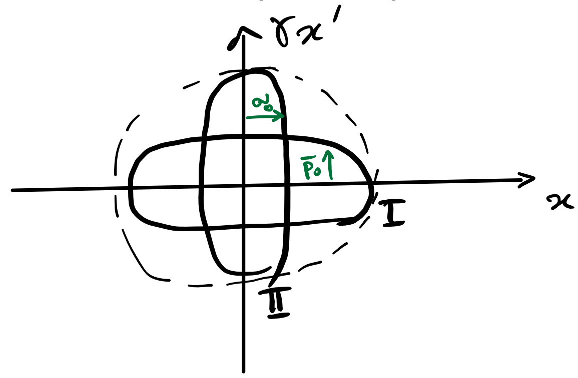
since $\alpha = 0$, see equations on pg 6

after a quarter turn, the initial beam size becomes momentum spread & vice versa: $\sigma_0 \rightarrow \gamma \sigma_0 k_\beta$

$$\bar{P}_0 \rightarrow \bar{P}_0 / k_\beta \gamma$$

If there is any energy spread, after a time characterized by $k_\beta/\Delta k_\beta$, where $k_\beta = \frac{1}{2}\frac{\Delta\gamma}{\gamma} k_\beta$, so after $\Delta\gamma/\gamma$ oscillations the entire phase space is filled

This means that some particles are in ellipse I and some are in ellipse II.



The area of the phase space is $\pi\epsilon$.

For a beam with initial spot size σ_0, $\pi\gamma\epsilon = \pi\sigma_0\gamma\sigma_0 k_\beta$

$$\epsilon_N = \gamma\sigma_0^2 k_\beta$$

For a beam with initial divergence σ_0', $\pi\gamma\epsilon = \pi\bar{P}_0 \frac{\bar{P}_0}{k_\beta\gamma}$

$$\epsilon_N = \frac{\bar{P}_0^2}{k_\beta\gamma}$$

$$\epsilon_{Nf} \sim \frac{1}{2}\left[\gamma\sigma_0^2 k_\beta + \frac{\bar{P}_0^2}{k_\beta\gamma}\right] = \frac{\sigma_0^2\bar{P}_0^2}{2}\left[\frac{\gamma k_\beta}{\bar{P}_0^2} + \frac{1}{k_\beta\gamma\sigma_0^2}\right]$$

$$\boxed{\epsilon_{NF} = \frac{\epsilon_{N_0}}{2}\left[\frac{\gamma k_\beta}{\bar{P}_0^2} + \frac{1}{k_\beta\gamma\sigma_0^2}\right]}$$

This turns out to be the exact expression

Appendix: evolution of spot size

We start from the definition of the spot size rms

$$\sigma^2 = \langle x^2 \rangle = \int dx\,dx'\; n(x,x')\; x^2 = \sum_{i=1}^{N} \frac{x_i^2}{N}$$

$$\sigma' = \left[\sqrt{\langle x^2 \rangle}\right]' = \frac{1}{\sqrt{\langle x^2 \rangle}} \cdot \frac{1}{2} \cdot 2\langle x x' \rangle = \frac{\langle x x' \rangle}{\sqrt{\langle x^2 \rangle}} = \frac{\langle x x' \rangle}{\sigma}$$

$$\sigma'' = \frac{\langle x'^2 \rangle}{\sqrt{\langle x^2 \rangle}} + \frac{\langle x x'' \rangle}{\sqrt{\langle x^2 \rangle}} - \frac{\langle x x' \rangle^2}{\langle x^2 \rangle^{3/2}}$$

$$= \frac{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}{\langle x^2 \rangle^{3/2}} + \frac{\langle x x'' \rangle}{\sqrt{\langle x^2 \rangle}}$$

$$= \frac{\epsilon^2}{\sigma^3} + \frac{\langle x x'' \rangle}{\sigma}$$

$$x'' = -K_\beta^2 x \qquad (\text{for } p' = 0)$$

or $$x'' = -K_\beta^2 x - \frac{p'}{p} x'$$

Let $p' \sim 0 \Rightarrow$ $$\sigma'' = \frac{\epsilon^2}{\sigma^3} - \frac{\langle x\, K_\beta^2 x \rangle}{\sigma}$$

$$\sigma'' = \frac{\epsilon^2}{\sigma^3} - \frac{K_\beta^2 \langle x^2 \rangle}{\sigma}$$

$$\boxed{\sigma'' = \frac{\epsilon^2}{\sigma^3} - K_\beta^2 \sigma}$$

Appendix 2: evolution of Courant-Snyder parameters

Consider the transport matrix:

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x'_0 \end{bmatrix}$$

$$G\epsilon\beta = \langle x^2 \rangle = \langle (M_{11} x_0 + M_{12} x'_0)(M_{11} x_0 + M_{12} x'_0) \rangle$$

$$= \langle M_{11}^2 x_0^2 + 2 M_{21} M_{12} x_0 x'_0 + M_{12}^2 x_0'^2 \rangle$$

$$= M_{11}^2 \langle x_0^2 \rangle + 2 M_{21} M_{12} \langle x_0 x'_0 \rangle + M_{12}^2 \langle x_0'^2 \rangle$$

$$= \epsilon \left[M_{11}^2 \beta_0 + M_{12}^2 \gamma_0 + 2 M_{12} M_{21} \alpha_0 \right]$$

Similar expressions can be obtained for propagation of γ and α.

In Matrix representation,

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} M_{11}^2 & -2 M_{11} M_{12} & M_{12}^2 \\ -M_{11} M_{21} & M_{11} M_{22} + M_{21} M_{12} & -M_{12} M_{22} \\ M_{21}^2 & -2 M_{21} M_{22} & M_{22}^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{bmatrix}$$