Money as Minimal Complexity*

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Abstract

We consider mechanisms that provide traders the *opportunity* to exchange commodity i for commodity j, for certain ordered pairs ij. Given any connected graph G of opportunities, we show that there is a unique mechanism M_G that satisfies some natural conditions of "fairness" and "convenience". Let $\mathfrak{M}(m)$ denote the class of mechanisms M_G obtained by varying G on the commodity set $\{1, \ldots, m\}$. We define the complexity of a mechanism M in $\mathfrak{M}(\mathfrak{m})$ to be a pair of integers $\tau(M), \pi(M)$ which represent the "time" required to exchange i for j and the "information" needed to determine the exchange ratio (each in the worst case scenario, across all $i \neq j$). This induces a quasiorder $\preceq$ on $\mathfrak{M}(m)$ by the rule

$$M \preceq M' \text{ if } \tau(M) \leq \tau(M') \textit{ and } \pi(M) \leq \pi(M').$$

We show that, for $m > 3$, there are precisely three $\preceq$-minimal mechanisms M_G in $\mathfrak{M}(m)$, where G corresponds to the star, cycle and complete graphs. The star mechanism has a distinguished commodity – the money – that serves as the sole medium of exchange and mediates trade between decentralized markets for the other commodities.

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Our main result is that, for *any* weights $\lambda, \mu > 0$, the star mechanism is the *unique* minimizer of $\lambda\tau(M) + \mu\pi(M)$ on $\mathfrak{M}(m)$ for large enough m.

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1 Introduction

We start with a Cournotian model of an exchange mechanism M on commodity set $\{1, \ldots, m\}$, in which the actions available to traders are of a very simple kind[1]. For certain ordered pairs ij, pre-specified by M, each trader may offer any quantity of commodity i in order to obtain commodity j. Once every offer is in, the mechanism M redistributes to the traders all the commodities it has received, holding back nothing. The returns to the traders are calculated by an algorithm[2] that is common knowledge. Thus a mechanism M is characterized by a collection of exchange *opportunities*, which form directed graph G on $\{1, \ldots, m\}$, and the algorithm. We assume throughout that G is *connected,* i.e., M permits *iterative* exchange of any i for any j.

At this level of generality, there are infinitely many mechanisms for any given graph G. However, we shall show that only one of them satisfies some natural conditions of "fairness" and "convenience" (see section 3). This special mechanism is denoted M_G and is described precisely in section 2. It is a striking property of M_G that it admits unique prices[3], which depend only on the aggregate offers by the traders on the various edges of G, and which mediate trade in the following strong sense: first, the return to any trader

[1] It is our purpose to see how far matters may develop within such an elementary Cournot mechanism. In particular, note that *ex ante* there are no "prices" to refer to, upon which a trader may condition his offers. We do show that prices can be "admitted", i.e., *defined*, but this happens *ex post* once unconditional offers for trade have come into the mechanism. Our mechanisms are thus a far cry from the more complex Bertrand mechanisms, in which traders use prices alongside quantities in order to make contingent statements to protect themselves against vagaries of the market (see,e.g., [4], [22]). An analysis analogous to ours might well be possible in the Bertrand setting, but that is a topic for future exploration.

[2] There is no presumption that the algorithm be "informationally decentralized". Indeed even the return to a simple offer of i, made only via the pair ij, may well depend on all the offers at *every* $kl \in G$; and may thus require a lot of information for its computation.

[3] Prices are rays in $\mathbb{R}^m_{++}$, i.e., invariant under multiplication by positive constants, and represent consistent exchange rates between commodities.

depends only on his own offers and the prices; second, the total value — under the prevailing prices — of every trader's offers is equal to that of his returns. The immediate upshot of price mediation is that the returns to any trader can be calculated in a transparent manner from the price ratios p_i/p_j and his own offers.

Thus we are led to consider the class $\mathfrak{M}(m)$ of mechanisms M_G, where G ranges over all directed, connected graphs on the vertex set $\{1, \ldots, m\}$. The cardinality of $\mathfrak{M}(m)$, though finite, grows super-exponentially in m. However we shall show in section 2 that if one invokes natural complexity considerations, based on the time needed to exchange any commodity i for j and the information needed to determine the exchange ratio p_i/p_j, then the welter of mechanisms in $\mathfrak{M}(m)$ is eliminated and we are left with only three mechanisms of minimal complexity, namely those that arise from the star, cycle and complete graphs (Theorem 1). Indeed, provided m is large enough, just the star mechanism remains (Theorem 2) in which one commodity emerges endogenously as money and mediates trade across decentralized markets for the other commodities[4].

2 The Emergence of Money

Let G be a directed and connected graph[5] with vertex set $\{1, \ldots, m\}$. We define a mechanism M_G as follows. Each trader can use every opportunity in M, i.e., place arbitrary weights on the edges ij of G, representing his offer of i for j. Let b_{ij} denote the total weight on ij (i.e., the aggregate amount of commodity i offered for j by all traders). We shall specify what happens when $b_{ij} > 0$ for every edge ij in G, i.e., when there is sufficient diversity in

[4]To be precise: the price of any commodity $1 \leq i \leq m-1$, in terms of money m, depends only on the aggregate offers on edges im and mi; and thus this pair of edges may be viewed as a decentralized market for i and m, with m mediating between the various markets.

[5]In this paper by a graph we mean a *directed simple graph.* Such a graph G consists of a finite *vertex* set V_G, togther with an *edge* set $E_G \subseteq V_G \times V_G$ that does not contain any loops, *i.e.,* edges of the form ii. For simplicity we shall often write $i \in G$, $ij \in G$ in place of $i \in V_G$, $ij \in E_G$ but there should be no confusion. By a *path* $ii_1i_2 \ldots i_kj$ from i to j we mean a nonempty sequence of edges in G of the form $ii_1, i_1i_2, \ldots, i_{k-1}i_k, i_kj$.If $k = 0$ then the path consists of the single edge ij, otherwise we insist that the *intermediate* vertices $i_1, \ldots, i_k$ be distinct from each other and from the endpoints i, j. However we do allow $i = j$, in which case the path is called a *cycle*. We say that G is *connected* if for any two vertices $i \neq j$ there is a *path* from i to j.

the population of traders so that each opportunity is active. Denote $b = (b_{ij})$ and let $\mathbb{R}^m_{++}/\sim$ be the set of rays in $\mathbb{R}^m_{++}$ representing prices. It is well-known that (with b_{ij} understood to be 0 if ij is not an edge in G) there is a unique ray $p = p(b)$ in $\mathbb{R}^m_{++}/\sim$ satisfying

$$\sum\nolimits_i p_i b_{ij} = \sum\nolimits_i p_j b_{ji} \text{ for all } j. \tag{1}$$

Note that the left side of (1) is the total value of all the commodities "chasing" j, while the right side is the total value of commodity j on offer; thus (1) is tantamount to "*value conservation*".

It is shown in [29] that the prices are given by the formula

$$p_i = \sum_{T \in \mathcal{T}_i} b_T \text{ where } b_H = \prod\nolimits_{ij \in H} b_{ij} \text{ for any subgraph } H \tag{2}$$

and $\mathcal{T}_i$ is the collection of all trees in G that are rooted at i (i.e., subgraphs of G in which there is a unique directed path from each $j \neq i$ to i).

The principle of value conservation, which determines prices, also determines trade. An individual who offers a_{ij} units of i via opportunity ij gets back r_j units of j, where $p_i a_{ij} = p_j r_j$. More generally, if a trader offers $a = (a_{ij}) \geq 0$ across all edges of G, he gets a return $r(a, b) \in \mathbb{R}^m_+$ whose components are given by

$$r_j(a, b) = \sum\nolimits_i (p_i/p_j) a_{ij} \tag{3}$$

for all j. Thus the return to a trader depends only on his offer a and the *price ratios* p_i/p_j, which are well-defined functions of b (unlike the price vector $p = (p_i)$ which is only defined up to a scalar multiple). It might be instructive to see the formulae for price ratios (and thereby also for returns, thanks to equation (3)) for specific mechanisms. Let us, from now on, identify two mechanisms if one can be obtained from the other by relabeling commodities. There are three mechanisms of special interest to us called the *star, cycle,* and *complete mechanisms;* with the following edge-sets and price ratios:

G	Star	Cycle	Complete
E_G	$\{mi, im : i < m\}$	$\{12, 23, \ldots, m1\}$	$\{ij : i \neq j\}$
p_i/p_j	$b_{mi}b_{jm}/b_{im}b_{mj}$	$b_{j,j+1}/b_{i,i+1}$	$*$

For the star and cycle mechanisms, the right-hand side of (2) involves a *single* tree and, in the ratio p_i/p_j, several factors cancel leading to the simple expressions in the table above. However, for the complete mechanism there is no cancellation and in fact here each price ratio depends on *every* b_{ij}.

The class of G-*mechanisms* is the set

$$\mathfrak{M}(m) = \{M_G : G \text{ is a directed, connected graph on } \{1, \ldots, m\}\}. \quad (4)$$

Although finite, $\mathfrak{M}(m)$ is rather large, indeed super-exponential in m. We shall see that some natural complexity considerations help cut down its size.

Consider a trader who interfaces with $M \in \mathfrak{M}(m)$ in order to exchange i for j. A natural concern for him would be: what is the minimum number of time periods $\tau_{ij}(M)$ needed to accomplish this exchange? We define the *time-complexity* of M to be

$$\tau(M) = \max_{i \neq j} \tau_{ij}(M). \quad (5)$$

It is evident that $\tau_{ij}(M)$ is the length of the shortest path in G from i to j and $\tau(M)$ is the *diameter* of the graph G.

The other concern of our trader would be: how much of commodity j can he get per unit of i? It follows from equation (3) that he can calculate this from the state b of the mechanism which determines the *price ratio*[6] p_i/p_j. Thus the question can be rephrased: how many components of b does he need to know[7] in order to calculate p_i/p_j? The table above indicates that it is easier to compute p_i/p_j for the star and cycle mechanisms than, say, the complete mechanism.

To make this notion precise, if f is a function of several variables $x = (x_{1,} \ldots, x_l)$, let us say that the component i of x is *influential* if there are two inputs x, x', differing only in the i-th place, such that $f(x) \neq f(x')$. Define $\pi_{ij}(M)$ to be the number of influential components of b in the price ratio function p_i/p_j. For example, from the expression for p_i/p_j for the star mechanism in the previous table, it is clear that $\pi_{ij}(M)$ is 4 unless one of i or j is m, in which case it is 2 We define the *price complexity* of M to be

$$\pi(M) = \max_{i \neq j} \pi_{ij}(M). \quad (6)$$

[6] If there is a continuum of traders (see Section 7), his own action has no affect on the price ratio. Otherwise it affects the aggregate offer and thereby the price ratio, which is but to be expected in an oligopolistic framework. In *either* case, equation (3) applies; and p_i/p_j is the exchange ratio between i and j.

[7] And, since he always knows his own offer, this is the same as asking: how many components does he need to know of the aggregate offer of the *others* ?

We now define a *quasiorder* $\preceq$ (reflexive and transitive) on $\mathfrak{M}(m)$ by

$$M \preceq M' \quad \Longleftrightarrow \quad \tau(M) \leq \tau'(M') \text{ and } \pi(M) \leq \pi'(M') \tag{7}$$

We are ready to state our main result[8].

Theorem 1 *If*[9] $m > 3$ *then the three special mechanisms are* precisely *the* $\preceq$*-minimal*[10] *elements of* $M(m)$*. Their complexities are as follows:*

	Star	*Cycle*	*Complete*
$\pi(M)$	4	2	$m(m-1)$
$\tau(M)$	2	$m-1$	1

This has the following immediate consequence.

Theorem 2 *Given any choice of strictly positive weights* $\lambda, \mu > 0$*, there exists an integer* m_0 *such that for* $m \geq m_0$ *the star mechanism is the* unique *minimizer in* $\mathfrak{M}(m)$ *of* $\lambda\pi(M) + \mu\tau(M)$*.*

Theorem 2 says that, so long as traders ascribe positive weight to *both* time and price complexity considerations, the star mechanism with money is the unique optimal mechanism as soon as the number of commodities is sufficiently large.

Remark 3 *In fact* m_0 *does not have to be too large. We only require* $4\lambda + 2\mu < 2\lambda + (m-1)\mu$ *and* $4\lambda + 2\mu < m(m-1)\lambda + \mu$ *for the star to beat the cycle and complete mechanisms, respectively; which may be rearranged*

$$m > 2\left(\frac{\lambda}{\mu}\right) + 3 \text{ and } m^2 - m > \frac{\mu}{\lambda} + 4$$

So, for example, if at least 10% *weight is accorded to both* π *and* μ*, then* λ/μ *and* μ/λ *can each be at most* 9 *and the above inequalities will hold if* $m > 18 + 3$ *and* $m^2 - m > 9 + 4$*; thus* $m_0 = 22$ *does the job.*

[8] A word about the numbering system used in this paper: all theorems, remarks, conditions, lemmas etc. are arranged in a *single* grand sequence. Thus the reader shall see, in order of appearance: Theorem 1, Theorem 2, Remark 3, Condition 4,.... This does *not* mean that Condition 4 is the fourth condition; in fact it is the first condition, but it has fourth place in the grand sequence (and, the marker 4 makes the remark easy to locate).

[9] When $m = 3$, we get a fourth mechanism with complexities $4, 2$ identical to the star mechanism. And when $m = 2$, we must change 4 to 2 in the table (the three graphs become identical with complexities $2, 2$ for each).

[10] M is said to be $\preceq$-*minimal* in $\mathfrak{M}(m)$ if there is no $M' \in \mathfrak{M}(m)$ for which $\tau(M') \leq \tau(M)$ and $\pi(M') \leq \pi(M)$, with strict inequality in at least one place.

3 Characterization of G-mechanisms

Our analysis above was carried out on the domain $\mathfrak{M}(m)$. We now show how to derive $\mathfrak{M}(m)$ from a more general standpoint. To this end, let us first define an *abstract* exchange mechanism on commodity set $\{1, \ldots, m\}$ and with trading opportunities given by a directed, connected graph G on $\{1, \ldots, m\}$. Such a mechanism allows individuals in $\{1, \ldots, n\}$ to trade by means of quantity offers in each commodity i across all edges ij in G. (Here m is fixed and n can be arbitrary.) The offer of any trader can thus be viewed as an $m \times m$ non-negative matrix in the space

$$S = \{a : a_{ij} = 0 \text{ if } ij \notin G,\, a_{ij} \geq 0 \text{ otherwise}\}$$

Define

$$S_+ = \{a \in S : a_{ij} > 0 \text{ if } ij \in G\}$$

Also define

$$\overline{a} = (\overline{a_1}, \ldots, \overline{a_m})$$

where $\overline{a_i} = \sum_j a_{ij}$ is the i-th row sum of a and denotes the total amount of commodity i involved in sending offer a_i. Let S^n be the n-fold Cartesian product of S with itself, and (with $\boldsymbol{a} = (\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n)$) let

$$S(n) = \left\{ \boldsymbol{a} \in S^n : \sum_{\alpha=1}^{n} \overline{\boldsymbol{a}}^\alpha \in S_+ \right\}$$

denote the n-tuples of offers that are positive on aggregate. Also let $C = \mathbb{R}^m_+$ denote the *commodity space*; and C^n its n-fold product.

An *exchange mechanism* M, for a given set $\{1, \ldots, m\}$ of commodities and with trading opportunities in accordance with the graph G, is a collection of maps (one for each positive integer n) from $S(n)$ to C^n such that, if $\boldsymbol{a} \in S(n)$ leads to returns $\mathbf{r} \in C^n$, then we have

$$\sum_{\alpha=1}^{n} \overline{\boldsymbol{a}}^\alpha = \sum_{\alpha=1}^{n} \boldsymbol{r}^\alpha,$$

i.e., there is *conservation of commodities*. It is furthermore understood, in keeping with our concept of opportunity ij, that for an offer $a \in S$ whose only non-zero components are $\{a_{ij} : j = \ldots\}$, the return will consist exclusively of commodity j.

We shall impose four conditions on the mechanisms which reflect "convenience" and "fairness" in trade. The first condition is that the mechanism must be blind to all other characteristics of a trader except for his offer (and rules out discrimination on irrelevant grounds):

Condition 4 (Anonymity) *Suppose $\boldsymbol{a} \in S(n)$ and $\boldsymbol{a}^{\alpha} = \boldsymbol{a}^{\beta}$. Let $\mathbf{r}$ denote the returns that accrue from $\boldsymbol{a}$. Then $\mathbf{r}^{\alpha} = \mathbf{r}^{\beta}$.*

The second condition is that if any trader pretends to be two different persons by splitting his offer, the returns to the others is unaffected. In its absence, traders would be faced with the complicated task of tracking everyone's offers. It is easier (and sufficient!) to state this condition for the "last" trader.

Condition 5 (Aggregation) *Suppose $\boldsymbol{a} \in S(n)$ and $\boldsymbol{b} \in S(n+1)$ are such that $\boldsymbol{a}^{\alpha} = \boldsymbol{b}^{\alpha}$ for $\alpha < n$ and $\boldsymbol{a}^{n} = \boldsymbol{b}^{n} + \boldsymbol{b}^{n+1}$. Let $\mathbf{r}$,$\mathbf{s}$ denote the returns that accrue from $\boldsymbol{a}$,$\boldsymbol{b}$ respectively. Then $\mathbf{r}^{\alpha} = \boldsymbol{s}^{\alpha}$ for $\alpha < n$.*

Anonymity and *Aggregation* immediately imply that, regardless of the size n of the population, the return to any trader may be written $r(a, b)$,where $a \in S$ is his own offer and $b \in S_{+}$ is the aggregate of all offers. Thus $\nu(a, b) = r(a, b) - \overline{a}$ denotes his *net trade*.

The third condition is *Invariance.* Its main content is that the *maps* which comprise M are invariant under a change of units in which commodities are measured. This makes the mechanism much simpler to operate in: one does not need to keep track of seven pounds or seven kilograms or seven tons, just the numeral 7 will do.

In what follows, we will consistently use a for an individual's offer and b for the positive aggregate offer; so, when we refer to the pair a, b it will be implicit that $a \in S$, $b \in S_{+}$ and $a \leq b$.

Condition 6 (Invariance) *$\nu(\lambda a, \lambda b) = \lambda \nu(a, b)$ for all a, b and any $m \times m$ strictly positive diagonal matrix λ.*

The fourth, and last, condition is that no trader can get strictly less than his offer (otherwise, such unfortunate traders would tend to abandon the mechanism).

Condition 7 (Non-dissipation) *If $\nu(a, b) \neq 0$, then $\nu_{i}(a, b) > 0$ for some component i.*

It turns out that these four conditions categorically determine a unique mechanism.

Theorem 8 *Let M be an exchange mechanism on commodity set $\{1, \ldots, m\}$ and let G be the (directed, connected) graph induced by the trading opportunities in M. If M satisfies* Anonymity*, Aggregation,* Invariance *and* Non-dissipation*, then $M = M_G$.*

3.0.1 Comments on the Conditions

Aggregation does not imply that if two individuals were to merge, they would be unable to enhance their "oligopolistic power". For despite the *Aggregation* condition, the merged individuals are free to *coordinate* their actions by jointly picking a point in the Cartesian product of their action spaces. Indeed all the mechanisms we obtain display this "oligopolistic effect", even though they also satisfy *Aggregation.*

It is worthy of note that the cuneiform tablets of ancient Sumeria, which are some of the earliest examples of written language and arithmetic, are in large part devoted to records and receipts pertaining to economic transactions. *Invariance* postulates the "numericity" property of the maps $r(a, b)$ (equivalently, $\nu(a, b)$) making them independent of the underlying choice of units, and this goes to the very heart of the quantitative measurement of commodities. In its absence, one would need to figure out how the maps are altered when units change, as they are prone to do, especially in a dynamic economy. This would make the mechanism cumbersome to use.

Non-dissipation (in conjunction with *Aggregation, Anonymity*, and the conservation of commodities) immediately implies *no-arbitrage:* for any a, b neither $\nu(a, b) \gneqq 0$ nor $\nu(a, b) \lneqq 0$. To check this, we need consider only the case $a \leq b$ and rule out $\nu(a, b) \gneqq 0$. Denote $c = b - a$. Then $\nu(a, b) + \nu(c, b) = \nu(a + c, b) = \nu(b, b) = 0$, where the first equality follows from *Aggregation*, and the last from conservation of commodities. But then $\nu(a, b) \gneqq 0$ implies $\nu(c, b) \lneqq 0$, contradicting *Non-dissipation.*

3.0.2 Alternative Characterizations of G-Mechanisms

The formula (3) for the return function of a G-mechanism immediately implies

$$p(b) = p(c) \Longrightarrow r(a, b) = r(a, c) \text{ for all } a \geq 0 \text{ and } b, c > 0 \qquad (8)$$

In [8], a mechanism was supposed to produce both trades and prices, based upon everyone's offers; and the property (8) was referred to as *Price Mediation.* It was shown in [8] that $\mathfrak{M}(m)$ is characterized by *Anonymity, Aggregation, Invariance, Price Mediation* and *Accessibility* (the last representing a weak form of continuity). An alternative characterization of $\mathfrak{M}(m)$, which assumes – as we do here – that a mechanism produces only trades (and no prices), was given in [9]. Here we have presented a simplified version of the analysis in [9], and established that M_G arises "naturally" *once* we assume that trading opportunities are restricted to pairwise exchange of commodities, i.e., correspond to the edges of a connected graph G. In contrast, in both [8] and [9], the opportunity structure G was *itself* an object of deduction, starting from a more abstract viewpoint.

3.1 Related Literature

The need for money in an exchange mechanism has been a topic of much discussion. We give a brief synopsis. (For a much fuller survey, see [35] and [36].)

Several search-theoretic models, involving random bilateral meetings between long-lived agents, have been developed following Jevons [17] (see, *e.g.*, [2], [16], [18], [19], [20], [21], [23], [38] and the references therein). These models turn on utility-maximizing behavior and beliefs of the agents in Nash equilibrium, and shed light on which commodities are likely to get adopted as money. A parallel, equally distinctive, strand of literature builds on partial or general equilibrium models with other kinds of frictions in trade, such as limited trading opportunities in each period, or transaction costs (see, *e.g.*, [11], [12], [13], [14], [15], [24], [25], [36], [37], [39]). In many of these models, a specific trading mechanism is exogenously fixed, and the focus is on activity within the mechanism that is induced by equilibrium, based again on the optimal behavior of utilitarian individuals.

Our approach complements this literature in two salient ways, and brings to light a new rationale for money that is different from those proposed earlier, but not inimical to them, in that the door is left fully open to incorporate their concerns within our framework. First, as we have emphasized, our focus is purely on mechanisms of trade with no regard to the characteristics of the individuals such as their endowments, production technologies, preferences or beliefs. Second, no specific trading mechanism is specified *ex-ante* by us. We start with a welter of mechanisms and cut them down by our four conditions and by complexity considerations, ultimately ending up with the star mechanism.

The model we present builds squarely upon [8], which provided an axiomatic characterization of the finite set of "G-mechanisms" (see section 2), bridging the gap between the Shapley-Shubik model of decentralized “trading posts”, i.e., the star mechanism (see [31], [32], [33]) and the Shapley model of centralized “windows”, i.e., the complete mechanism (see [30]). Various strategic market games, based upon trading posts, have been analyzed, with commodity or fiat money in [5], [26], [27], [28], [31], [32], [33], [34]; most of these papers also discuss the convergence of Nash equilibria (NE) to Walras equilibria (WE) under replication of traders. For a continuum-of-traders version, with details on explicit properties of the commodity money (its distribution and desirability) or of fiat money (its availability and the harshness of default penalties), under which we obtain equivalence (or near-equivalence) of NE and WE, see [7], [10]; and, for an axiomatic approach to the equivalence phenomenon, see [6].

Strategic market games differ in a fundamental sense from the Walras equilibrium model, despite the equivalence of NE and WE. In the WE framework, agents always optimize generating supply and demand, but markets do not clear except at equilibrium. We are left in the dark as to what happens outside of equilibrium. In sharp contrast markets always clear, producing prices and trades based on agents' strategies, in the market games; but agents do not optimize except at equilibrium. The very formulation of a game demands that the “game form”, *i.e.*, the map from strategies to outcomes, must be defined prior to the introduction of agents' preferences on outcomes; thus disentangling the physics of trade from its psychology. Our mechanisms are firmly in this genre, and indeed form the bases upon which many market games are built. To be precise: game forms arise from our mechanisms by introducing private endowments, along with the constraints that these impose on individuals' offers; and strategic market games then arise by further introducing preferences.

4 Proofs

4.1 Graphs with complexity ≤ 4

Let G be a connected graph on $\{1, \ldots, m\}$ as in section 2, and write

$$p_i(G) = p_i(M_G), \quad p_{ij}(G) = p_{ij}(M_G) \text{ and } \pi(G) = \pi(M_G)$$

If G consists of a single vertex then $\pi(G) = 0$ by definition.

Lemma 9 *If G is a cycle then $\pi(G) = 2$.*

Proof. Each vertex i in a cycle has a unique outgoing edge, and we denote its weight by[11] a_i. For each i we have $p_i = b_G/a_i$ where $b_G = \prod_{ij \in G} b_{ij} = \prod_i a_i$ as in (2); hence $p_i/p_j = a_j/a_i$ and the result follows. ■

By a *chorded cycle* we mean a graph that is a union $G = C \cup P$ where C is a cycle and P, the chord, is a path that connects two distinct vertices of C, but which is otherwise disjoint from C.

Lemma 10 *If $G = C \cup P$ is a chorded cycle then $\pi(G) = 4$.*

Proof. Let i be the initial vertex of the path P, then i has two outgoing edges, ij and ik say, on the cycle and path respectively. Any vertex $l \neq i$ has a unique outgoing edge, and we denote its weight by a_l as before. Let x be the terminal vertex of the path P. If $x = j$ then G has two j-trees, otherwise there is a unique j-tree; similarly if $x = k$ then there are two k-trees, otherwise there is a unique k-tree. Thus we get the following table:

	$x = j$	$x = k$	$x \neq j, k$
p_j/b_G	$a_j^{-1}\left(b_{ik}^{-1} + b_{ij}^{-1}\right)$	$a_j^{-1}b_{ik}^{-1}$	$a_j^{-1}b_{ik}^{-1}$
p_k/b_G	$a_k^{-1}b_{ij}^{-1}$	$a_k^{-1}\left(b_{ik}^{-1} + b_{ij}^{-1}\right)$	$a_k^{-1}b_{ij}^{-1}$

In every case, the ratio p_j/p_k depends on all 4 variables a_j, a_k, b_{ij}, b_{ik}, thus $\pi(G) \geq 4$.

On the other hand, since all vertices other than i have a unique outgoing edge, it follows that if x is any vertex then every x-tree contains all the outgoing edges except perhaps the edges b_{ij}, b_{ik} and a_x (if $x \neq i$); thus p_x is divisible by all other weights. It follows that for any two vertices x, y the ratio p_x/p_y can only depend on the variables b_{ij}, b_{ik}, a_x, a_y. Thus we get $\pi(G) \leq 4$ and hence $\pi(G) = 4$ as desired. ■

[11] This is a departure from our convention heretofore that a shall refer to an individual's offer, and b to the aggregate offer; but there should be no confusion.

Remark 11 *A special case of a chorded cycle is a graph T_0 with three vertices that we call a chorded triangle.*

3		
$\uparrow\downarrow$	$\nwarrow$	
1	$\longrightarrow$	2

p_1	$b_{23}b_{31}$
p_2	$b_{12}b_{31}$
p_3	$b_{23}\left(b_{12}+b_{13}\right)$

p_1/p_2	b_{23}/b_{12}
p_2/p_3	$b_{12}b_{31}/b_{23}\left(b_{12}+b_{13}\right)$
p_3/p_1	$\left(b_{12}+b_{13}\right)/b_{31}$

For future use we note that for each index j there is an i such that $\pi_{ij} \geq 3$.

By a *k-rose* we mean a graph that is a union $C_1 \cup \cdots \cup C_k$, where the C_i are cycles that share a single vertex j, but which are otherwise disjoint. Thus a 0-rose is a single vertex and a 1-rose is a cycle. If G is a k-rose for some $k \geq 2$ then we will simply say that G is a *rose*.

If each cycle in a rose G has exactly two vertices, *i.e.,* is a bidirected edge, then we say that G is a *star*.

Lemma 12 *If G is a rose then $\pi(G) = 4$.*

Proof. Let G be the union of cycles $C_1 \cup \cdots \cup C_k$ with common vertex j as above. Let $a_1, \ldots, a_k$ be the weights of the outgoing edges from j in cycles $C_1, \ldots, C_k$ respectively, and for all other vertices x let b_x denote the weight of the unique outgoing edge at x. It is easy to see that there for each vertex v of G there is a unique v-tree, and thus the price vectors are given as follows:

$$p_j = \prod_{x \neq j} b_x, \quad p_x = \frac{a_i p_j}{b_x} \text{ if } x \neq j \text{ is a vertex of } C_i$$

Thus we get

$$p_j/p_x = b_x/a_i, \quad p_y/p_x = b_x a_l / b_y a_i \text{ if } y \neq j \text{ is a vertex of } C_l$$

Taking $i \neq l$, we see that p_y/p_x depends on 4 variables, and $\pi(G) = 4$. ■

Our main result is a classification of connected graphs with $\pi(G) \leq 4$.

Theorem 13 *If G is not a chorded cycle or a k-rose, then $\pi(G) \geq 5$.*

We give a brief sketch of the proof of this theorem, which will be carried out in the rest of this section. The actual proof is organized somewhat differently, but the main ideas are as follows.

We say that a graph H is a *minor* of G, if H can be obtained from G by removing some edges and vertices, and collapsing certain kinds of edges. Our first key result is that the property $\pi(G) \leq 4$ is a *hereditary* property, in the sense that connected minors of such graphs also satisfy the property. The usual procedure for studying a hereditary property is to identify the *forbidden minors*, namely a set Γ of graphs such that G fails to have the property iff it contains one of the graphs from Γ. We identify a finite collection of such graphs. The final step is to show that if G is not a chorded cycle or a k-rose then it contains one of the forbidden minors.

We note the following immediate consequence of the results of this section.

Corollary 14 *If G is not a cycle then $\pi_{ij}(G) \geq 4$ for some ij.*

4.2 Subgraphs

Throughout this section G denotes a connected graph. We say that a graph H is a *subgraph* of G if H is obtained from G by deleting some edges and vertices.

Proposition 15 *If G' is a connected subgraph of G then $\pi(G) \geq \pi(G')$.*

Proof. For a vertex i in G' let p'_i and p_i denote its price in G' and G respectively; we first relate p'_i to a certain specialization of p_i.

Let E, E' be the edge sets of G, G' respectively, and let E_0 (resp. E_1) denote the edges in $E \setminus E'$ whose source vertex is inside (resp. outside) G'. Let $\bar{p}_i$ be the specialization of p_i obtained by setting the edge weights in E_0 and E_1 to 0 and 1 respectively. Then we claim that

$$p'_i = |F|\,\bar{p}_i, \tag{9}$$

where F is the set of directed forests ϕ in G such that

1. the root vertices of ϕ are contained in G',

2. the non-root vertices of ϕ consist of *all* G-vertices not in G'.

Indeed, consider the expression of p_i as a sum of i-trees in G. The specialization $\bar{p}_i$ assigns zero weight to all trees with an edge from E_0. The remaining i-trees in G are precisely of the from $\tau \cup \phi$ where τ is an i-tree in G' and $\phi \in F$, and these get assigned weight $wt(\tau)$. Formula (9) is an immediate consequence.

Now if i, j are vertices in G', then formula (9) gives

$$\frac{p'_i}{p'_j} = \frac{\bar{p}_i}{\bar{p}_j}$$

Thus the ij price ratio in G' is obtained by a *specialization* of the ratio in G. Consequently the former cannot involve *more* variables. Taking the maximum over all i, j we get $\pi(G) \geq \pi(G')$ as desired. ∎

4.3 Collapsible edges

We write $out(k)$ for the number of outgoing edges at the vertex k. In a connected graph we have $out(k) \geq 1$ for all vertices, and we will say k is *ordinary* if $out(k) = 1$ and *special* if $out(k) > 1$. Among special vertices, we will say that k is *binary* if $out(k) = 2$ and *tertiary* if $out(k) = 3$.

Definition 16 *We say that an edge ij of a graph G is* collapsible *if*

1. *i is an ordinary vertex*
2. *ji is not an edge of G*
3. *there is no vertex k such that ki and kj are both edges of G.*

Definition 17 *If G has no collapsible edges we will say G is* rigid.

If G is a connected graph with a collapsible edge ij, we define the *ij-collapse* of G to be the graph G' obtained by deleting the vertex i and the edge ij, and replacing any edges of the form li with edges lj. The assumptions on ij imply that the procedure does not introduce any loops or double edges, hence G' is also simple (and connected). Moreover each vertex $k \neq i$ has the same outdegree in G' as in G.

Lemma 18 *If G' is the ij-collapse of G as above, then $\pi(G) \geq \pi(G')$.*

Proof. Let k be any vertex of G' then k is also a vertex of G. Since i is ordinary every k-tree in G must contain the edge ij; collapsing this edge gives a k-tree in G' and moreover every k-tree in G' arises uniquely in this manner. Thus we have a factorization

$$p_k(G) = a_{ij} p_k(G').$$

Thus for any two vertices k, l of G' we get $p_k(G)/p_l(G) = p_k(G')/p_l(G')$ and the result follows. ■

We will say that H is a *minor* of G if it is obtained from G by a *sequence* of steps of the following kind: a) passing to a connected subgraph, b) collapsing some collapsible edges. By Proposition 15 and Lemma 18 we get

Corollary 19 *If H is a minor of G then $\pi(H) \leq \pi(G)$.*

4.4 Augmentation

Throughout this section G denotes a connected graph.

Notation 20 *We write $H \trianglelefteq G$ if H is a* connected *subgraph of G, and write $H \triangleleft G$ to mean $H \trianglelefteq G$ and $H \neq G$.*

We say that $H \triangleleft G$ can be *augmented* if there is a path P in G whose endpoints are in H, but which is otherwise completely disjoint from H. We refer to P as an augmenting path of H, and to $K = H \cup P$ as an augmented graph of H; note that K is also connected, *i.e.* $K \trianglelefteq G$. It turns out that augmentation is always possible.

Lemma 21 *If $H \triangleleft G$ then H can be augmented.*

Proof. If G and H have the same vertex set then any edge in $G \setminus H$ comprises an augmenting path. Otherwise consider triples (k, P_1, P_2) where k is a vertex not in H, P_1 is a path from some vertex in H to k, and P_2 is a path from k to some vertex in H. Among all such triples choose one with $e(P_1) + e(P_2)$ as small as possible. Then P_1 and P_2 cannot share any *intermediate* vertices with H or with each other, else we could construct a smaller triple. It follows that $P = P_1 \cup P_2$ is an augmenting path. ■

We are particularly interested in augmenting paths for H that consist of one or two edges; we refer to these as *short augmentations* of H.

Corollary 22 *If $H \triangleleft G$ then G has a minor that is a short augmentation of H.*

Proof. Let $K = H \cup P$ be an augmentation of H. If P has more than two edges, then we may collapse the first edge of P in K. The resulting graph is a minor of G, which is again an augmentation of H. The result follows by iteration. ■

Lemma 23 *If $K = H \cup P$ with $P = \{jk, kl\}$, then for any vertex i of H we have $\pi_{ik}(K) = \pi_{ij}(H) + 2$.*

Proof. The edges (j, k) and (k, l) are the unique incoming and outgoing edges at k. It follows that every i-tree in K is obtained by adding the edge kl to an i-tree in H, and every k-tree in K is obtained by adding the edge jk to a j-tree in H. Thus if a_{jk} and a_{kl} are the respective weights of the two edges in the path P then we have

$$p_i(K) = a_{kl} p_i(H), p_k(K) = a_{jk} p_j(H) \implies \frac{p_i(K)}{p_k(K)} = \frac{a_{kl}}{a_{jk}} \frac{p_i(H)}{p_j(H)}$$

Thus the price ratio in question depends on two additional variables, and the result follows. ■

Corollary 24 *If G contains the chorded triangle T_0 as a proper subgraph then $\pi(G) \geq 5$.*

Proof. By Corollary 22, G has a minor $K = T_0 \cup P$, which is a short augmentation of T_0, and it is enough to show that $\pi(K) \geq 5$. If P consists of two edges $\{jk, kl\}$ then by Remark 11 we can choose i such that $\pi_{ij}(T_0) = 3$; now by Lemma 23, we have $c_{ik}(K) = 5$ and hence $\pi(K) \geq 5$. If P consists of a single edge then K is necessarily as below, and once again $\pi(K) \geq 5$.

2		
↑↓	↘	
1	⇆	3

p_1/p_3
$\dfrac{b_{31}(b_{21} + b_{23})}{b_{23}b_{12} + b_{23}b_{13} + b_{21}b_{13}}$

■

4.5 The circuit rank

As usual G denotes a simple connected graph, and we will write $e(G)$ and $v(G)$ for the numbers of edges and vertices of G.

Definition 25 *The circuit rank of G is defined to be*

$$c(G) = e(G) - v(G) + 1$$

The circuit rank is also known as the *cyclomatic number*, and it counts the number of independent cycles in G, see *e.g.* [3].

Example 26 *If G is a k-rose then $c(G) = k$, and if G is a chorded cycle then $c(G) = 2$.*

We now prove a crucial property of $c(G)$.

Proposition 27 *If $H \lhd G$ then there is some $K \trianglelefteq G$ such that $H \lhd K$ and $c(K) = c(H) + 1$.*

Proof. Let $K = H \cup P$ be an augmentation of H. If P consists of m edges, then K has $e(H) + m$ edges and $v(H) + m - 1$ vertices; hence $c(K) = c(H) + 1$. ■

Corollary 28 *Let G be a connected graph.*

1. *If $H \lhd G$ then $c(H) < c(G)$.*
2. *$c(G) = 0$ iff G is a single vertex.*
3. *$c(G) = 1$ iff G is a cycle.*
4. *$c(G) = 2$ iff G is a chorded cycle or a 2-rose.*

Proof. The first part follows from Proposition 27, the other parts are completely straightforward. ■

Lemma 29 *If G is not a rose and $c(G) > 3$, then there is some $K \lhd G$ such that K is not a rose and $c(K) = 3$.*

Proof. Let R be a k-rose in G with $c(R) = k$ as large as possible, then $R \lhd G$ by assumption. If $c(R) \leq 2$ then any $K \lhd G$ with $c(K) = 3$ is not a rose. Thus we may assume that $c(R) > 2$, and in particular R has a unique special vertex i and at least three loops. Since $R \neq G$, R can be augmented, and $S = R \cup P$ is an augmentation, then P cannot both begin and end at i, else $R \cup P$ would be a rose, contradicting the maximality of R. Since there are at most two endpoints of P, we can choose two *distinct* loops L_1 and L_2 of R, such that $L_1 \cup L_2$ contains these endpoints of P. Then $K = L_1 \cup L_2 \cup P$ is the desired graph. ■

4.6 Covered vertices

Definition 30 *Let i be an ordinary vertex of G with outgoing edge ij. We say that a vertex k* covers *i, if one of the following holds:*

1. *the edges ki and kj belong to G*

2. *$j = k$ and the edge ki belongs to G*

If there is no such k then we say that i is an uncovered *vertex.*

We emphasize that the terminology covered/uncovered is only applicable to ordinary vertices in a graph G. The main point of this definition is the following simple observation.

Remark 31 *An ordinary vertex is uncovered iff its outgoing edge is collapsible.*

Lemma 32 *Suppose G is a connected graph .*

1. *If $v(G) \geq 3$ then an ordinary vertex cannot cover another vertex.*

2. *If $v(G) \geq 4$ then a binary vertex can cover at most one vertex.*

3. *A tertiary vertex can cover at most three vertices.*

4. *If G is a rigid graph with $c(G) = 3$, then $v(G) \leq 4$.*

Proof. If k is an ordinary vertex covering i then G must contain the edges ki and ik. Thus i and k do not have any other outgoing edges, and if G has a third vertex j then there is no path from k or i to j, which contradicts the connectedness of G, thereby proving the first statement.

If k is a binary vertex covering the ordinary vertices i and j then G must contain the edges ki, kj, ij, ji. The vertices i, j, k cannot have any other outgoing edges, so a fourth vertex would contradict the connectedness of G as before. This proves the second statement.

If a vertex k covers i then there must be an edge from k to i. Thus if $out(k) = 3$ then k can cover at most three vertices.

If $c(G) = 3$ then G has either 2 binary vertices or 1 tertiary vertex, with the remaining vertices being ordinary. If $v(G) > 4$ then by previous two paragraphs G would have an uncovered vertex, which is a contradiction. ■

4.7 Proof of Theorem 13

Proposition 33 *If $c(G) \geq 3$ and G is not a rose, then $\pi(G) \geq 5$.*

Proof. By Proposition 15 and Lemma 29 we may assume that $c(G) = 3$. By Lemma 18, we may further assume that G is rigid, and thus by Lemma 32 that $v(G) \leq 4$. We now divide the argument into three cases.

First suppose that G contains a 3-cycle C. We claim that at least one of the edges of C must be a bidirected edge in G, so that G properly contains a chorded triangle T_0, whence $\pi(G) \geq 5$ by Corollary 24. Indeed if G has no other vertices outside C, then G must have 5 edges and 3 vertices and the claim is obvious. Thus we may suppose that there is an outside vertex l. We further claim that C contains two vertices i, j such that i covers j. Granted this, it is immediate that G contains either the bidirected edge ij and ji, or the bidirected edge jk and kj where k is the third vertex of C. To prove the "further" claim we note that the special vertices of G consist of either a) one tertiary vertex, or b) two binary vertices. In case a) the connectedness of G implies that the tertiary vertex must be in C, and hence it must cover both the ordinary vertices in C. In case b) either C contains both binary vertices, one of which must cover the unique ordinary vertex of C; or C contains one binary vertex, which must cover one of the two ordinary vertices of C.

Next suppose that G does not contain a 3-cycle, but does contain a 4-cycle labeled 1234, say. Now G has two additional edges, which cannot be the diagonals $13, 31, 24, 42$, since otherwise G would have a 3-cycle; therefore G must have two bidirected edges. The bidirected edges cannot be adjacent else G would have a collapsible vertex, therefore G must be the first graph below, which has $\pi(G) \geq 5$.

2	$\longrightarrow$	3
$\uparrow\downarrow$		$\uparrow\downarrow$
1	$\longleftarrow$	4

p_1/p_3
$\dfrac{b_{21}b_{34}b_{41}}{b_{23}b_{12}\left(b_{41}+b_{43}\right)}$

2	$\rightleftarrows$	3
$\uparrow\downarrow$		$\uparrow\downarrow$
1		4

p_1/p_4
$\dfrac{b_{21}b_{32}b_{43}}{b_{34}b_{23}b_{12}}$

Finally suppose G has no 3-cycles or 4-cycles. Then every edge must be a bidirected edge, and G must be a tree with all bidirected edges. Since G is not a star, this only leaves the second graph above, which has $\pi(G) \geq 6$. ■

We can now finish the proof of Theorem 13.

Proof of Theorem 13. If $c(G) \leq 2$ then, by Corollary 28, G is a single vertex, a cycle, chorded cycle or a 2-rose. If $c(G) \geq 3$ then the result follows by Proposition 33. ■

5 Proof of Theorem 1

In this section, after a couple of preliminary results, we apply Theorem 13 to prove Theorem 1.

Lemma 34 *If G is a chorded cycle on 4 or more vertices, then $\tau(G) \geq 3$.*

Proof. We can express G as a union of two paths P, Q from 1 to 2, say and a third path R from 2 to 1. At least one of the first two paths, say P must have an intermediate vertex, say 3. Since $m \geq 4$ there is an additional intermediate vertex 4 on one of the paths.

If $m = 4$ then we get three possible graphs depending on the location of the vertex 4.

$$\begin{array}{|ccc|}\hline 3 & \rightarrow & 4 \\ \uparrow & & \downarrow \\ 1 & \leftrightarrows & 2 \\ \hline \end{array} \qquad \begin{array}{|ccc|}\hline 3 & \rightarrow & 2 \\ \uparrow & \nearrow & \downarrow \\ 1 & \leftarrow & 4 \\ \hline \end{array} \qquad \begin{array}{|ccc|}\hline 3 & \rightarrow & 2 \\ \uparrow & \swarrow & \uparrow \\ 1 & \rightarrow & 4 \\ \hline \end{array}$$

For these graphs we have $\tau_{24} = 3, \tau_{42} = 3$ and $\tau_{34} = 3$, respectively. Thus $\tau(G) \geq 3$ in all three cases.

If $m > 4$ then G can be realized as one of these graphs, albeit with additional intermediate vertices on one or more of the paths P, Q, R. These additional vertices are ordinary uncovered vertices, with collapsible outgoing edges. Collapsing one of these edges does not increase time complexity, and produces a smaller chorded cycle G'. Arguing by induction on m we conclude $\tau(G) \geq \tau(G') \geq 3$. ■

Lemma 35 *If G is the complete graph, then $\pi_{ij}(G) = m(m-1)$ for all $i \neq j$.*

Proof. Fix a pair of vertices $i \neq j$ in G. Then we claim that the price ratio $p_{ij}(G)$ depends on each of the $m(m-1)$ edge weights b_{kl}. Indeed if H is any "spanning" connected subgraph of G then $p_{ij}(H)$ is obtained from $p_{ij}(G)$ by specializing to 0 the weights of all edges outside H. Therefore it suffices to find a connected subgraph H such that $p_{ij}(G)$ depends on b_{kl}.

We consider two cases. If $\{i, j\} = \{k, l\}$ then exchanging i, j if necessary we may assume $i = k, j = l$. Let H be an m-cycle two of whose edges are ij and hi (say); then $p_i/p_j = b_{hi}/b_{ij}$ depends on $b_{kl} = b_{ij}$.

If $\{i, j\} \neq \{k, l\}$ then let H be an 2-rose with loops C_1 and C_2 such that

1. k is the special vertex, and kl is an edge in C_1
2. i belongs to C_1 and j belongs to C_2

Then p_i and p_j are each given by unique directed trees T_i and T_j. Moreover T_i involves kl while T_j does not. Hence $p_{ij}(H)$ depends on b_{kl}. ■

Proof of Theorem 1. Let $\mathfrak{S}$ denote the set consisting of the three special mechanisms: star, cycle and complete. We need to show that $\mathfrak{M}_{\preceq} = \mathfrak{S}$, where $\mathfrak{M}_{\preceq}$ denotes the set of $\preceq$-minimal elements of $\mathfrak{M} = \mathfrak{M}(\mathfrak{m})$.

Let us say that G is a minimal graph if M_G is a minimal mechanism of $\mathfrak{M}$. Now the star mechanism has complexity $(\tau, \pi) = (2, 4)$. Therefore if G is any minimal graph then either $\tau(G) = 1$ or $\pi(G) \leq 4$. For $\tau(G) = 1$ we get the complete graph, which has complexity $(\tau, \pi) = (1, m(m-1))$ by Lemma 35. The graphs with $\pi(G) \leq 4$ are characterized by Theorem 13, and we have three possibilities for G.

1. *Chorded cycle.* In this case we have $(\tau, \pi) = (3^+, 4)$ by Lemma 34, and so G is not minimal.
2. *Cycle.* In this case we have $(\tau, \pi) = (m-1, 2)$ by Lemma 9.
3. *k-rose,* $k \geq 2$. If each petal of G has exactly 2 edge then G is the star mechanism. Otherwise after collapsing edges, we obtain the following minor with $\tau_{12} = 3$

$$\boxed{\begin{array}{lllll} 1 & & & & \\ \downarrow & \nwarrow & & & \\ \cdot & \rightarrow & \cdot & \leftrightarrows & 2 \end{array}}$$

   Thus G has complexity $(\tau, \pi) = (3^+, 4)$ and so is not minimal.

Thus the three graphs in the statement of Theorem 1 are the only *possible* minimal graphs, and have the indicated complexities. Since they are incomparable with each other, each is minimal. Thus we conclude $\mathfrak{M}_{\preceq} = \mathfrak{S}$ as desired. ■

Remark 36 *For $m = 3$, Lemma 34 does not hold and we have an additional strongly minimal mechanism with $(\tau, \pi) = (2, 4)$, namely the chorded triangle*

$$\boxed{\begin{array}{lll} \cdot & & \\ \downarrow & \nwarrow & \\ \cdot & \leftrightarrows & \cdot \end{array}}$$

6 Proof of Theorem 8

Note that a mechanism is determined uniquely by its *net trade* function $\nu(a,b) := r(a,b) - \overline{a}$ which, although initially defined for $a \leq b$, admits a natural extension as follows.

Proposition 37 *The function ν admits a unique extension to $S \times S_+$ satisfying*

$$\nu(\lambda a + \lambda' a', b) = \lambda \nu(a,b) + \lambda' \nu(a', b), \quad \nu(a, \lambda b) = \nu(a,b) \text{ for } \lambda, \lambda' > 0$$

Proof. Since $\nu(a,b) := r(a,b) - \overline{a}$, it suffices to show

$$r(\lambda a + \lambda' a', b) = \lambda r(a,b) + \lambda' r(a', b), \quad r(a, \lambda b) = r(a,b) \text{ for } \lambda, \lambda' > 0 \quad (10)$$

But this is just Lemma 1 of [8], whose proof we now reproduce for the sake of completeness.

First observe that, by the conservation of commodities, $r(a,b) \leq \overline{b}$ for all $a \leq b$; moreover if a and a' in S are such that $a + a' \leq b$, then *Aggregation* implies the functional (Cauchy) equation $r(a + a', b) = r(a,b) + r(a', b)$.

From Corollary 2 in [1] we conclude that, for all non-negative λ and λ' such that $\lambda a + \lambda' a' \leq b$, the first inequality of (10) holds.

Next let $a \leq b$ and choose $\lambda \geq 1$. Then the argument just given shows that $r(\lambda a, \lambda b) = \lambda r(a, \lambda b)$. On the other hand, *Invariance* implies that the left side equals $\lambda(a,b)$. Comparing these expressions we obtain the second inequality of (10).

Thus even for a *not* less than b, we may define $r(a,b)$ via (10) by choosing λ sufficiently large. This extends r to all of $S \times S_+$. ∎

In view of the above result, we drop the restriction $a \leq b$ when considering $\nu(a,b)$.

The net trade vector can have negative and positive components, and hence belongs to $\mathbb{R}^m$. The next definition pertains to such vectors in $\mathbb{R}^m$.

Definition 38 *By an i-vector, we mean a vector whose ith component is positive and all other components are zero. By an $\bar{\imath}j$-vector we mean a vector that has a negative i-component, a positive j-component and zeros in all other components.*

Proposition 39 *For $b \in S_+$ and any $i \neq j$ there is $a \in S$ such that $\nu(a,b)$ is an $\bar{\imath}j$-vector.*

Proof. Since the graph G underlying the mechanism is connected, there is a directed path from i to j. Denote the nodes on the path by $i = 1, \ldots, t = j$. Let w^1 be an i-vector which can be offered on edge 12 to get a return $w^2 \neq 0$ consisting only of commodity 2 (here $w^2 \neq 0$ by *Non-dissipation*); then w^2 can be offered on edge 23 to get $w^3 \neq 0$ consisting only of commodity 3, and so on. This yields a sequence $w^1, \ldots, w^t$ such that

$$w^i + \nu\left(w^i, b\right) = w^{i+1} \text{ for } i = 1, \ldots, t-1$$

If $w = \sum w^i$ then by Proposition 37 we have

$$\nu(w, b) = \sum \nu\left(w^i, b\right) = w^t - w^1$$

which is an $\bar{\imath}j$-vector. ■

It will be convenient to write an $\bar{\imath}j$-vector in the form $(-x, y)$ after suppressing the other components. In the context of the above proposition if $\nu(a, b) = (-x, y)$ then by linearity $\nu(a/x, b) = (-1, y/x)$, and we will say that the offer a (or a/x) achieves an *ij-exchange ratio* of y/x at b.

Proposition 39 shows that there exists at least one offer a to achieve an $\bar{\imath}j$-vector in trade, at any given b. But a is by no means unique. There may be many paths from i to j, along which i can be exchanged exclusively for j; and, also, there may be more complicated trading strategies, that use edges no longer confined to any single path, to accomplish such an exchange. These could give rise to offers different from a and yield (for the fixed aggregate b) other $\bar{\imath}j$-vectors in trade. But, as the following lemma shows, the *same* exchange ratio obtains under all circumstances.

Lemma 40 *If a', a'' achieve ij-exchange ratios α', α'' at b, then $\alpha' = \alpha''$.*

Proof. By Proposition 39 there exists an a such that $\nu(a, b)$ is a $\bar{\jmath}i$-vector; if α is the corresponding exchange ratio then by rescaling a, a', a'' we may assume that

$$\nu(a, b) = (1, -\alpha), \nu(a', b) = (-1, \alpha'), \nu(a'', b) = (-1, \alpha'').$$

By Proposition 37 we get

$$\nu(a + a', b) = (0, \alpha - \alpha')$$

Now by *Non-dissipation* we get $\alpha \geq \alpha'$, and exchanging the roles of i and j we conclude that $\alpha' \geq \alpha$ and hence that $\alpha = \alpha'$. Arguing similarly we get $\alpha = \alpha''$ and hence that $\alpha' = \alpha''$ ■

Lemma 41 *Denote the net trade function of* M *by* ν*. Then there is a unique map* $p : \mathbb{R}^K_{++} \to \mathbb{R}^m_{++}/\sim$ *satisfying* $p(b) \cdot \nu(a, b) = 0$.

Proof. Fix $b \in S_+$ and consider the vector

$$p = (1, p_2, \ldots, p_m)$$

where p_j^{-1} is the $1j$-exchange ratio at b, as in Lemma 40. We will show that p satisfies the budget balance condition, *i.e.* that

$$p \cdot \nu(a, b) = 0 \text{ for all } a. \tag{11}$$

We argue by induction on the number $d(a, b)$ of non-zero components of $\nu(a, b)$ in positions $2, \ldots, m$. If $d(a, b) = 0$ then $\nu(a, b) = 0$ by *Non-dissipation* and (11) is obvious. If $d(a, b) = 1$ then $\nu(a, b)$ is either an $\bar{1}j$-vector or a $\bar{j}1$ vector, which by the definition of p_j and Lemma 40 is necessarily of the form

$$\left(-x, xp_j^{-1}\right) \text{ or } \left(x, -xp_j^{-1}\right);$$

for such vectors (11) is immediate. Now suppose $d(a, b) = d > 1$ and fix j such that $\nu_j(a, b) \neq 0$. Then we can choose a' such that $\nu(a', b)$ is a $\bar{1}j$ or a $\bar{j}1$- vector such that $\nu_j(a, b) = -\nu_j(a', b)$. It follows that $d(a + a', b) < d$ and by linearity we get

$$p \cdot \nu(a, b) = p \cdot \nu(a + a', b) - p \cdot \nu(a', b).$$

By the inductive hypothesis the right side is zero, hence so is the left side.

Finally the uniqueness of the price function is obvious, because the return function of the mechanism dictates how many units of j may be obtained for one unit of i, yielding just one possible candidate for the exchange rate for every pair ij. ■

We can now prove Theorem 8

Proof. (of Theorem 8) To prove that $M = M_G$ it is enough to show that p and r satisfy (1) and (3).

Let us write, as before,

$$b = \sum a^\alpha, p = p(b) \text{ and } \nu(a, b) = r(a, b) - \overline{a}.$$

Consider replacing trader α by m traders $\alpha_1, \ldots, \alpha_m$, where trader α_j makes only the offers $\left\{a^\alpha_{ij} : 1 \leq i \leq m\right\}$ in a^α that entitle α to the return of commodity j. By *Aggregation* this will have no effect on traders other than α;

and hence α_j will get precisely the return $r_j(a^\alpha, b)$. By Lemma 41, applied to each such trader α_j, we have

$$p_j r_j(a^\alpha, b) = \sum_i p_i a_{ij}^\alpha \tag{12}$$

which is just (3).

Now (1) follows by summing (12) over all α. ■

7 A Continuum of Traders

Our analysis easily extends to the case where the set of individuals T is the unit interval $[0, 1]$, endowed with a nonatomic population measure [12]. Let $\mathcal{S}$ denote the collection of all integrable functions $\boldsymbol{a} : T \mapsto S$ such that $\int_T \boldsymbol{a} \in S_+$. (An element of $\mathcal{S}$ represents a choice of offers by the traders in T which are positive on aggregate.) In the same vein, let $\mathcal{R}$ denote the collection of all integrable functions from T to C, whose elements $\mathbf{r} : T \mapsto C$ represent returns to T. An *exchange mechanism* M, on a given set of m commodities, is a map from $\mathcal{S}$ to $\mathcal{R}$ such that, if M maps $\boldsymbol{a}$ to $\mathbf{r}$ then we have (reflecting conservation of commodities):

$$\int_T \boldsymbol{a} = \int_T \mathbf{r}$$

We wrap the *Aggregation* and *Anonymity* conditions into one, and directly postulate that the return to any individual depends only on his own offer and the integral of everyone's offers, and that this return function is the same for everyone. Thus we have a function r from $S \times S_+$ to C such that $\mathbf{r}(t) = r(a, b)$, where $a = \mathbf{a}(t)$ and $b = \int_T \boldsymbol{a}$. The following lemma is essentially from [6].

Proposition 42 *$r(a, b)$ is linear in a (for fixed b) and $r(a, \lambda b) = r(a, b)$ for any a, b and positive scalar λ.*

Proof. We will first show that if $a, c \in S$ and $0 < \lambda < 1$, then

$$r(\lambda a + (1 - \lambda)c, b) = \lambda r(a, b) + (1 - \lambda) r(c, b)$$

[12] Denote the measure μ. And since μ is to be held fixed throughout, we may suppress it, abbreviating $\int_T \mathbf{f}\ (t) d\mu(t)$ by $\int_T \mathbf{f}$ for any measurable function $\mathbf{f}$ on $[0, 1]$.

There clearly exists an integrable map $\mathbf{d}$ from $T = [0,1]$ to space of offers S such that (i) positive mass of traders choose a in $\mathbf{d}$; (ii) positive mass of traders choose c in $\mathbf{d}$; and (iii) the integral of $\mathbf{d}$ on T is b. So $\int_T r(\mathbf{d}^\alpha,b)d\mu(\alpha) = \int_T r(\mathbf{d},b) = \bar{b}$ since commodities are conserved. Shift $\varepsilon\lambda$ mass from a to $\lambda a+(1-\lambda)c$ and $(1-\lambda)\varepsilon$ mass from c to $\lambda a+(1-\lambda)c$, letting the rest be according to $\mathbf{d}$. This yields a new function (from T to S) which we call $\mathbf{e}$. Clearly the integral of $\mathbf{e}$ on T is also b. Therefore, once again by conservation of commodities, we must have $\int_T r(\mathbf{e},b) = \bar{b}$, hence $\int_T r(\mathbf{d},b) = \int_T r(\mathbf{e},b)$. But this can only be true if the displayed equality holds, proving that (every coordinate of) r is affine in a for fixed b.

Now $r(0,b) \geq 0$ by assumption. Suppose $r(0,b) \gneqq 0$. Partition T into two non-null sets T_1 and T_2. Consider the case where all the individuals in T_1 offer 0, and all in T_2 offer $b/\mu(T_2)$. Then, since everone in T_1 gets the return $r(0,b) \gneqq 0$, by conservation of commodities everyone in T_2 gets $\bar{b} - \mu(T_1)$ $r(0,b) \lneqq b/\mu(T_2)$, contradicting *non-dissipation.* So $r(0,b) = 0$, showing r is linear.

Finally $\lambda r(a,b) = r(\lambda a, \lambda b) = \lambda r(a, \lambda b)$,where the first equality comes from *Invariance* and the second from linearity. ■

Remark 43 *As mentioned in the introduction, when there is a continuum of traders, the star mechanism leads to equivalence (or, near-equivalence) of Nash and Walras equilibria under suitable postulates regarding the commodity or fiat money. (See [7] for a detailed discussion.)*

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