Anomalies, Entanglement and Boundary Geometry in Conformal Field Theory

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Abstract of the Dissertation

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A conformal field theory embedded in a curved spacetime background can be characterized by the trace anomaly coefficients of the stress tensor. We first derive general vacuum stress tensors of even-dimensional conformal field theories using Weyl anomalies. We then consider some aspects of conformal field theory in space-time dimensions higher than two with a codimension-one boundary. We discuss how boundary effect plays an important role in the study of quantum entanglement. We also obtain universal relationships between boundary trace anomalies and stress-tensor correlation functions near the boundary. A non-supersymmetric graphene-like conformal field theory with a four-dimensional bulk photon and a three-dimensional boundary electron is found to have two boundary central charges that depend on an exactly marginal direction, namely the gauge coupling.

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Publications

Publications related to this dissertation

[1] C. P. Herzog and **K.-W. Huang**,
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Phys. Rev. D **87**, 081901 (2013). doi:10.1103/PhysRevD.87.081901
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[2] **K.-W. Huang**,
"Weyl Anomaly Induced Stress Tensors in General Manifolds,"
Nucl. Phys. B **879**, 370 (2014). doi:10.1016/j.nuclphysb.2013.12.013
[arXiv:1308.2355 [hep-th]].

[3] C. P. Herzog, **K.-W. Huang** and K. Jensen,
"Universal Entanglement and Boundary Geometry in Conformal Field Theory,"
JHEP **1601**, 162 (2016). doi:10.1007/JHEP01(2016)162
[arXiv:1510.00021 [hep-th]].

[4] C. P. Herzog and **K.-W. Huang**,
"Boundary Conformal Field Theory and a Boundary Central Charge,"
JHEP **1710**, 189 (2017). doi:10.1007/JHEP10(2017)189
[arXiv:1707.06224 [hep-th]].

[5] C. P. Herzog, **K.-W. Huang** and K. Jensen,
"Displacement Operators and Constraints on Boundary Central Charges,"
Phys. Rev. Lett. **120**, no. 2, 021601 (2018). doi:10.1103/PhysRevLett.120.021601
[arXiv:1709.07431 [hep-th]].

Other publications

[6] C. P. Herzog, **K.-W. Huang** and R. Vaz,
"Linear Resistivity from Non-Abelian Black Holes,"

JHEP **1411**, 066 (2014). doi:10.1007/JHEP11(2014)066
[arXiv:1405.3714 [hep-th]].

[7] **K.-W. Huang**,
"Central Charge and Entangled Gauge Fields,"
Phys. Rev. D **92**, no. 2, 025010 (2015). doi:10.1103/PhysRevD.92.025010
[arXiv:1412.2730 [hep-th]].

[8] **K.-W. Huang**,
"Boundary Anomalies and Correlation Functions,"
JHEP **1608**, 013 (2016). doi:10.1007/JHEP08(2016)013
[arXiv:1604.02138 [hep-th]].

[9] C. P. Herzog and **K.-W. Huang**,
"Boundary Fluctuations and A Reduction Entropy,"
Phys. Rev. D **95**, no. 2, 021901 (2017). doi:10.1103/PhysRevD.95.021901
[arXiv:1610.08970 [hep-th]].

[10] **K.-W. Huang**, R. Roiban and A. A. Tseytlin,
"Self-dual 6d 2-form fields coupled to non-abelian gauge field: quantum corrections,"
JHEP **1806**, 134 (2018). doi:10.1007/JHEP06(2018)134
[arXiv:1804.05059 [hep-th]].

[11] C. P. Herzog, **K.-W. Huang**, I. Shamir and J. Virrueta,
"Superconformal Models for Graphene and Boundary Central Charges,"
[arXiv:1807.01700 [hep-th]].

Chapter 1

Introduction

As fixed points of renormalization group flow, conformal field theories play a cornerstone role in our understanding of quantum field theories. A classically conformal field theory has an action invariant under conformal transformations and the conformal invariance implies a vanishing trace of the stress tensor.[1] The well-known trace anomaly represents a quantum phenomenon that breaks the conformal symmetry and the trace of the "expectation value" of the stress tensor becomes non-zero. In general, a conformal field theory can be characterized by the trace anomaly coefficients or central charges of the stress tensor. In addition to their well-known important roles in determining correlation functions, these central charges also provide a way of ordering quantum field theories under renormalization group flow. In two dimensions, the classic c-theorem [12] states that the central charge c decreases through the renormalization group flow from the ultraviolet to the infrared. In four-dimensions, the corresponding trace anomaly is defined by two types of central charge, c_4 and a_4. The conjectural a-theorem which stated that the four-dimensional Euler central charge a_4 could be the analog of c in 2D [13] was proven only recently using dilaton fields to probe the trace anomaly [14]. The possibility of a 6D a-theorem was explored in [15].

While there are thousands of papers discussing topics related to the conformal anomaly, the correpsonding discussions in field theories with a *boundary* have been far less explored, in particular in spacetime dimensions higher than two. One might naively wonder that the reason for the relative lack of this research direction might be that boundary effects play only minor roles and will not lead to interesting consequences. Here we would like to instead emphasize that boundary effects in fact have becoming a unifying theme in several areas where there has been significant progress in modern theoretical physics. Indeed, a boundary is essential for understanding condensed matter systems such as impurity models

[1]The Weyl transformation is defined by:

$$\bar{g}_{\mu\nu}(x) = e^{2\sigma(x)} g_{\mu\nu}(x) = \Omega^2 g_{\mu\nu}(x) \ . \tag{1.1}$$

For a conformally flat background, $\bar{g}_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$.

and topological insulators. D-branes, which gave us insight into non-perturbative properties of string theory, are the boundaries of fundamental strings. In gauge-gravity duality, which has provided us a glimpse of connections between quantum gravity and strongly-interacting field theories, quantum fields fluctuate on the boundary of anti-de Sitter space. As a measure of quantum information, entanglement entropy is often defined with respect to spatial regions where an "entangling boundary" plays a crucial role. The concept of entanglement has deepened our understanding of black hole thermodynamics and has given us new insight into renormalization group flow in relativistic quantum field theories. Lately, a fascinating picture is that spacetime geometry might spring up from quantum entanglement. A natural question arises: "might these developments have been obvious if we simply had understood quantum field theory and gravity in the presence of a boundary better to begin with?" The renewed research on boundary physics is therefore timely and needed in view of modern research developments.

This thesis is devoted to exploring anomalies and boundary effects in field theories, in particular those in conformal field theories. We shall start with the more familar case where spacetime has no boundary. In Chapter 2, using trace anomalies, we determine the vacuum stress tensors of even-dimensional conformal field theories in conformally flat backgrounds, adopting the dimensional regularization scheme. A simple relation between the Casimir energy on the real line times a sphere and the type A anomaly coefficient will be demonstrated. This relation generalizes earlier results in two and four dimensions. These field theory results for the Casimir are shown to be consistent with holographic predictions in two, four, and six dimensions. In Chapter 3, we obtain stress tensors from Weyl anomalies in more general (non-conformally flat) backgrounds. The spacetime remains no boundary in this chapter. The results of type A anomaly-induced stress tensors in four and six-dimensions generalize the previous results in Chapter 2 calculated in a conformally flat background. We emphasize that regulators are needed to have well-defined type B anomaly-induced stress tensors. We also discuss ambiguities related to type D anomalies and order of limit issues.

In Chapter 4, we compute the universal contribution to the vacuum entanglement entropy (EE) across a sphere in even-dimensional conformal field theory by employing a conformal map. Previous attempts to derive the EE in this way were hindered by a lack of knowledge of the appropriate boundary terms in the trace anomaly. We will show that the universal part of the EE can be treated as a purely boundary effect. As a byproduct of our computation, we derive an explicit form for the A-type anomaly contribution to the Wess-Zumino term for the trace anomaly, now including boundary terms. In d=4 and 6, these boundary terms generalize earlier bulk actions derived in the literature. Furthermore, a complete classification of d=4 conformal anomalies with a boundary is given.

Motivated by boundary terms of the conformal anomaly, in Chapter 5 we will study the structure of current and stress tensor two-point functions in conformal field theory with a boundary. The main result of this chapter is a relation between a boundary central charge and the coefficient of a displacement operator correlation function. The boundary central charge under consideration is the coefficient of the product of the extrinsic curvature and the Weyl curvature in the conformal anomaly. Along the way, we describe several auxiliary results. Three of the more notable are as follows: (1) we give the bulk and boundary conformal blocks for the current two-point function; (2) we show that the structure of these current and stress tensor two-point functions is essentially universal for all free theories; (3) we introduce a class of interacting conformal field theories where the interactions are confined to the boundary. The most interesting example we consider can be thought of as the infrared fixed point of graphene. This particular interacting conformal model in four dimensions provides a counterexample of a previously conjectured relation between a boundary central charge and a bulk central charge. The model also demonstrates that the boundary central charge can change in response to marginal deformations.

Finally, in Chapter 6 we constrain all the boundary central charges in three and four dimensional conformal field theories in terms of two- and three-point correlation functions of the displacement operator. We provide a general derivation by comparing the trace anomaly with scale dependent contact terms in the correlation functions. We conjecture a relation between the a-type boundary charge in three dimensions and the stress tensor two-point function near the boundary. We check our results for several free theories. The thesis ends with some interesting open questions.

Chapter 2

Stress Tensors from Trace Anomalies: Conformally Flat Spacetime

This chapter is an edited version of my publication [1], written in collaboration with Christopher Herzog.

A conformal field theory (CFT) embedded in a curved spacetime background can be characterized by the trace anomaly coefficients of the stress tensor. Here we only consider even dimensional CFTs because there is no trace anomaly in odd dimensions. The anomaly coefficients (or central charges) a_d and c_{dj} show up in the trace as follows,

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^{d/2}} \left(\sum_j c_{dj} I_j^{(d)} - (-)^{\frac{d}{2}} a_d E_d \right) . \tag{2.1}$$

Here E_d is the Euler density in d dimensions and $I_j^{(d)}$ are independent Weyl invariants of weight $-d$. The subscript "j" is used to index the Weyl invariants. Our convention for the Euler density is that

$$E_d = \frac{1}{2^{d/2}} \delta^{\nu_1 \cdots \nu_d}_{\mu_1 \cdots \mu_d} R^{\mu_1 \mu_2}{}_{\nu_1 \nu_2} \cdots R^{\mu_{d-1} \mu_d}{}_{\nu_{d-1} \nu_d} . \tag{2.2}$$

We will not need the explicit form of the $I_j^{(d)}$ in what follows, although we will discuss their form in $d \leq 6$.

Note that we are working in a renormalization scheme where the trace anomaly is free of the so-called type D anomalies which are total derivatives that can be changed by adding local covariant but not Weyl-invariant counter-terms to the effective action. For example, in four space-time dimensions, a $\Box R$ in the trace can be eliminated by adding an R^2 term to the effective action.

In this chapter, we show how to compute $\langle T^{\mu\nu} \rangle$ in terms of a_d and curvatures for a conformally flat background.

The properties of central charges in the 6D case are of particular interest; the (2,0) theory, which describes the low energy behavior of M5-branes in M-theory, is a 6D CFT. From the AdS/CFT correspondence, it has been known for over a decade that quantities such as the thermal free energy [16] and the central charges [17] have an N^3 scaling for a large number N of M5-branes. However, a direct field theory computation has proven difficult. Any results calculated from the field theory side of the 6D CFT without referring to AdS/CFT should be interesting. Such results also provide a non-trivial check of the holographic principle.

We would like to study the general relation between the stress tensor and the trace anomaly of a CFT in a conformally flat background. The main result of this chapter (2.21) is an expression for the vacuum stress tensor of an even dimensional CFT in a conformally flat background in terms of a_d and curvatures. (By vacuum, we have in mind a state with no spontaneous symmetry breaking, where the expectation values of the matter fields vanish.)

We pay special attention to the general relation between the Casimir energy (ground state energy) and a_d. Let ϵ_d be the Casimir energy on $\mathbb{R} \times S^{d-1}$. The well known 2D CFT result is [18]

$$\epsilon_2 = -\frac{c}{12\ell} = -\frac{a_2}{2\ell} , \tag{2.3}$$

where ℓ is the radius of S^1. This result is universal for an arbitrary 2D CFT, independent of supersymmetry or other requirements. For general $\mathbb{R} \times S^{d-1}$, we will find

$$\epsilon_d = \frac{1 \cdot 3 \cdots (d-1)}{(-2)^{d/2}} \frac{a_d}{\ell} . \tag{2.4}$$

2.1 Stress Tensor and Conformal Anomaly

We would like to determine the contribution of the anomaly to the stress tensor of a field theory in a conformally flat background. The general strategy we use was originally developed in [19]. (See also [20, 21, 22, 23] for related discussion.) The conformal (Weyl) transformation is parametrized by $\sigma(x)$ in the standard form

$$\bar{g}_{\mu\nu}(x) = e^{2\sigma(x)} g_{\mu\nu}(x) . \tag{2.5}$$

Denote the partition function as $Z[g_{\mu\nu}]$. The effective potential is given by

$$\Gamma[\bar{g}_{\mu\nu}, g_{\mu\nu}] = \ln Z[\bar{g}_{\mu\nu}] - \ln Z[g_{\mu\nu}] . \tag{2.6}$$

The expectation value of the stress tensor $\langle T^{\mu\nu} \rangle$ is defined by the variation of the effective potential with respect to the metric. Here we consider a conformally flat background, $\bar{g}_{\mu\nu}(x) = e^{2\sigma(x)} \eta_{\mu\nu}$, and we normalize the stress tensor in the flat spacetime to be zero. The (renormalized) stress tensor is given by

$$\langle T^{\mu\nu}(x) \rangle = \frac{2}{\sqrt{-\bar{g}}} \frac{\delta \Gamma[\bar{g}_{\alpha\beta}]}{\delta \bar{g}_{\mu\nu}(x)} , \tag{2.7}$$

which implies

$$\sqrt{-\bar{g}}\langle T^{\lambda}_{\lambda}(x)\rangle = 2\bar{g}_{\mu\nu}(x)\frac{\delta\Gamma[\bar{g}_{\alpha\beta}]}{\delta\bar{g}_{\mu\nu}(x)} = \frac{\delta\Gamma[\bar{g}_{\alpha\beta}]}{\delta\sigma(x')} \ . \tag{2.8}$$

We rewrite

$$\frac{\delta(\sqrt{-\bar{g}}\langle T^{\mu}_{\nu}(x)\rangle)}{\delta\sigma(x')} = 2\bar{g}_{\lambda\rho}(x')\frac{\delta}{\delta\bar{g}_{\lambda\rho}(x')}2\bar{g}_{\nu\gamma}(x)\frac{\delta\Gamma[\bar{g}_{\alpha\beta}]}{\delta\bar{g}_{\mu\gamma}(x)} \ . \tag{2.9}$$

Then we use the following commutative property

$$\left[\bar{g}_{\lambda\rho}(x')\frac{\delta}{\delta\bar{g}_{\lambda\rho}(x')}, \bar{g}_{\nu\gamma}(x)\frac{\delta}{\delta\bar{g}_{\mu\gamma}(x)}\right] = 0 \tag{2.10}$$

to obtain the following differential scale equation

$$\frac{\delta\sqrt{-\bar{g}}\langle T^{\mu\nu}(x)\rangle}{\delta\sigma(x')} = 2\frac{\delta\sqrt{-\bar{g}}\langle T^{\lambda}_{\lambda}(x')\rangle}{\delta\bar{g}_{\mu\nu}(x)} \ . \tag{2.11}$$

This equation determines the general relation between the stress tensor (and hence the Casimir energy) and the trace anomaly.

Next we would like to re-write the trace anomaly $\langle T^{\mu}_{\mu}\rangle$ in terms of a Weyl exact form, $\langle T^{\mu}_{\mu}\rangle = \frac{\delta}{\delta\sigma}$(something), so that we can factor out the sigma variation in (2.11) to simplify the calculation. The integration constant is fixed to zero by taking $\langle T^{\mu\nu}\rangle = 0$ in flat space. We use dimensional regularization and work in $n = d + \epsilon$ dimensions. While we do not alter E_d in moving away from d dimensions, we will alter the form of the $I^{(d)}_j$. Let $\lim_{n\to d}\mathcal{I}^{(d)}_j = I^{(d)}_j$ where $\mathcal{I}^{(d)}_j$ continues to satisfy the defining relation $\delta_\sigma\mathcal{I}^{(d)}_j = -d\,\mathcal{I}^{(d)}_j$. We assume that in general $\mathcal{I}^{(d)}_j$'s exist such that

$$\begin{aligned}\frac{\delta}{(n-d)\delta\sigma(x)}\int d^nx'\sqrt{-\bar{g}}E_d(x') &= \sqrt{-\bar{g}}E_d \ , &(2.12)\\ \frac{\delta}{(n-d)\delta\sigma(x)}\int d^nx'\sqrt{-\bar{g}}\mathcal{I}^{(d)}_j(x') &= \sqrt{-\bar{g}}\mathcal{I}^{(d)}_j \ . &(2.13)\end{aligned}$$

We now make a brief detour to discuss the existence of $\mathcal{I}^{(d)}_j$ in $d = 2$, 4 and 6 [24, 25] and also a general proof of the variation (2.12). In 2D, there are no Weyl invariants $I^{(2)}_j$ and we can ignore (2.13). In 4D, we have the single Weyl invariant $I^{(4)}_1 = C^{(n=4)}_{\mu\nu\lambda\rho}C^{(n=4)\,\mu\nu\lambda\rho}$ where $C^{(4)\mu\nu\lambda\rho}$ is the 4D Weyl tensor. If we define the n-dimensional Weyl tensor

$$C^{(n)\mu\nu}{}_{\lambda\sigma} \equiv R^{\mu\nu}{}_{\lambda\sigma} - \frac{1}{n-2}\left[2(\delta^{\mu}_{[\lambda}R^{\nu}_{\sigma]} + \delta^{\nu}_{[\sigma}R^{\mu}_{\lambda]}) + \frac{R\,\delta^{\mu\nu}_{\lambda\sigma}}{(n-1)}\right] \ , \tag{2.14}$$

then we find $\mathcal{I}^{(4)}_1 = C^{(n)}_{\mu\nu\lambda\rho}C^{(n)\,\mu\nu\lambda\rho}$ defined in terms of the n-dimensional Weyl tensor satisfies the eigenvector relation (2.13).

At this point, our treatment differs somewhat from ref. [19] where the authors vary instead $I_1^{(4)}$ with respect to σ. While ref. [19] allows for an additional total derivative $\Box R$ term in the trace anomaly, here we choose a renormalization scheme where the trace anomaly takes the minimal form. It turns out that this scheme is the one used to match holographic predictions as we will discuss shortly. A $\Box R$ can be produced by varying $(n-4)R^2$ with respect to σ. Such an R^2 term appears in the difference between $\mathcal{I}_1^{(4)}$ and $I_1^{(4)}$ in [19].

In 6D, there are three Weyl invariants

$$
\begin{aligned}
I_1^{(6)} &= C^{(6)}_{\mu\nu\lambda\sigma}\, C^{(6)\nu\rho\eta\lambda}\, C_{\rho}^{(6)\mu\sigma}{}_{\eta}\,, && (2.15)\\
I_2^{(6)} &= C_{\mu\nu}^{(6)\lambda\sigma}\, C_{\lambda\sigma}^{(6)\rho\eta}\, C_{\rho\eta}^{(6)\mu\nu}\,, && (2.16)\\
I_3^{(6)} &= C^{(6)}_{\mu\nu\lambda\sigma}\Big(\Box\delta^{\mu}_{\rho} + 4R^{\mu}_{\rho} - \frac{6}{5}R\delta^{\mu}_{\rho}\Big)C^{(6)\rho\nu\lambda\sigma} + D_\mu J^\mu\,. && (2.17)
\end{aligned}
$$

To produce the $\mathcal{I}_j^{(6)}$ when $j=1,2$, we replace the six dimensional Weyl tensor with its n-dimensional cousin as in the 4D case. The variation (2.13) is then straightforward to show. For $j=3$, [26] demonstrated the corresponding Weyl transformation for a linear combination of the three $\mathcal{I}_j^{(6)}$, there denoted H. The full expression for $\mathcal{I}_3^{(6)}$ and the n-dimensional version of J^μ is not important; we refer the reader to [26, 27] for details. For $d>6$, we assume the Weyl invariants can be engineered in a similar fashion; see [28] for the $d=8$ case.

To vary E_d, we write the corresponding integrated Euler density as

$$
\int d^n x\sqrt{-g}E_d = \int \frac{\left(\bigwedge_{j=1}^{n} dx^{\mu_j}\right)}{2^{d/2}(n-d)!} R^{a_1a_2}{}_{\mu_1\mu_2}\cdots R^{a_{d-1}a_d}{}_{\mu_{d-1}\mu_d} e^{a_{d+1}}_{\mu_{d+1}}\cdots e^{a_n}_{\mu_n}\epsilon_{a_1\cdots a_n}\,. \qquad (2.18)
$$

Recall that the variation of a Riemann curvature tensor with respect to the metric is a covariant derivative acting on the connection. After integration by parts, these covariant derivatives act on either the vielbeins e^a_μ or the other Riemann tensors and hence vanish by metricity or a Bianchi identity. Thus, in varying the integrated Euler density, we need only vary the vielbeins. We use the functional relation $2\delta/\delta g^\nu_\mu = e^a_{(\nu}\delta/\delta e^a_{\mu)}$. One finds

$$
\frac{\delta}{\delta\bar{g}^\nu_\mu(x)}\int d^n x'\sqrt{-\bar{g}}E_d = \frac{\sqrt{-\bar{g}}}{2^{\frac{d}{2}+1}} R^{\nu_1\nu_2}{}_{\mu_1\mu_2}\cdots R^{\nu_{d-1}\nu_d}{}_{\mu_{d-1}\mu_d}\,\delta^{\mu_1\cdots\mu_d\mu}_{\nu_1\cdots\nu_d\nu}\,. \qquad (2.19)
$$

From this expression, the desired relation (2.12) follows after contracting with δ^ν_μ.

Given the variations (2.12, 2.13), we can factor out the sigma variation in (2.11) to obtain

$$
\begin{aligned}
\langle T^{\mu\nu}\rangle = \langle X^{\mu\nu}\rangle &\equiv \lim_{n\to d}\frac{1}{(n-d)}\frac{2}{\sqrt{-\bar{g}}(4\pi)^{d/2}} \qquad (2.20)\\
&\times\frac{\delta}{\delta\bar{g}_{\mu\nu}(x)}\int d^n x'\sqrt{-\bar{g}}\left(\sum_j c_{dj}\mathcal{I}_j^{(n)} - (-)^{\frac{d}{2}}a_d E_d\right).
\end{aligned}
$$

(While we specialize to conformally flat backgrounds, under a more general conformal transformation one has $\langle T^{\mu\nu}(\bar{g})\rangle - \langle X^{\mu\nu}(\bar{g})\rangle = e^{-(d+2)\sigma}\left(\langle T^{\mu\nu}(g)\rangle - \langle X^{\mu\nu}(g)\rangle\right)$.) Comparing with

(2.7), we see that the effective action must contain terms proportional to $\langle T^\mu_\mu \rangle$. Indeed, these are precisely the counter terms that must be added to regularize divergences coming from placing the CFT in a curved space time [29]. We next perform the metric variation for a conformally flat background. The metric variation of the Weyl tensors $\mathcal{I}_j^{(d)}$ vanishes for conformally flat backgrounds because the $\mathcal{I}_j^{(d)}$ are all at least quadratic in the n-dimensional Weyl tensor. (Conformal flatness is used only after working out the metric variation.) Thus the stress tensor in a conformally flat background may be obtained by varying only the Euler density:

$$\langle T^\mu_\nu \rangle = -\frac{a_d}{(-8\pi)^{d/2}} \lim_{n\to d} \frac{1}{n-d} R^{\nu_1\nu_2}{}_{\mu_1\mu_2} \cdots R^{\nu_{d-1}\nu_d}{}_{\mu_{d-1}\mu_d}\, \delta^{\mu_1\cdots\mu_d\mu}_{\nu_1\cdots\nu_d\nu} \, . \tag{2.21}$$

Note that in a conformally flat background, employing (2.14), the Riemann curvature can be expressed purely in terms of the Ricci tensor and Ricci scalar:

$$R^{\nu_1\nu_2}{}_{\mu_1\mu_2} = \frac{1}{n-2}\left[2(\delta^{\nu_1}_{[\mu_1} R^{\nu_2}_{\mu_2]} + \delta^{\nu_2}_{[\mu_2} R^{\nu_1}_{\mu_1]}) - \frac{R\,\delta^{\nu_1\nu_2}_{\mu_1\mu_2}}{n-1}\right] \, .$$

Contracting a $\delta^{\nu_j}_{\mu_j}$ with the antisymmetrized Kronecker delta $\delta^{\mu_1\cdots\mu_d\mu}_{\nu_1\cdots\nu_d\nu}$ eliminates the factor of $(n-d)$ in (2.21).

In 2D and 4D, we can use (2.21) to recover results of [19]. In 2D, the right hand side of $\langle T^\mu_\nu \rangle$ is proportional to $R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu$ which vanishes in 2D. Thus we first must expand the Einstein tensor in terms of the Weyl factor σ where $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$ before taking the $n \to 2$ limit. The result is [19]

$$\langle T^{\mu\nu} \rangle \;=\; \frac{a_2}{2\pi}\left(\sigma^{,\mu;\nu} + \sigma^{,\mu}\sigma^{,\nu} - g^{\mu\nu}\left(\sigma_{,\lambda}{}^{;\lambda} + \frac{1}{2}\sigma_{,\lambda}\sigma^{,\lambda}\right)\right) \, . \tag{2.22}$$

In 4D, we obtain

$$\langle T^{\mu\nu} \rangle = \frac{-a_4}{(4\pi)^2}\left[g^{\mu\nu}\left(\frac{R^2}{2} - R^2_{\lambda\rho}\right) + 2R^{\mu\lambda}R^\nu_\lambda - \frac{4}{3}RR^{\mu\nu}\right] \, . \tag{2.23}$$

In 6D, we obtain (to our knowledge) a new result

$$\begin{aligned} \langle T^{\mu\nu} \rangle \;=\; & -\frac{a_6}{(4\pi)^3}\left[\frac{3}{2}R^\mu_\lambda R^\nu_\sigma R^{\lambda\sigma} - \frac{3}{4}R^{\mu\nu}R^\lambda_\sigma R^\sigma_\lambda - \frac{1}{2}g^{\mu\nu}R^\sigma_\lambda R^\lambda_\rho R^\rho_\sigma \right. \\ & \left. -\frac{21}{20}R^{\mu\lambda}R^\nu_\lambda R + \frac{21}{40}g^{\mu\nu}R^\sigma_\lambda R^\lambda_\sigma R + \frac{39}{100}R^{\mu\nu}R^2 - \frac{1}{10}g^{\mu\nu}R^3\right] \, . \end{aligned} \tag{2.24}$$

As we work in Weyl flat backgrounds, there is no contribution from B type anomalies. These $\langle T^{\mu\nu} \rangle$ are covariantly conserved, as they must be since they were derived from a variational principle.

2.2 Casimir Energy and Central Charge

Next we would like to relate a_d to the Casimir energy defined as

$$\epsilon_d = \int_{S^{d-1}} \langle T^{00} \rangle \,\mathrm{vol}(S^{d-1}), \tag{2.25}$$

on $\mathbb{R} \times S^{d-1}$. In preparation, let us first calculate E_d for the sphere S^d. For S^d with radius ℓ, the Riemann tensor is $R^{\nu_1\nu_2}{}_{\mu_1\mu_2} = \delta^{\nu_1\nu_2}_{\mu_1\mu_2}/\ell^2$. It follows from (2.2) that $E_d = \frac{d!}{\ell^d}$. We conclude that the trace of the vacuum stress tensor on S^d takes the form

$$\langle T^\mu_\mu \rangle = -\frac{a_d\, d!}{(-4\pi\ell^2)^{d/2}} \ . \tag{2.26}$$

Let us now calculate $\langle T^\mu_\nu \rangle$ for $S^1 \times S^{d-1}$. The Riemann tensor on $S^1 \times S^{d-1}$ is zero whenever it has a leg in the S^1 direction and looks like the corresponding Riemann tensor for S^{d-1} in the other directions. We can write $R^{i_1 i_2}{}_{j_1 j_2} = \delta^{i_1 i_2}_{j_1 j_2}/\ell^2$, where i and j index the S^{d-1}. The computation of $\langle T^0_0 \rangle$ and $\langle T^i_j \rangle$ proceeds along similar lines to the computation of E_d:

$$\langle T^0_0 \rangle = -\frac{a_d (d-1)!}{(-4\pi\ell^2)^{d/2}} \ , \quad \langle T^i_j \rangle = \frac{a_d (d-2)!}{(-4\pi\ell^2)^{d/2}} \delta^i_j \ . \tag{2.27}$$

Note that $\langle T^\mu_\nu \rangle$ is traceless, consistent with a result of [22]. Using the definition (2.25), we compute the Casimir energy ϵ_d. We find that (for d even)

$$\epsilon_d = \frac{a_d (d-1)!}{(-4\pi\ell^2)^{d/2}} \mathrm{Vol}(S^{d-1}) = \frac{1 \cdot 3 \cdots (d-1)}{(-2)^{d/2}} \frac{a_d}{\ell} \ . \tag{2.28}$$

In 2D, 4D and 6D, the ratios between the Casimir energy and a_d are $-\frac{1}{2\ell}$, $\frac{3}{4\ell}$ and $-\frac{15}{8\ell}$, respectively.

2.3 Holography and Discussion

Here we would like to use the AdS/CFT correspondence to check our relation between ϵ_d and a_d for $d = 2$, 4 and 6. For CFTs with a dual anti-de Sitter space description, the stress-tensor can be calculated from a classical gravity computation [30, 31, 32]. The Euclidean gravity action is

$$\begin{aligned}
S &= S_{\mathrm{bulk}} + S_{\mathrm{surf}} + S_{\mathrm{ct}} \ , \qquad (2.29)\\
S_{\mathrm{bulk}} &= -\frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{G} \left(\mathcal{R} + \frac{d(d-1)}{L^2} \right), \\
S_{\mathrm{surf}} &= -\frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{g} K, \\
S_{\mathrm{ct}} &= \frac{1}{2\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{g} \Big[\frac{2(d-1)}{L} + \frac{L}{d-2} R + \\
&\quad \frac{L^3}{(d-4)(d-2)^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{d}{4(d-1)} R^2 \right) + \ldots \Big].
\end{aligned}$$

The Ricci tensor $R_{\mu\nu}$ is computed with respect to the boundary metric $g_{\mu\nu}$ while $\mathcal{R}$ is the Ricci Scalar computed from the bulk metric G_{ab}. The object $K_{\mu\nu}$ is the extrinsic curvature of the boundary $\partial\mathcal{M}$. The counter-terms S_{ct} render S finite, and we keep only as many as we need. The metrics with $S^{d-1} \times S^1$ conformal boundary,

$$ds^2 = L^2(\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, d\Omega_{d-1}) \ , \tag{2.30}$$

and S^d boundary,

$$ds^2 = L^2(dr^2 + \sinh^2 r\, d\Omega_d) \ , \tag{2.31}$$

satisfy the bulk Einstein equations. Note that the S^{d-1} and S^d spheres have radius $\ell = \frac{L}{2}e^{r_0}$ at some large reference r_0 while we take the S^1 to have circumference β (hence the range of t is $0 < t < \beta/\ell$). We compute the stress tensor from the on-shell value of the gravity action using (2.7), making the identification $\Gamma = -S$ and using the boundary value of the metric in place of $\bar{g}_{\mu\nu}$. One has [31]:

d	Γ_{S^d}	$\Gamma_{S^1\times S^{d-1}}$
2	$\frac{4\pi L}{\kappa^2}\log\ell$	$\frac{\pi\beta L}{\kappa^2\ell}$
4	$-\frac{4\pi^2 L^3}{\kappa^2}\log\ell$	$-\frac{3\pi^2\beta L^3}{4\kappa^2\ell}$
6	$\frac{2\pi^3 L^5}{\kappa^2}\log\ell$	$\frac{5\pi^3\beta L^5}{16\kappa^2\ell}$

We include only the leading log term of Γ_{S^d}. From (2.7), it follows that $\langle T^0_0\rangle\,\text{Vol}(S^{d-1}) = \partial_\beta\Gamma_{S^1\times S^{d-1}}$ and $\langle T^\mu_\mu\rangle\,\text{Vol}(S^d) = \partial_\ell\Gamma_{S^d}$ For a conformally flat manifold, we have from (2.1) that $\langle T^\mu_\mu\rangle = -a_d(-4\pi)^{-d/2}E_d$ which allows us to calculate a_d from $\langle T^\mu_\mu\rangle$ [17]. Defining the Casimir energy with respect to a time $\tilde{t} = \ell t$ whose range is the standard $0 < \tilde{t} < \beta$, we can deduce from (2.25) that $\epsilon_d = -\partial_\beta\Gamma_{S^1\times S^{d-1}}$ (see also [33]). We have

	$\langle T^0_0\rangle$	ϵ_d		$\langle T^\mu_\mu\rangle$	E_d	a_d
$S^1\times S^1$	$\frac{L}{2\kappa^2\ell^2}$	$-\frac{\pi L}{\kappa^2\ell}$	S^2	$\frac{L}{\kappa^2\ell^2}$	$\frac{2}{\ell^2}$	$\frac{2\pi L}{\kappa^2}$
$S^1\times S^3$	$-\frac{3L^3}{8\kappa^2\ell^4}$	$\frac{3\pi^2 L^3}{4\kappa^2\ell}$	S^4	$-\frac{3L^3}{2\kappa^2\ell^4}$	$\frac{24}{\ell^4}$	$\frac{\pi^2 L^3}{\kappa^2}$
$S^1\times S^5$	$\frac{5L^5}{16\kappa^2\ell^6}$	$-\frac{5\pi^3 L^5}{16\kappa^2\ell}$	S^6	$\frac{15L^5}{8\kappa^2\ell^6}$	$\frac{720}{\ell^6}$	$\frac{\pi^3 L^5}{6\kappa^2}$

Comparing the ϵ_d and a_d columns, we can confirm the results from earlier, namely that

$$\epsilon_2 = -\frac{a_2}{2\ell} \ ; \quad \epsilon_4 = \frac{3a_4}{4\ell} \ ; \quad \epsilon_6 = -\frac{15a_6}{8\ell}. \tag{2.32}$$

In the 4D case, such a gravity model arises in type IIB string theory by placing a stack of N D3-branes at the tip of a 6D Calabi-Yau cone. In this case, we can make the further identification [34, 17]: $a_4 = \frac{N^2}{4}\frac{\text{Vol}(S^5)}{\text{Vol}(SE_5)}$ where SE_5 is the 5D base of the cone. These constructions are dual to 4D quiver gauge theories with $\mathcal{N} = 1$ supersymmetry. In 6D, such a gravity model arises in M-theory by placing a stack of N M5-branes in flat space. In this case, we can make the further identification [17, 27] (see also [35]): $a_6 = \frac{N^3}{9}$. The dual field theory is believed to be the non-abelian (2,0)-theory.

We would like to also comment briefly on the Casimir energy calculated in the weak coupling limit.[1] In typical regularization schemes, for example zeta-function regularization, the Casimir energy will not be related to the conformal anomaly via (2.4) because of the presence of total derivative terms (D type anomalies) in the trace of the stress tensor. For a conformally coupled scalar in 4D, ref. [29] tells us $a_4 = 1/360$. Our result (2.4) would imply then that $\epsilon_4 = 1/480L$, but naive zeta-function regularization yields instead $\epsilon_4 = 1/240L$. The discrepancy can be resolved either by including a $\Box R$ term in the trace, thus changing (2.4) [22], or by adding an R^2 counter-term to the effective action, thereby changing ϵ_4. Amusingly in zeta-function regularization, the effect of the total derivative terms on ϵ_4 cancels for the full $\mathcal{N} = 4$ SYM multiplet, and the weak coupling results for ϵ_4 and a_4 are related via (2.4) [36, 37]. In contrast, for the (2,0) multiplet in 6D, the total derivative terms do not cancel [27]. The resulting discrepancy [38] in the relation between a_6 and ϵ_6 can presumably be cured either by adding counter-terms to the effective action to eliminate the total derivatives or by improving (2.4) to include the effect of these derivatives. Generalizing our results to include the contribution of D type anomalies to the stress tensor would allow a more straightforward comparison of weak coupling Casimir energies obtained via zeta-function regularization and the conformal anomaly a_d.

There are two other obvious calculations for future study: i) Determine how $\langle T^{\mu\nu} \rangle$ transforms in non-conformally flat backgrounds. Such transformations would involve the type B anomalies. ii) Check the full 6D stress tensor (2.24) for any conformally flat background by the holographic method. A 4D check of (2.23) was performed in [32].

[1] We thank J. Minahan for discussions on this issue.

Chapter 3

Stress Tensors from Trace Anomalies: Non-conformally Flat Spacetime

This chapter is an edited version of my publication [2].

In the previous chapter, we have shown that the stress tensors of conformal field theories in a conformally flat background can be obtained from the trace anomalies without the knowledge of a Lagrangian. In this chapter, we generalize these results to arbitrary general (non-conformally flat) backgrounds.

In Sec. 3.1, we first review the main strategy of obtaining the stress tensor in a conformally flat background discussed in the previous chapter. We then discuss the main issue of having a well-defined dimensional regularization method when the spacetime is not conformally flat; our main formula will be given in Sec. 3.2.1. In Sec. 3.2.2, we will obtain the corresponding stress tensors from type A anomalies in 4D and 6D in general backgrounds. These results generalize the previous results calculated in a conformally flat background ([1], [19], [22]). In Sec. 3.2.3, we obtain the 4D type B anomaly-induced stress tensor in general backgrounds. We also discuss the appearance of the term $\sim \Box R$ from the type B anomaly. We will comment on various ambiguities related to Weyl invariants in Sec. 3.2.4, where the 4D type D anomaly-induced stress tensor is also given. In the final discussion section, we compare our 4D results with the literature.

3.1 Review

Let us first review the strategy of obtaining the stress tensors in conformally flat backgrounds. Again let $Z[g_{\mu\nu}]$ be the partition function. The effective potential is $\Gamma[\bar{g}_{\mu\nu}, g_{\mu\nu}] = \ln Z[\bar{g}_{\mu\nu}] - \ln Z[g_{\mu\nu}]$. We normalize the stress tensor in the flat spacetime to be zero. The (renormalized) stress tensor is $\langle T^{\mu\nu}(x)\rangle = \frac{2}{\sqrt{-\bar{g}}}\frac{\delta\Gamma[\bar{g}_{\alpha\beta}]}{\delta\bar{g}_{\mu\nu}(x)}$. From the previous chapter, we found that the following equation determines the general relation between the stress tensor and the trace anomalies:

$$\frac{\delta\sqrt{-\bar{g}}\langle\bar{T}^{\mu\nu}(x)\rangle}{\delta\sigma(x')} = 2\frac{\delta\sqrt{-\bar{g}}\langle\bar{T}^{\lambda}_{\lambda}(x')\rangle}{\delta\bar{g}_{\mu\nu}(x)} \ . \tag{3.1}$$

In the scheme with no type D anomalies, we further assumed that we could always re-write the anomalies as $\sigma-$exact forms using the identities

$$\frac{\delta}{(n-d)\delta\sigma(x)}\int d^n x'\sqrt{-g}E_d(x') \ = \ \sqrt{-g}E_d \ , \tag{3.2}$$

$$\frac{\delta}{(n-d)\delta\sigma(x)}\int d^n x'\sqrt{-g}\mathcal{I}^{(d)}_j(x') \ = \ \sqrt{-g}I^{(d)}_j \ . \tag{3.3}$$

While we did not alter E_d in moving away from d dimensions but we altered the form of the $I^{(d)}_j$: let $\lim_{n\to d}\mathcal{I}^{(d)}_j = I^{(d)}_j$ where $\mathcal{I}^{(d)}_j$ continues to satisfy the defining relation $\delta_\sigma\mathcal{I}^{(d)}_j = -d\,\mathcal{I}^{(d)}_j$. (We ignored $\lim_{n\to d}$ in (3.3) just for the simplicity of the expression.) The n-dimensional Weyl tensor (denoted as C in the previous chapter) is

$$W^{(n)\mu\nu}{}_{\lambda\sigma} \equiv R^{\mu\nu}{}_{\lambda\sigma} - \frac{1}{n-2}\left[2(\delta^{\mu}_{[\lambda}R^{\nu}_{\sigma]} + \delta^{\nu}_{[\sigma}R^{\mu}_{\lambda]}) + \frac{R\,\delta^{\mu\nu}_{\lambda\sigma}}{(n-1)}\right] \ . \tag{3.4}$$

Factoring out the sigma variation in (3.1) and setting the integration constant to zero in flat spacetime, we obtained

$$\begin{aligned}\langle\bar{T}^{\mu\nu}\rangle &= \lim_{n\to d}\frac{1}{(n-d)}\frac{2}{\sqrt{-\bar{g}}(4\pi)^{d/2}} \\ &\times\frac{\delta}{\delta\bar{g}_{\mu\nu}(x)}\int d^n x'\sqrt{-\bar{g}}\left(\sum_j c_{dj}\mathcal{I}^{(n)}_j - (-)^{\frac{d}{2}}a_d E_d\right)|_{\bar{g}} \ .\end{aligned} \tag{3.5}$$

The type B anomalies do not contribute to the stress tensors in a conformally flat background. The stress tensor in a conformally flat background could be obtained by varying only the Euler density and we found

$$\langle\bar{T}^{\mu}_{\nu}\rangle = -\frac{a_d}{(-8\pi)^{d/2}}\lim_{n\to d}\frac{1}{n-d}\left[R^{\nu_1\nu_2}{}_{\mu_1\mu_2}\cdots R^{\nu_{d-1}\nu_d}{}_{\mu_{d-1}\mu_d}\,\delta^{\mu_1\cdots\mu_d\mu}_{\nu_1\cdots\nu_d\nu}\right]|_{\bar{g}} \ , \tag{3.6}$$

where the factor of $(n-d)$ would be eliminated when using the conformal flatness condition by contracting with $\delta^{\nu_j}_{\mu_j}$.

3.2 Generalization to Non-Conformally Flat Backgrounds

3.2.1 General Strategy

Using (3.5), we saw in (3.6) that the $\frac{1}{n-d}$ could be cancelled by a factor of $(n-d)$ in the conformally flat case after the metric variation. Thus, the limit $n\to d$ is well-defined.

However, for general (non-conformally flat) backgrounds, we need to check that the limit $n \to d$ can be still well-defined.

In the type A case, we do not have this issue because the type A anomaly is a topological quantity.[1] This means that in the type A anomaly part, after the metric variation in (3.5), it always gives us the form $\frac{0}{0}$ in the limit $n \to d$, thus we can adopt L'Hospital's rule to obtain meaningful results. We will use the following identity for the type A anomalies:

$$\frac{\delta}{(n-d)\delta\sigma(x)}\mathcal{A}^{(d)} \equiv \frac{\delta}{(n-d)\delta\sigma(x)}\Big[\int d^n x' \sqrt{-g} E_d(x')\Big] = \sqrt{-g} E_d \ . \tag{3.8}$$

In the type B case, we will need a regulator to have a well-defined limit $n \to d$. (Notice that type B anomalies are generally not invariant under the metric variation.) Let us consider the following identities:

$$\frac{\delta}{(n-d)\delta\sigma(x)}\mathcal{B}_i^{(d)} \equiv \frac{\delta}{(n-d)\delta\sigma(x)}\Big[\int d^n x' \sqrt{-g}\mathcal{I}_j^{(d)}(x') - \int d^d x' \sqrt{-g} I_j^{(d)}(x')\Big] = \sqrt{-g} I_j^{(d)} \ , \tag{3.9}$$

where we add a term that is essentially the type B anomaly in a given dimension, which is by definition a Weyl invariant quantity. The method to get rid of the infinite contribution is as follows. After the metric variation, the parts without the additional term in (3.9) could be written symbolically as

$$\lim_{n\to d}\left\{\frac{1}{(n-d)}[(n-d)f^{(n)}(R,W) + g^{(n)}(R,W)]\right\} \ . \tag{3.10}$$

The function $g(R,W)$ that causes the infinite contribution will be combined with the additional term's contribution, $-\frac{1}{(n-d)}[g^{(d)}(R,W)]$. Treating the additional term as a regulator, we could use L'Hospital's rule

$$\lim_{n\to d}\frac{g^{(n)}(R,W) - g^{(d)}(R,W)}{(n-d)} = \lim_{n\to d}\frac{d}{dn}\Big[g^{(n)}(R,W)\Big] \ . \tag{3.11}$$

Thus, the stress tensors from the type B anomalies contain the following two finite parts:

$$f^{(d)}(R,W) + \lim_{n\to d}\frac{d}{dn}\Big[g^{(n)}(R,W)\Big] \ . \tag{3.12}$$

[1]One might think the fact that the variation of the Euler density with respect to the metric vanishes in integer dimensions would imply type A anomalies must give terms all proportional to $(n-d)$ to some positive powers after the metric variation. But it is not true. Let's take 4D as an explicit example: in 4D, the metric variation on the type A anomaly in fact would give additional terms that are not proportional to $(n-4)$:

$$\sim (g^{ab} W_{cdef} W^{cdef} - 4 W^{acde} W^{b}{}_{cde}) + \mathcal{O}(n-4) \ . \tag{3.7}$$

In 4D only, the above expression vanishes as an identity. Hence the metric variation of the 4D Euler density indeed vanishes. A similar structure would apply for higher dimensional type A anomalies.

The fact that the regulator is needed for a well-defined effective action of the type B anomaly agrees with [23, 24], but here we use a different kind of effective action that is given by re-writing trace anomaly as a σ-exact form.

Let us now express the full formula more precisely. Denote

$$\mathcal{K}_g = \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\sum_j c_{dj} \mathcal{B}_j^{(d)} - (-)^{\frac{d}{2}} a_d \mathcal{A}^{(d)} \right)_g \ . \tag{3.13}$$

Then we factor out the sigma variation (from (3.1)) to get

$$\sqrt{-\bar{g}}\langle \bar{T}^{\mu\nu}\rangle - \sqrt{-g}\langle T^{\mu\nu}\rangle = \lim_{n\to d} \frac{1}{(n-d)} \frac{2}{(4\pi)^{d/2}} \mathcal{K}_{\bar{g}} - \lim_{n\to d} \frac{1}{(n-d)} \frac{2}{(4\pi)^{d/2}} \mathcal{K}_g \ . \tag{3.14}$$

We further re-write the above expression as

$$\delta\langle T^{\mu\nu}\rangle \equiv \langle \bar{T}^{\mu\nu}\rangle - \Omega^{-d}\langle T^{\mu\nu}\rangle = \lim_{n\to d} \frac{1}{\sqrt{-\bar{g}}(n-d)} \frac{2}{(4\pi)^{d/2}} \mathcal{K}_{\bar{g}} - \Omega^{-d}\Big[.....\Big]|_{\bar{g}\to g} \ , \tag{3.15}$$

where

$$\Big[.....\Big]|_{\bar{g}\to g} \equiv \lim_{n\to d} \frac{1}{\sqrt{-g}(n-d)} \frac{2}{(4\pi)^{d/2}} \mathcal{K}_g \ , \tag{3.16}$$

simply denotes the same curvature tensor forms but only with $\bar{g}$ replaced by g. (Eq. (3.15) is the main formula that we will be using in the following sections.)

3.2.2 Type A

In 4D we obtain

$$\begin{aligned} \delta\langle T^{ab}\rangle^{(A)} \ &= \ \langle T^{ab(A)}\rangle(c.f)|_{\bar{g}} - \frac{a_4}{(4\pi)^2}\Big[4R^{cd}W^{a\ \ b}_{\ c\ d} + \lim_{n\to 4}\frac{1}{(n-4)}(g^{ab}W_{cdef}W^{cdef} - 4W^{acde}W_{bcde})\Big]|_{\bar{g}} \\ &\quad -\Omega^{-4}\Big[.....\Big]|_{\bar{g}\to g} \ , \end{aligned} \tag{3.17}$$

where $(c.f)$ denotes the conformally flat case. The 4D stress tensor in a conformally flat background was obtained in the previous chapter:

$$\langle T^{ab}\rangle^{(A)}(c.f) = \frac{-a_4}{(4\pi)^2}\Big[g^{ab}\Big(\frac{R^2}{2} - R^2_{cd}\Big) + 2R^{ac}R^b_c - \frac{4}{3}RR^{ab}\Big] \ . \tag{3.18}$$

Notice that (3.17) is obtained by rewritting Riemann tensors into Weyl tensors in order to factor out the $(n-4)$ factors. After rewritting Riemann/Weyl tensors into Weyl/Riemann tensors, we should treat the remaining tensors as dimension-independent variables. The topological nature of the type A anomalies implies that we can use the L'Hospital's rule on $\lim_{n\to 4}\frac{1}{(n-4)}(g_{ab}W_{cdef}W^{cdef} - 4W_{acde}W_b^{cde})$, which gives zero. Thus, the result is

$$\delta\langle T^{ab}\rangle^{(A)}_{n=4} \ = \ \Big[\langle T^{ab(A)}\rangle(c.f) - \frac{a_4}{(4\pi)^2}4R^{cd}W^{a\ \ b}_{\ c\ d}\Big]|_{\bar{g}} - \Omega^{-4}\Big[.....\Big]|_{\bar{g}\to g} \ , \tag{3.19}$$

where the extra term $\sim R^{cd}W^{a\ \ b}_{\ c\ d}$ vanishes once traced.[2] In this type A case,

$$[\lim_{n\to 4}, Tr]\delta\langle T^{ab}\rangle^{(A)} = -\frac{a_4}{(4\pi)^2}(I^{(4)}|_{\bar{g}} - \Omega^{-4}I^{(4)}|_g) \ , \tag{3.20}$$

where

$$I_1^{(4)} = W^{(n=4)}_{\mu\nu\lambda\rho}W^{(n=4)\,\mu\nu\lambda\rho} \ , \tag{3.21}$$

is the only Weyl invariant in 4D. Note (3.20) in fact gives zero because of the nature of $I^{(4)}$ which transforms covariantly. We also have

$$[\lim_{n\to 4}, \lim_{W\to 0}]\delta\langle T^{ab}\rangle^{(A)} = 0 \ , \tag{3.22}$$

since $R^{cd}W^{a\ \ b}_{\ c\ d}$ vanishes in a conformally flat background.

Let us next consider the stress tensor derived from the 6D type A anomaly in general backgrounds. We obtain a new result in 6D that (to our knowledge) was not computed before:

$$\begin{aligned}
\delta\langle T^{ab}\rangle^{(A)}_{n=6} &= \langle T^{ab}\rangle^{(A)}(c.f)|_{\bar{g}} + \frac{a_6}{(4\pi)^3}\Big[\frac{12}{5}RR^{cd}W^{a\ \ b}_{\ c\ d} - 3R^{de}R^{bc}W^{a}_{\ dce} - 3R^{e}_{c}R^{cd}W^{a\ \ b}_{\ d\ e} \\
&+6R^{bc}W^{adef}W_{cdef} + \frac{3}{2}g^{ab}R^{cd}R^{ef}W_{cdef} - 12R^{cd}W^{aebf}W_{cedf} - \frac{3}{2}R^{ab}W^{cdef}W_{cdef} \\
&+\frac{27}{20}g^{ab}RW^{cdef}W_{cdef} - 6g^{ab}R^{cd}W_{c}^{\ efg}W_{defg} - \frac{27}{5}RW^{acde}W^{b}_{\ cde} - 3R^{ac}R^{de}W^{b}_{\ dce} + \\
&6R^{cd}W^{a\ ef}_{\ c}W^{b}_{\ def} + 6R^{ac}W_{cdef}W^{bdef} + 12R^{cd}W^{ae\ \ f}_{\ \ c}W^{b}_{\ edf}\Big]|_{\bar{g}} - \Omega^{-6}\Big[.....\Big]|_{\bar{g}\to g} \ ,
\end{aligned} \tag{3.23}$$

where the 6D stress tensor in a conformally flat background was obtained in the previous chapter:

$$\begin{aligned}
\langle T^{\mu\nu}\rangle^{(A)}(c.f) &= \frac{a_6}{(4\pi)^3}[-\frac{3}{2}R^{\mu}_{\lambda}R^{\nu}_{\sigma}R^{\lambda\sigma} + \frac{3}{4}R^{\mu\nu}R^{\lambda}_{\sigma}R^{\sigma}_{\lambda} + \frac{1}{2}g^{\mu\nu}R^{\sigma}_{\lambda}R^{\lambda}_{\rho}R^{\rho}_{\sigma} \\
&+\frac{21}{20}R^{\mu\lambda}R^{\nu}_{\lambda}R - \frac{21}{40}g^{\mu\nu}R^{\sigma}_{\lambda}R^{\lambda}_{\sigma}R - \frac{39}{100}R^{\mu\nu}R^2 + \frac{1}{10}g^{\mu\nu}R^3] \ .
\end{aligned} \tag{3.24}$$

In this case we find

$$[\lim_{n\to 6}, Tr]\delta\langle T^{ab}\rangle^{(A)} \ = \ -\frac{a_6}{(4\pi)^3}\Big[\Big(8I_1^{(6)} + 2I_2^{(6)}\Big)|_{\bar{g}} - \Omega^{-6}\Big(8I_1^{(6)} + 2I_2^{(6)}\Big)|_g\Big] \ , \tag{3.25}$$

where $I_1^{(6)}$ and $I_2^{(6)}$ are the first two kinds of 6D Weyl invariant tensors given by

$$I_1^{(6)} \ = \ W^{(6)}_{\mu\nu\lambda\sigma}\ W^{(6)\nu\rho\eta\lambda}\ W^{(6)\mu\sigma}_{\rho\qquad\eta} \ , \tag{3.26}$$

$$I_2^{(6)} \ = \ W^{(6)\lambda\sigma}_{\mu\nu}\ W^{(6)\rho\eta}_{\lambda\sigma}\ W^{(6)\mu\nu}_{\rho\eta} \ , \tag{3.27}$$

$$I_3^{(6)} \ = \ W^{(6)}_{\mu\nu\lambda\sigma}\Big(\Box\delta^{\mu}_{\rho} + 4R^{\mu}_{\rho} - \frac{6}{5}R\delta^{\mu}_{\rho}\Big)W^{(6)\rho\nu\lambda\sigma} + D_{\mu}J^{\mu} \ . \tag{3.28}$$

We see again that (3.25) is zero because of the nature of $I_1^{(6)}$ and $I_2^{(6)}$ that transform covariantly. Finally, similar to 4D, we have

$$[\lim_{n\to 6}, \lim_{W\to 0}]\delta\langle T^{ab}\rangle^{(A)} = 0 \ . \tag{3.29}$$

[2] This result computed in a new way agrees with [21].

3.2.3 Type B

As mentioned earlier, the type B anomaly is not metric variation invariant so we need a regulator to have the form $\frac{0}{0}$ when taking the $\lim_{n\to d}$. After the metric variation, the result from the 4D type B anomaly reads

$$\delta\langle T^{ab}\rangle^{(B)}_{n=4} = \frac{c_4}{(4\pi)^2}\Big[-4R^{cd}W^{a\ \ b}_{\ c\ \ d} - g^{ab}R_{cd}R^{cd} + 4R^{ac}R^{b}_{\ c}$$
$$-\frac{14}{9}RR^{ab} + \frac{7}{18}g^{ab}R + \frac{8}{9}D^aD^bR - 2D^2R^{ab} + \frac{1}{9}g^{ab}D^2R\Big]|_{\bar{g}} - \Omega^{-4}\Big[.....\Big]|_{\bar{g}\to g}\ . \tag{3.30}$$

In this case,

$$[\lim_{n\to 4}, Tr]\delta\langle T^{ab}\rangle^{(B)} \ = \ \frac{c_4}{(4\pi)^2}\Big(\frac{2}{3}D^2R|_{\bar{g}} - \Omega^{-4}\ \frac{2}{3}D^2R|_g\Big)\ . \tag{3.31}$$

When the $\frac{2}{3}D^2R$ term appears in the 4D trace anomaly, one can relate it to an R^2 term in the effective action. However, here it shows up as an artifact of dimensional regularization. By taking the $n\to 4$ limit, we have used

$$\lim_{n\to 4}\Big[\frac{\delta}{(n-4)\delta\sigma(x)}\int d^nx'\sqrt{-g}W^2(n)(x')\Big] = \sqrt{-g}W^2(4)\ , \tag{3.32}$$

where $W(n)$ is defined in (3.4). We factored out the σ variation and obtained the stress tensor after the metric variation. We then found a $\frac{2}{3}D^2R$ in (3.31) after taking the trace. This process could be formally re-expressed as

$$Tr\ \frac{\delta}{\delta g_{\mu\nu}}\lim_{n\to 4}\Big[\frac{1}{(n-4)}\int d^nx'\sqrt{-g}W^2(n)(x')\Big]\ , \tag{3.33}$$

which gives

$$Tr\ \frac{\delta}{\delta g_{\mu\nu}}\Big[\Big(\frac{1}{(n-4)}\int d^4x'\sqrt{-g}W^2(4)\Big)|_{n\to 4} + \int d^4x'\sqrt{-g}\frac{\partial W^2(n)}{\partial n}|_{n\to 4}\Big]\ . \tag{3.34}$$

The divergent first term will be cancelled by the regulator. It is the second term that gives $\frac{2}{3}D^2R$. (One can further check that the orders of taking the metric variation and $n\to 4$ expansion commute.) Therefore, we see that the $\frac{2}{3}D^2R$ has another origin besides adding an R^2 term in the effective action. It should be stressed that these two ways of producing a D^2R term will give different contributions to the stress tensor, although they both lead to $\frac{2}{3}D^2R$ when traced.[3]

In 6D, there are three kinds of type B anomalies so that three regulators are needed. One can derive the corresponding transformed stress tensors following the same method we developed here. But the results are very lengthy so we do not present then here. Moreover, we will soon comment on ambiguities related to the type B anomalies in the following sections.

[3] We notice that there were also several related discussions in AdS/CFT regarding this $\frac{2}{3}D^2R$ term. For instance, [39] discussed this term on page 5 in the context of the holographic c-theorem. [30] mentioned this kind of ambiguity on page 16. In [40], they included the $\frac{2}{3}D^2R$ term on page 30 to study entanglement entropy.

3.2.4 Type D and Ambiguities

In 4D, there is only one kind of the type D anomaly given by:

$$\langle T^{\mu}_{\mu}\rangle^{(D)} = \frac{\gamma}{(4\pi)^2} D^2 R \ , \tag{3.35}$$

where $\gamma \equiv d_4$ represents the corresponding type D central charge. This anomaly can be generated by using the following identity

$$\frac{\delta}{(n-4)(4\pi)^2\delta\sigma(x)}\Big[\int d^n x'\sqrt{-g}(n-4)\frac{-\gamma}{12}R^2(x')\Big] \ = \ \frac{\gamma}{(4\pi)^2}D^2 R \ . \tag{3.36}$$

Obviously, there is no $n \to d$ problem here. The stress tensor corresponding to this anomaly is therefore given by the metric variation on the R^2 term and we have

$$\delta\langle T^{ab}\rangle^{(D)}_{n=4} = -\frac{\gamma}{6(4\pi)^2}\Big(2D^a D^b R - 2g^{ab}D^2 R - 2RR^{ab} + \frac{1}{2}g^{ab}R^2\Big)|_{\bar{g}} - \Omega^{-4}\Big[.....\Big]|_{\bar{g}\to g} \ . \tag{3.37}$$

Since one could introduce a counterterm in the effective action to cancel this anomaly, this contribution is arbitrary. We will not consider 6D type D anomalies, which would presumably lead to lengthy expressions; we refer readers to [41] for all possible type D anomalies in 6D.

Going back to the case of 4D type B anomaly, (3.30), one might ask if there is an $\lim_{n\to4}$ and $\lim_{W\to0}$ order of limits issue since we consider $\lim_{W\to0}\langle T^{(B)}_{ab}\rangle = 0$ under the scheme that the type B central charge does not contribute to the stress tensor in a conformally flat background.[4] Our answer to this question is that there is no definite contribution to the stress tensor from the type B central charge because of various ambiguities related to Weyl tensors. Recall that the main strategy in the dimensional regularization approach is to rewrite the trace anomaly into a σ-exact term. However, one has some arbitrariness that can be added in the effective action: (1) $(n-4)\times\int d^4x\sqrt{-g}R^2$ with an arbitrary coefficient. This term only modifies the coefficient of the type D anomaly, which is arbitrary as mentioned before; (2) σ-variation invariant terms such as $(n-4)\times\int d^4x\sqrt{-g}$ *type A/B anomaly* with an arbitrary coefficient. But notice that the type A anomaly is topological, so it will not contribute to the stress tensor. By using the first kind of arbitrariness, it is found that if we instead use the following identity

$$\begin{aligned}\lim_{n\to4}\frac{\delta}{(n-4)\delta\sigma(x)}\Big[&\int d^n x'\sqrt{-g}\mathcal{I}^{(4)}_j(x') - \int d^4x'\sqrt{-g}I^{(4)}(x') \\ &-(n-4)\Big(\frac{1}{18}\int d^4x'\sqrt{-g}R^2(x')\Big)\Big] = \sqrt{-g}[I^{(4)}_j + \frac{2}{3}D^2R] \ ,\end{aligned} \tag{3.38}$$

[4] Note that (3.30) is the result after taking $\lim_{n\to4}$. If we instead take $\lim_{W\to0}$ first, we have symbolically $\lim_{W\to0}\frac{\delta}{\delta g_{\mu\nu}}\int W^2$, which simply is already zero because of the squared Weyl tensor.

we could modify (3.30) by adding contributions from the metric variation on the R^2 term. We then have the following 4D result:

$$\delta\langle T^{ab}\rangle^{(B)}_{n=4} = -4\frac{c_4}{(4\pi)^2}\Big(D_c D_d W^{cadb} + \frac{1}{2}R_{cd}W^{cadb}\Big)|_{\bar{g}} - \Omega^{-4}\Big[.....\Big]|_{\bar{g}\to g} = 0 \ . \quad (3.39)$$

Note that $\sqrt{-g}\Big(D_c D_d W^{cadb} + \frac{1}{2}R_{cd}W^{cadb}\Big)$ is conformal invariant and traceless. In this case, we trivially get

$$[\lim_{n\to 4}, \lim_{W\to 0}]\delta\langle T_{ab}\rangle^{(B)} = 0 \ , \quad (3.40)$$

with the same result as (3.31). Regarding the second kind of arbitrariness, we note that because of the following identity:

$$-4\sqrt{-g}\Big(D_c D_d W^{cadb} + \frac{1}{2}R_{cd}W^{cadb}\Big) = \frac{\delta}{\delta g_{ab}}\int d^4x\sqrt{-g}W_{abcd}W^{abcd} \ . \quad (3.41)$$

One could generate the form $\Big(D_c D_d W^{cadb} + \frac{1}{2}R_{cd}W^{cadb}\Big)$ with an arbitrary coefficient. But since this term transforms covariantly, it always give zero contribution to the transformed stress tensor.

Let us make a remark on the orders of different limits: in the previous chapter, which is based on [1], we followed the same argument in [19] that the type B anomalies do not contribute to the stress tensors in a conformally flat background because of the (at least) squared Weyl tensors. This implies that we were actually adopting the order

$$\lim_{n\to 4}\ \lim_{W\to 0} \ . \quad (3.42)$$

For the order $\lim_{W\to 4}\ \lim_{n\to 4}$, one should argue firstly why the $n\to 4$ limit is well-defined then use the argument of the squared Weyl tensors for the conformally flat case. The latter consideration is included in this chapter. In fact, using the order $\lim_{n\to 4}\ \lim_{W\to 0}$ was the hidden reason why $\frac{2}{3}D^2R$ in $c(W^2+\frac{2}{3}D^2R)$ in the trace anomaly gives a separated contribution to the stress tensor in [19]. In the previous chapter (or in [1]), we simply ignored $c\frac{2}{3}D^2R$ as the scheme to match with AdS/CFT results. Under the order $\lim_{W\to 4}\ \lim_{n\to 4}$ the regulator is needed since the type B anomaly is not a topological quantity. However, this time we will need $c\frac{2}{3}D^2R$ to have a result that vanishes in $W=0$. It might be most natural to adopt the scheme that one always introduces the regulator instead of considering the order $\lim_{n\to d}\lim_{W\to 0}$ on the type B anomaly.

Now, let us discuss yet another ambiguity by observing the following identity[5]:

$$\frac{\delta}{\delta\sigma(x)}\Big[\frac{1}{8}\int d^4x'\sqrt{-\bar{g}}\bar{W}^2(x')\ln\bar{g}(x')\Big] = \sqrt{-\bar{g}}\bar{W}^2(x) \ . \quad (3.43)$$

[5]Note the basic result $\delta\bar{g}_{\mu\nu} = 2\bar{g}_{\mu\nu}\delta\sigma$ implies $\frac{\delta g_{\mu\nu}}{\delta\sigma} = 0$ by considering a fixed $g_{\mu\nu}$ with respect to the conformal factor.

After the metric variation, one obtains

$$\delta\langle T^{ab}\rangle^{(B)}_{n=4} = -\frac{c_4}{(4\pi)^2}\Big[\Big(D_c D_d W^{cadb} + \frac{1}{2}R_{cd}W^{cadb}\Big)\ln g - \frac{1}{4}W^2 g_{ab}\Big]|_{\bar{g}} - \Omega^{-4}\Big[.....\Big]|_{\bar{g}\to g} \ , \tag{3.44}$$

in contrast to (3.39). This case gives nicely that

$$[\lim_{n\to 4}, Tr]\delta\langle T^{(B)}_{ab}\rangle = 0 = [\lim_{n\to 4}, \lim_{W\to 0}]\delta\langle T^{(B)}_{ab}\rangle \ . \tag{3.45}$$

Moreover, the identity implies the following σ invariant form:

$$\alpha\ \frac{\delta}{(n-4)\delta\sigma(x)}\Big[\int d^n x'\sqrt{-g}\mathcal{I}^{(4)} - \int d^4 x'\sqrt{-g}I^{(4)} - \frac{1}{8}\int d^4 x'(n-4)\sqrt{-g}I^{(4)}\ln\bar{g}(x')\Big] = 0 \ , \tag{3.46}$$

that can be freely added into (3.38) with an arbitrary coefficient α. In total it gives non-zero contribution to the stress tensor after the metric variation. As mentioned before, we might further introduce an $\alpha\frac{1}{18}R^2$ term that makes the result become the form $(DDW + 1/2RW)$ when combined with the first two terms in (3.46). Note that α will lead to a different coefficient of D^2R in the trace anomaly. Hence it would change the scheme. Fixing the coefficient of D^2R under a given scheme is needed to completely fix α.

3.3 Remarks

Let us relate our results with [21], where a general (trial) solution to the differential equation (3.1) was given by

$$\begin{aligned}\langle\bar{T}^\mu_\nu\rangle &= \Omega^{-4}\langle T^\mu_\nu\rangle - \frac{a_4}{(4\pi)^2}\Big[(4\bar{R}^\lambda_\rho\bar{W}^{\rho\mu}{}_{\lambda\nu} - 2\bar{H}^\mu_\nu) - \Omega^{-4}(4R^\lambda_\rho W^{\rho\mu}{}_{\lambda\nu} - 2H^\mu_\nu)\Big] \\ &- \frac{\gamma}{6(4\pi)^2}\Big[I^\mu_\nu - \Omega^{-4}I^\mu_\nu\Big] - 8\frac{c_4}{(4\pi)^2}\Big[\bar{D}^\rho\bar{D}_\lambda(\bar{W}^{\rho\mu}{}_{\lambda\nu}\ln\Omega) + \frac{1}{2}\bar{R}^\lambda_\rho\bar{W}^{\rho\mu}{}_{\lambda\nu}\ln\Omega\Big] \ ,\end{aligned} \tag{3.47}$$

where we have expressed it under the same convention, and

$$H_{\mu\nu} \equiv -\frac{1}{2}\Big[g_{\mu\nu}\Big(\frac{R^2}{2} - R^2_{\lambda\rho}\Big) + 2R^\lambda_\mu R_{\nu\lambda} - \frac{4}{3}RR_{\mu\nu}\Big] \ , \tag{3.48}$$

$$I_{\mu\nu} \equiv 2D_\mu D_\nu R - 2g_{\mu\nu}D^2R - 2RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^2 \ . \tag{3.49}$$

The corresponding results from the type A and type D anomaly parts agree with the results obtained from the dimensional regularization. The only mismatch part comes from the type B anomaly. The following is our explanation, which is again coming from the ambiguity. We note that the result (3.47) could be derived by varying the effective action given in (2.2 − 2.4) in [42] with respect to the metric. (One might call those actions as dilaton effective actions.)

That is, we can re-produce (3.47) by simply adopting these dilaton actions in our formulation. However, there might be some potential issues. The first issue is that these dilaton actions were written down with the explicitly given σ. One uses these dilaton actions because their σ variation could give trace anomalies. However, in the context of the dimensional regularization, we see it is certainly not the only way to re-write the anomalies into σ-exact forms. Allowing the explicit σ to appear will generate more ambiguities. Moreover, there is another issue that was already mentioned in [42] (in the paragraph between $eq(2.20-2.24)$): they needed to impose certain assumption on a spacetime background in order to deal with the metric variation on the σ. Finally, if we adopt the dilaton action it might lose the spirit of the dimensional regularization where the results are expressed in terms of curvature tensors.

Chapter 4

Universal Entanglement and Boundary Geometry in Conformal Field Theory

This chapter is an edited version of my publication [3], written in collaboration with Christopher Herzog and Kristan Jensen.

Entanglement entropy has played an increasingly important role in theoretical physics. Invented as a measure of quantum entanglement, it has been successfully applied in a much broader context. Entanglement entropy can serve as an order parameter for certain exotic phase transitions [43, 44]. It is likely very closely related to black hole entropy [45, 46]. Certain types of entanglement entropy order quantum field theories under renormalization group flow [47, 48, 49, 14]. It is the last result which is most relevant to this chapter. In even space-time dimension, the connection between entanglement entropy and renormalization group flow is tied up in the existence of the trace anomaly [47, 49, 14]. In fact, certain universal terms in the entanglement entropy can be extracted from the anomaly. The moral of this chapter will be that to use the anomaly correctly, one should understand how to write it down on a manifold with a codimension one boundary.

To define entanglement entropy, we assume that the Hilbert space can be factorized, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ corresponds to the Hilbert space for a spatial region A of the original quantum field theory.[1] Given such a factorization one can construct the reduced density matrix $\rho_A = \mathrm{tr}_B\, \rho$ by tracing over the degrees of freedom in the complementary region B, where ρ is the initial density matrix. The entanglement entropy is the von Neumann

[1]This factorization is a nontrivial assumption. The boundary between A and B, ∂A, plays an important role in recent discussons regarding the entanglement entropy of gauge theory [50, 51, 52, 7]. The boundary terms associated with ∂A we find in this chapter suggests that the factorization is not always a clean and unambiguous procedure even for non-gauge theories.

entropy of the reduced density matrix:

$$S_E \equiv -\operatorname{tr}(\rho_A \ln \rho_A) \ . \tag{4.1}$$

Only when $\rho = |\psi\rangle\langle\psi|$ is constructed from a pure state $|\psi\rangle$ does S_E measure the quantum entanglement. Otherwise, it is contaminated by the mixedness of the density matrix ρ.

In a quantum field theory context, the definition of S_E presents a challenge because the infinite number of short distance degrees of freedom render S_E strongly UV divergent. Consider for example a d-dimensional conformal field theory (CFT) in the vacuum. Let d be even so that the theory may have a Weyl anomaly, and let A be a $(d-1)$-dimensional ball of radius ℓ. In this case, the entanglement entropy has an expansion in a short distance cut-off δ of the form

$$S_E = \alpha \frac{\operatorname{Area}(\partial A)}{\delta^{d-2}} + \ldots + 4a(-1)^{d/2} \ln \frac{\delta}{\ell} + \ldots \tag{4.2}$$

The constant α multiplying the leading term is sensitive to the definition of the cut-off δ and thus has no physical meaning. The fact that the leading term scales with the area of the boundary of A, however, is physical and suggests that most of the correlations in the vacuum are local.

Most important for this chapter, the subleading term in eq. (4.2) proportional to the logarithm is "a," the coefficient multiplying the Euler density in the trace anomaly [24]

$$\langle T^\mu{}_\mu \rangle = \sum_j c_j I_j - (-1)^{d/2} \frac{4a}{d! \operatorname{Vol}(S^d)} E_d + \mathrm{D}_\mu J^\mu \,, \tag{4.3}$$

with D_μ the covariant derivative. In this expression, E_d is the Euler density normalized such that integrating E_d over an S^d yields $d! \operatorname{Vol}(S^d)$. See section 4.2 for more details about the definition of E_d. The I_j are curvature invariants which transform covariantly with weight $-d$ under Weyl rescalings. There is also a total derivative $\mathrm{D}_\mu J^\mu$ whose precise form depends on the particular regularization scheme used in defining the partition function.[2]

Our motivation is a puzzle described in ref. [40]. The authors describe several different methods for verifying the logarithmic contribution to the entanglement entropy in (4.2). One is to conformally map the causal development of the ball, $\mathcal{D}$, to the static patch of de Sitter spacetime, and then exploit the trace anomaly (4.3). Another method runs into difficulties. They attempt to compute S_E by mapping $\mathcal{D}$ to hyperbolic space. Here, the authors were not able to use the anomaly to obtain the expected results. As we shall explain, and as was anticipated in ref. [40], getting the correct answer requires a careful treatment of boundary terms in the anomaly.

To our knowledge, the relation between these boundary terms and entanglement entropy has not been considered carefully before.[3] In $d = 2$, the boundary contribution to the trace

[2]In the terminology of ref. [24], the Euler term is a type-A anomaly and the Weyl-covariants I_j are type-B.

[3]In a somewhat different vein, there is a discussion of entanglement entropy on spaces with boundary in ref. [53].

anomaly is textbook material [54]. In $d = 4$ and $d = 6$, the bulk anomaly induced dilaton effective actions are written down in refs. [14] and [15] respectively. (See also [55] for $d = 4$.) Given the importance of the dilaton effective action in understanding the a-theorem [14], and the recent "b-theorem" in $d = 3$ [56], it seems conceivable the boundary correction terms may be useful in a more general context.

In this chapter we generalize these dilaton effective actions with boundary terms for a manifold with codimension one boundary and we show that these boundary terms are crucial in computing entanglement entropy. We also provide a general procedure, valid in any even dimension, for computing these boundary terms.

We begin with the two-dimensional case in section 4.1, where we illustrate our program and use an anomaly action with boundary terms to recover the well-known results of the interval Rényi entropy [57, 58] and the Schwarzian derivative. In section 4.2, we construct the boundary terms in the trace anomaly in $d > 2$ and present an abstract formula for the anomaly action in arbitrary even dimension. We demonstrate the result satisfies Wess-Zumino consistency. In section 4.3, we compute the anomaly action in four and six dimensions, keeping careful track of the boundary terms. (In six dimensions, our boundary action is only valid in a conformally flat space time, while in four dimensions, the answer provided is completely general.) In section 4.4, we resolve the puzzle of how to compute the entanglement entropy of the ball through a map to hyperbolic space in general dimension. The resolution of this puzzle constitutes the main result of this chapter. Finally, we conclude in section 4.5. We relegate various technical details to appendices. Appendix 4.6.1 reviews some useful differential geometry for manifolds with boundary. Appendix 4.6.2 contains a detailed check of Wess-Zumino consistency in four dimensions. Appendix 4.6.3 contains details of the derivation of the anomaly action in four and six dimensions.

4.1 The Two Dimensional Case and Rényi entropy

In two dimensions, the stress tensor has the well known trace anomaly

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{c}{24\pi} R \,, \tag{4.4}$$

where we have replaced the anomaly coefficient a with the more common central charge $c = 12a$ which appears in the two-point correlation function of the stress tensor. Eq. (4.4) is the Ward identity for the anomalous Weyl symmetry. It is equivalent to the variation of the generating functional $W[g_{\mu\nu}] = -\ln Z[g_{\mu\nu}]$ under a Weyl variation $\delta g_{\mu\nu} = 2 g_{\mu\nu} \delta\sigma$. However, on a manifold with boundary, the anomalous variation of W may contain a boundary term. In this section, we show how to construct the anomaly effective action with boundary terms for the simplest case, $d = 2$. We will reproduce the classic entanglement entropy result using the boundary term in the anomaly action. We also show that the boundary term correctly recovers the universal term in the single-interval $d = 2$ Rényi entropy.

4.1.1 Anomaly Action with Boundary and Entanglement Entropy

In $d = 2$, the most general result for the Weyl variation of the partition function consistent with Wess-Zumino consistency is [54]

$$\delta_\sigma W = -\frac{c}{24\pi}\left[\int_M \mathrm{d}^2x\sqrt{g}R\,\delta\sigma + 2\int_{\partial M}\mathrm{d}y\sqrt{\gamma}K\,\delta\sigma\right] . \tag{4.5}$$

To write this expression, we have introduced some notation. In $d = 2$, the notation is overkill, but we need the full story in what follows in $d > 2$. We denote bulk coordinates as x^μ and boundary coordinates as y^α. Let n^μ be the unit-length, outward pointing normal vector to ∂M and $\gamma_{\alpha\beta}$ the induced metric on ∂M. We can define K in two equivalent ways. First, locally near the boundary we can extend n^μ into the bulk. We can choose to extend it in such a way that $n^\mu \mathrm{D}_\mu n_\nu = 0$, in which case the extrinsic curvature is defined to be $K_{\mu\nu} \equiv \mathrm{D}_{(\mu}n_{\nu)}$. The trace of the extrinsic curvature is $K = K^\mu{}_\mu$. Alternatively, we can also define K purely from data on the boundary. The bulk covariant derivative D_μ induces a covariant derivative $\mathring{\nabla}_\alpha$ on the boundary. It can act on tensors with bulk indices, boundary indices, or mixed tensors with both. We specify the boundary through a map $\partial M \to M$, which amounts to a set of d embedding functions $X^\mu(y^\alpha)$. The $\partial_\alpha X^\mu$ are tensors on the boundary, and their derivative gives the extrinsic curvature as $K_{\alpha\beta} = -n_\mu \mathring{\nabla}_\alpha \partial_\beta X^\mu$, and its trace $K = \gamma^{\alpha\beta}K_{\alpha\beta}$. For more details on differential geometry of manifolds with boundary, see appendix 4.6.1.

Observe that, for a constant Weyl rescaling $\delta\sigma = \lambda$, the Weyl anomaly (4.5) is equivalent to

$$\delta_\lambda W = -\frac{c}{6}\chi\lambda\,, \tag{4.6}$$

where χ is the Euler characteristic of M. That is, the boundary term in the Weyl anomaly is simply the boundary term in the Euler characteristic.

Recall that the stress tensor is defined as

$$\langle T^{\mu\nu}\rangle = -\frac{2}{\sqrt{g}}\frac{\delta W}{\delta g_{\mu\nu}}\,, \tag{4.7}$$

in which case (4.5) leads to a boundary term in the trace of the stress tensor,

$$\langle T^\mu{}_\mu\rangle = \frac{c}{24\pi}\left(R + 2K\delta(x^\perp)\right) , \tag{4.8}$$

where $\delta(x^\perp)$ is a Dirac delta function with support on the boundary.

We now wish to write down a local functional which reproduces the variation (4.5). To do so we introduce an auxiliary "dilaton" field τ which transforms under a Weyl transformation $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$ as $\tau \to \tau + \sigma$. The quantity

$$\hat{g}_{\mu\nu} \equiv e^{-2\tau}g_{\mu\nu}\,, \tag{4.9}$$

is invariant under this generalized Weyl scaling and so too the effective action $\hat{W} \equiv W[e^{-2\tau} g_{\mu\nu}] = W[\hat{g}_{\mu\nu}]$. Then

$$\mathcal{W}[g_{\mu\nu}, e^{-2\tau} g_{\mu\nu}] \equiv W - \hat{W}\,, \tag{4.10}$$

will vary to reproduce the anomaly, $\delta_\sigma \mathcal{W} = \delta_\sigma W$. In what follows, we refer to $\mathcal{W}$ as a "dilaton effective action," given its similarities with the dilaton effective action presented in refs. [14, 15]. However, unlike those works we are only considering conformal fixed points and not renormalization group flows, and so this name is a bit of a misnomer. More precisely, $\mathcal{W}$ is a Wess-Zumino term for the Weyl anomaly, or alternatively an anomaly effective action. Analytically continuing to Lorentzian signature, it computes the phase picked up by the partition function under the Weyl rescaling from a metric $g_{\mu\nu}$ to $e^{-2\tau} g_{\mu\nu}$.

What exactly is $\mathcal{W}$ in $d = 2$? The first quick guess is

$$\mathcal{W}_0 = -\frac{c}{24\pi}\left[\int_M \mathrm{d}^2x \sqrt{g} R\tau + 2\int_{\partial M} \mathrm{d}y \sqrt{\gamma} K\tau\right]\,. \tag{4.11}$$

But the metric scales, and we should take into account that under Weyl scaling in $d = 2$,

$$\begin{aligned} R[e^{2\sigma} g_{\mu\nu}] &= e^{-2\sigma}(R[g_{\mu\nu}] - 2\Box\sigma)\,, \\ K[e^{2\sigma} g_{\mu\nu}] &= e^{-\sigma}(K[g_{\mu\nu}] + n^\mu \partial_\mu \sigma)\,. \end{aligned} \tag{4.12}$$

To cancel these variations, we add a $(\partial\tau)^2 \equiv (\partial_\mu\tau)(\partial^\mu\tau)$ term to the effective action. The total effective anomaly action is then

$$\mathcal{W} = -\frac{c}{24\pi}\left[\int_M \mathrm{d}^2x \sqrt{g}\left(R[g_{\mu\nu}]\tau - (\partial\tau)^2\right) + 2\int_{\partial M} \mathrm{d}y \sqrt{\gamma}\, K[g_{\mu\nu}]\tau\right] + (\text{invariant})\,. \tag{4.13}$$

The right-hand side is computed with the original unscaled metric $g_{\mu\nu}$.[4] In writing (4.13), we have allowed for the possibility of additional terms invariant under the Weyl symmetry. There are only two such terms with dimensionless coefficients,

$$\int_M \mathrm{d}^2x \sqrt{\hat{g}} \hat{R}\,, \qquad \int_{\partial M} \mathrm{d}y \sqrt{\hat{\gamma}} \hat{K}\,. \tag{4.14}$$

However, now we use that by definition $\mathcal{W} = 0$ when $\tau = 0$. Thus neither of these terms can appear in $\mathcal{W}$, so

$$\mathcal{W} = -\frac{c}{24\pi}\left[\int_M \mathrm{d}^2x \sqrt{g}\left(R[g_{\mu\nu}]\tau - (\partial\tau)^2\right) + 2\int_{\partial M} \mathrm{d}y \sqrt{\gamma}\, K[g_{\mu\nu}]\tau\right]\,. \tag{4.15}$$

The second step, which involved adding by hand a $(\partial\tau)^2$ term to cancel some unwanted pieces of the Weyl variation, seemed to involve some guess work which could become a

[4] This action corrects a typo in eq. (1.2) of ref. [59], as well as accounts for the boundary term.

problem in $d > 2$ where the expressions are much more complicated. In fact, there are several constructive algorithms which remove this element of guesswork. One method involves integrating the anomaly [60, 22, 61]:

$$
\begin{aligned}
\mathcal{W} &= -\frac{c}{24\pi}\int_0^1 \mathrm{d}t\,\left[\int_M \mathrm{d}^2x\,\sqrt{g'}\,R[g'_{\mu\nu}]\tau + 2\int_{\partial M}\mathrm{d}y\,\sqrt{\gamma'}\,K[g'_{\mu\nu}]\tau\right]\Bigg|_{g'_{\mu\nu}=e^{-2t\tau}g_{\mu\nu}} \\
&= -\int_0^1 dt\int_M \mathrm{d}^2x\sqrt{g'}\,\langle T^{\mu}{}_{\mu}[g'_{\nu\rho}]\rangle\tau\Bigg|_{g'_{\mu\nu}=e^{-2t\tau}g_{\mu\nu}} .
\end{aligned}
\tag{4.16}
$$

Thus, given the trace anomaly $\langle T^{\mu}{}_{\mu}\rangle$, it is straightforward albeit messy to reconstruct $\mathcal{W}$.

The second method (which we elaborate in this chapter) is dimensional regularization [19, 62]. We define $\widetilde{W}[g_{\mu\nu}]$ in $n = 2+\epsilon$ dimensions:

$$
\widetilde{W}[g_{\mu\nu}] \equiv -\frac{c}{24\pi(n-2)}\left[\int_M \mathrm{d}^n x\,\sqrt{g}\,R + 2\int_{\partial M}\mathrm{d}^{n-1}y\,\sqrt{\gamma}\,K\right] , \tag{4.17}
$$

where R, K, $g_{\mu\nu}$, and $\gamma_{\alpha\beta}$ are dimensionally continued in the naive way. We claim then that

$$
\mathcal{W} = \lim_{n\to 2}\left(\widetilde{W}[g_{\mu\nu}] - \widetilde{W}[e^{-2\tau}g_{\mu\nu}]\right) , \tag{4.18}
$$

as one may verify after a short calculation, using the more general rules for the Weyl transformations in n dimensions,

$$
\begin{aligned}
R[e^{2\sigma}g_{\mu\nu}] &= e^{-2\sigma}\left(R[g_{\mu\nu}] - 2(n-1)\Box\sigma - (n-2)(n-1)(\partial\sigma)^2\right) , \\
K[e^{2\sigma}g_{\mu\nu}] &= e^{-\sigma}\left(K[g_{\mu\nu}] + (n-1)n^{\mu}\partial_{\mu}\sigma\right) .
\end{aligned}
\tag{4.19}
$$

In all three cases, we are computing the same difference between two effective actions. It would be preferable to have access to the effective actions themselves. There are two problems here. The full actions depend on more than the anomaly coefficients. They are also likely to be ultraviolet and perhaps also infrared divergent. If we focus just on the anomaly dependent portion, it could easily be that some of this anomaly dependence is invariant under Weyl scaling and drops out of the difference we have computed. Interestingly, the dimensional regularization procedure offers a regulated candidate $\widetilde{W}[g_{\mu\nu}]$ for the anomaly dependent portion of $W[g_{\mu\nu}]$.

Let us try to extract some information from the regulated candidate action in flat space:

$$
\widetilde{W}[\delta_{\mu\nu}] = -\frac{c}{12\pi(n-2)}\int_{\partial M}\mathrm{d}^{n-1}y\sqrt{\gamma}K . \tag{4.20}
$$

A simple case, which also turns out to be relevant for the entanglement entropy calculations we would like to perform, is where M is a large ball of radius Λ with a set of q smaller, non-intersecting balls of radius δ_j removed. For each ball, we can work in a local coordinate system where r is a radial coordinate. For the smaller balls, $\sqrt{\gamma}K = -r^{n-2}$ while for the large ball $\sqrt{\gamma}K = r^{n-2}$. It then follows that

$$\widetilde{W}[\delta_{\mu\nu}] = -\frac{c}{6}\left[\frac{1}{n-2}(1-q) + \frac{q+1}{2}(\gamma + \ln \pi) + \ln \Lambda - \sum_{j=1}^{q} \ln \delta_j + O(n-2)\right] . \quad (4.21)$$

The leading divergent contribution is proportional to the Euler characteristic $\chi = 1 - q$ of the surface. We claim that the $\ln \delta_j$ pieces of the expression (4.21) can be used to identify a universal contribution to the entanglement entropy of a single interval in flat space. We will justify the computation through a conformal map to the cylinder, but in brief, the computation goes as follows. For an interval on the line with left endpoint u and right endpoint v, to regulate the UV divergences in the entanglement entropy computation we place small disks around the points u and v with radius δ. The entanglement entropy then turns out to be the logarithmic contribution of these disks to $-\widetilde{W}[\delta_{\mu\nu}]$:

$$S_E \sim -\frac{c}{3} \ln \delta \ . \quad (4.22)$$

As the underlying theory is conformal, the answer can only depend on the conformal cross ratio of the two circles $4\delta^2/|u-v|^2$. Thus we find the classic result [63, 58]

$$S_E \sim \frac{c}{3} \ln \frac{|v-u|}{\delta} \ . \quad (4.23)$$

Here and henceforth, the $\sim$ indicates that the LHS has a logarithmic dependence given by the RHS. We neglect the computation of the constant quantity in S_E, as it depends on the precise choice of regulator and so is unphysical.

A more thorough justification of this computation occupies the next two subsections. In broad terms, the same result turns out to be valid in even dimensions $d > 2$, a fact whose demonstration will occupy most of the remainder of this chapter. More specifically, we mean that the logarithmic contribution to $\widetilde{W}[\delta_{\mu\nu}]$ for flat space with $D \times S^{d-2}$ removed, where D is a small disk of radius δ, yields a universal contribution to entanglement entropy for a ball shaped region in flat space.

To return to $d = 2$, we describe the plane to the cylinder map and its relevance for entanglement entropy in section 4.1.3. The demonstration however requires we also know how the stress tensor transforms under conformal transformations. The transformation involves the Schwarzian derivative which can be found in most textbooks on conformal field theory. In an effort to be self contained we will use our effective anomaly action to derive the Schwarzian derivative in section 4.1.2. In $d = 2$, the effective action turns out to be useful to compute not only the entanglement entropy but also the single interval Rényi entropies. A calculation of the Rényi entropies is provided in section 4.1.4.

4.1.2 The Schwarzian Derivative

To calculate the change in the stress tensor under a Weyl scaling from $g_{\mu\nu}$ to $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$, we begin with a variation of $\mathcal{W} = W - \hat{W}$ with respect to the metric $g_{\mu\nu}$,

$$\begin{aligned} \delta\mathcal{W} =& \delta W - \delta\hat{W} \\ =& -\frac{1}{2}\int_M \mathrm{d}^2x \left(\sqrt{g}\,\delta g_{\mu\nu}\langle T^{\mu\nu}\rangle_g - \sqrt{\hat{g}}\,\delta\hat{g}_{\mu\nu}\langle T^{\mu\nu}\rangle_{\hat{g}}\right) \\ =& -\frac{1}{2}\int_M \mathrm{d}^2x\sqrt{g}\,\delta g_{\mu\nu}\left(\langle T^{\mu\nu}\rangle_g - e^{-4\tau}\langle T^{\mu\nu}\rangle_{\hat{g}}\right) , \end{aligned} \tag{4.24}$$

where in the last line we have used that $\sqrt{\hat{g}}\delta\hat{g}_{\mu\nu} = \sqrt{g}\,e^{-(d+2)\tau}\delta g_{\mu\nu}$ in d dimensions. The subscript g on the expectation value refers to $\langle T^{\mu\nu}\rangle$ on the space with metric g, and similarly for $\hat{g}$. Using the explicit expression for $\mathcal{W}$ in (4.15), we compute its variation

$$\begin{aligned} \delta\mathcal{W} = & -\frac{c}{24\pi}\int_M \mathrm{d}^2x\sqrt{g}\,\delta g_{\mu\nu}\left[\partial^\mu\tau\partial^\nu\tau + \mathrm{D}^\mu\partial^\nu\tau - g^{\mu\nu}\left(\frac{1}{2}(\partial\tau)^2 + \Box\tau\right)\right] \\ & - \frac{c}{24\pi}\int_{\partial M}\mathrm{d}y\sqrt{\gamma}\,\delta g_{\mu\nu}h^{\mu\nu}n^\rho\partial_\rho\tau , \end{aligned} \tag{4.25}$$

where $h^{\mu\nu}$ is the projector to the boundary,

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu . \tag{4.26}$$

In obtaining (4.25) we have used that in two dimensions the Einstein tensor $R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}$ vanishes, and that the variation of the Ricci tensor is a covariant derivative $\delta R_{\mu\nu} = \mathrm{D}_\rho\delta\Gamma^\rho{}_{\mu\nu} - \mathrm{D}_\nu\delta\Gamma^\rho{}_{\mu\rho}$. Putting (4.25) together with (4.24), we find

$$\langle T_{\mu\nu}\rangle_{\hat{g}} = \langle T_{\mu\nu}\rangle_g - \frac{c}{12\pi}\left[\partial_\mu\tau\partial_\nu\tau + \mathrm{D}_\mu\partial_\nu\tau - g_{\mu\nu}\left(\frac{1}{2}(\partial\tau)^2 + \Box\tau\right)\right] - \frac{c}{12\pi}\delta(x^\perp)h_{\mu\nu}n^\rho\partial_\rho\tau . \tag{4.27}$$

Suppose we consider a Weyl rescaling which takes us from flat space, $g_{\mu\nu} = \delta_{\mu\nu}$, to the new metric $\hat{g}_{\mu\nu} = e^{-2\tau}\delta_{\mu\nu}$. The stress tensor for a conformal theory in vacuum on the plane is usually defined to vanish. Thus the stress tensor on the manifold with metric $e^{-2\tau}\delta_{\mu\nu}$ will be

$$\langle T_{\mu\nu}\rangle = -\frac{c}{12\pi}\left[\partial_\mu\tau\partial_\nu\tau + \partial_\mu\partial_\nu\tau - \delta_{\mu\nu}\left(\frac{1}{2}(\partial\tau)^2 + (\Box\tau)\right)\right] \tag{4.28}$$

(dropping the boundary contribution). The Schwarzian derivative describes how the stress tensor transforms under a conformal transformation, i.e. a combination of a Weyl rescaling and a diffeomorphism that leaves the metric invariant. If the complex plane is parametrized initially by z and $\bar{z}$, we introduce new variables $w(z)$ and $\bar{w}(\bar{z})$ and require that the Weyl rescaling satisfies

$$e^{-2\tau} = \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial\bar{w}}{\partial\bar{z}}\right) . \tag{4.29}$$

Start with the stress tensor in the w-plane, and perform a diffeomorphism to go to the z variables. That transformed stress tensor should be related by a Weyl rescaling by $e^{-2\tau}$ to the stress tensor on the flat complex z-plane. Recalling that $g_{zz} = 0$, we find that

$$\begin{aligned}(\partial_z w)^2 \langle T_{ww}(w)\rangle = \langle T_{zz}(z)\rangle_{e^{-2\tau}\delta_{\mu\nu}} &= -\frac{c}{12\pi}\left[(\partial_z\tau)^2 + (\partial_z^2\tau)\right] \\ &= \frac{c}{48\pi}\frac{2(\partial_z^3 w)(\partial_z w) - 3(\partial_z^2 w)^2}{(\partial_z w)^2} ,\end{aligned} \tag{4.30}$$

which is the usual result for the Schwarzian derivative.

4.1.3 Entanglement Entropy from the Plane and Cylinder

We now consider the entanglement entropy of an interval with left endpoint u and right endpoint v. The information necessary to compute the entropy is contained in the causal development of this interval, i.e. the diamond shaped region bounded by the four null lines $x \pm t = u$ and $x \pm t = v$.

We will indirectly deduce the entanglement entropy by conformally mapping to a thermal cylinder, keeping careful track of the phase picked up by the partition function under the transformation.

Consider the following change of variables

$$e^{2\pi w/\beta} = \frac{z-u}{z-v} , \tag{4.31}$$

where $z = x - t = x + it_{\rm E}$, and correspondingly for $\bar{z}$ and $\bar{w}$. If we let $w = \sigma^1 + i\sigma^2$, then σ^2 is periodic with periodicity β, $\sigma^2 \sim \sigma^2 + \beta$. In other words, the theory on the w-plane is naturally endowed with a temperature $1/\beta$. The other nice property of this map is that the the interval at time $t = 0$ is mapped to the real line $\mathrm{Re}(w)$. Thus the reduced density matrix ρ_A associated with the interval is related by a unitary transformation to the thermal density matrix ρ_β on the line. As the entanglement entropy is invariant under unitary transformations, the entanglement entropy of the interval is the thermal entropy associated with the cylinder, that is the thermal entropy on the infinite line. If we let

$$\rho = \frac{e^{-\beta H}}{\mathrm{tr}\, e^{-\beta H}} , \tag{4.32}$$

where H is the Hamiltonian governing evolution on the line, then

$$S_E = -\,\mathrm{tr}(\rho \ln \rho) = \beta\,\mathrm{tr}(\rho H) + \ln \mathrm{tr}(e^{-\beta H}) = \beta\langle H\rangle - W_{\rm cyl} , \tag{4.33}$$

where $W_{\rm cyl} \equiv -\ln \mathrm{tr}\, e^{-\beta H}$ is the partition function on the cylinder. This entropy is infinite because the cylinder is infinitely long in the σ^1 direction, and we need to regulate the divergence. The natural way to regulate is to cut off the cylinder such that $-\Lambda < \sigma^1 < \Lambda$.

In the $z = x + it_\mathrm{E}$ plane, these cut-offs correspond to putting small disks of radius δ around the endpoints u and v, where now

$$\frac{\delta}{v-u} = e^{-2\pi\Lambda/\beta} \ . \tag{4.34}$$

We have two quantities to compute, $\beta\langle H\rangle$ and W_cyl. We can use the Schwarzian derivative from the previous subsection to compute

$$\beta\langle H\rangle = \int_\mathrm{cyl} \langle T^{00}\rangle \mathrm{d}\sigma^1 \ , \tag{4.35}$$

where we have analytically continued $\sigma^0 = -i\sigma^2$. From the transformation rules (4.30) and (4.31), the ww component of the stress tensor on the cylinder is

$$\langle T_{ww}(w)\rangle \ = \ \frac{\pi c}{12\beta^2} \ . \tag{4.36}$$

In Cartesian coordinates, $T^{22} = -\frac{1}{4}(T^{ww} + T^{\bar{w}\bar{w}})$. Thus we have, analytically continuing to real time $\sigma^0 = -i\sigma^2$, a positive thermal energy $\langle T^{00}\rangle = \frac{\pi c}{6\beta^2}$ from which follows the first quantity of interest

$$\beta\langle H\rangle = \frac{\pi c}{3\beta}\Lambda = \frac{c}{6}\ln\frac{|v-u|}{\delta} \ . \tag{4.37}$$

Toward the goal of computing W_cyl, we first compute the difference in anomaly actions $\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]$ where the dilaton τ is derived from the plane to cylinder map

$$\tau = -\frac{1}{2}\ln\left[\frac{\beta}{2\pi}\left(\frac{1}{v-z} - \frac{1}{u-z}\right)\right] + c.c. \tag{4.38}$$

Given the dilaton, we can compute the bulk contribution to the difference in effective actions

$$\int \mathrm{d}^2x\sqrt{g}\,(\partial\tau)^2 = \left(\frac{\pi}{\beta}\right)^2 \int_\mathrm{cyl} \mathrm{d}w\,\mathrm{d}\bar{w}\left|\coth\frac{\pi w}{\beta}\right|^2 = \frac{8\pi^2}{\beta}\Lambda \ , \tag{4.39}$$

and the boundary contribution

$$-2\int \mathrm{d}y\sqrt{\gamma}\,K\tau \sim 8\pi\ln\delta \sim -\frac{16\pi^2}{\beta}\Lambda \ . \tag{4.40}$$

Assembling the pieces, the difference in anomaly actions is then

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}] \sim -\frac{\pi c}{3\beta}\Lambda = -\frac{c}{6}\ln\frac{|v-u|}{\delta} \ . \tag{4.41}$$

The last component we need is the universal contribution to $W[\delta_{\mu\nu}]$, which we claimed was actually equal to the universal contribution to single interval entanglement entropy. Indeed, everything works as claimed since the contributions from $\beta\langle H\rangle$ and $\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]$ cancel out:

$$S_E = \beta\langle H\rangle + \mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}] - W[\delta_{\mu\nu}] \sim -\widetilde{W}[\delta_{\mu\nu}] \sim \frac{c}{3}\ln\frac{|v-u|}{\delta} \ . \tag{4.42}$$

4.1.4 Rényi Entropies from the Annulus

In $d = 2$, the anomaly action also allows us to compute the Rényi entropies of an interval A,

$$S_n \equiv \frac{1}{1-n} \ln \operatorname{tr} \rho_A^n \ . \tag{4.43}$$

We use the replica trick to compute S_n. We can replace $\operatorname{tr} \rho_A^n$ with a certain ratio of Euclidean partition functions

$$\operatorname{tr} \rho_A^n = \frac{Z(n)}{Z(1)^n} \ , \tag{4.44}$$

where $Z(n)$ is the path integral on an n-sheeted cover of flat space, branched over the interval A. In the present case, we can use the coordinate transformation,

$$w = \frac{z-u}{z-v} \ , \tag{4.45}$$

to put the point u at the origin and the point v at infinity. As is familiar from the computation in the previous subsection, we need to excise small disks around the points u and v, or correspondingly restrict to an annulus in the w plane of radius $r_{\text{min}} < r < r_{\text{max}}$.

To get the Rényi entropies, we would like to compare the partition function on the annulus to an n-sheeted cover of the annulus. In two dimensions, these two metrics are related by a Weyl transformation. We take the metric on the annulus to be

$$g = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \ , \tag{4.46}$$

while on the n-sheeted cover we have

$$\hat{g} = e^{-2\tau} g = \mathrm{d}\rho^2 + n^2 \rho^2 \mathrm{d}\theta^2 \ , \tag{4.47}$$

with $e^{-\tau} = n r^{n-1}$ and $\rho = r^n$. With this choice of τ, the difference in anomaly actions becomes

$$\begin{aligned} \mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu},] &= \frac{c}{12}\left[\int_{r_{\text{min}}}^{r_{\text{max}}} (\partial\tau)^2 r \, \mathrm{d}r - 2\tau|_{r_{\text{min}}}^{r_{\text{max}}}\right] \\ &= \frac{c}{12}(n^2-1) \ln \frac{r_{\text{max}}}{r_{\text{min}}} \ . \end{aligned} \tag{4.48}$$

Now to isolate the universal contribution to $W[e^{-2\tau}\delta_{\mu\nu}]$, we should remove the universal contribution from $W[\delta_{\mu\nu}]$:

$$W[e^{-2\tau}\delta_{\mu\nu}] \sim -\frac{c}{12}(n^2+1) \ln \frac{r_{\text{max}}}{r_{\text{min}}} \sim -\frac{c}{12}\left(n + \frac{1}{n}\right) \ln \frac{\rho_{\text{max}}}{\rho_{\text{min}}} \ . \tag{4.49}$$

We can tentatively identity this quantity with $-\ln Z(n)$. To compute the Rényi entropies, we need to subtract off $n \ln Z(1)$. There is an issue here, however: both $\ln Z(n)$ and $\ln Z(1)$ are divergent quantities, and in comparing them we must arrange for the cutoffs to be congruous. We claim that in order to compare $\ln Z(n)$ with $\ln Z(1)$ we ought to use the ρ-cutoffs so that we excise discs of the same radius in each case. Thus, we need to subtract $nW[\delta_{\mu\nu}]$ using the cut-offs in the ρ coordinate system,

$$\ln Z(n) - n \ln Z(1) \sim \frac{c}{12}\left(-n + \frac{1}{n}\right) \ln \frac{\rho_{\max}}{\rho_{\min}} \; . \tag{4.50}$$

Using the definition (4.43) of the Rényi entropy, we find that

$$S_n \sim \frac{c}{12}\left(1 + \frac{1}{n}\right) \ln \frac{\rho_{\max}}{\rho_{\min}} \; . \tag{4.51}$$

Translating back to the z plane, this result recovers the classic result [57, 58][5]

$$S_n \sim \frac{c}{6}\left(1 + \frac{1}{n}\right) \ln \frac{|v-u|}{\delta} \; . \tag{4.52}$$

Taking $n \to 1$, it reduces to the previous entanglement entropy result (4.23). Note that in $d > 2$, one still has an n-sheeted cover of an annulus, but it is less clear what to do with the remaining $d-2$ dimensions.

4.2 Anomaly Actions in More than Two Dimensions

The trace anomaly (4.3) and effective anomaly action $\mathcal{W}$ have an increasingly complicated structure as the dimension increases. Several issues need to be addressed for a complete treatment of the effective action. Before embarking, we warn the reader that this section is technical. The chief results are 1) the boundary term in the a-type anomaly (4.61) and (4.68), 2) two equivalent forms for the a-type anomaly action in (4.69) and (4.113), and 3) a demonstration that the a-type anomaly, including the boundary term we obtain, is Wess-Zumino consistent in any dimension in subsection 4.2.3. Finally, 4) in (4.108) we present the most general form of the trace anomaly in $d = 4$, including boundary central charges.

4.2.1 Boundary Term of the Euler Characteristic

As we are motivated by the problem of universal contributions to the entanglement entropy across a sphere in flat space, our main focus is on how the a contribution to the anomaly action is modified in the presence of a boundary. Regarding the other issues, we make a few preliminary comments which will be developed minimally in the rest of the chapter.

The presence of a boundary affects the c_j contributions to the trace anomaly (4.3) trivially. Let us dispose of this issue immediately. The I_j are, by definition, covariant under

[5]The calculation we have just presented is very similar in spirit if not in detail to ones in refs. [64, 65].

Weyl scaling. In fact the $\sqrt{g}I_j$ are invariant under Weyl scaling and so the c_j contributions to $\mathcal{W}[g_{\mu\nu}, e^{-2\tau}g_{\mu\nu}]$ are simply

$$\mathcal{W}_c \equiv -\sum_j c_j \int_M \mathrm{d}^d x \sqrt{g}\, \tau I_j \ , \tag{4.53}$$

with no additional boundary term.

The total derivative term in the trace anomaly (4.3) depends on the choice of scheme. As we focus on universal aspects of the trace anomaly, with some exceptions we shall largely ignore this object in what follows. A fourth issue we have little to say about, with one exception, is the possible existence of additional terms in the trace anomaly associated purely with the boundary. These additional terms are best understood when the bulk CFT is odd-dimensional, so that the trace anomaly only has boundary terms. Those boundary terms can include the boundary Euler density as well as Weyl-covariant scalars [66, 67], in analogy with the trace anomaly of even-dimensional CFT. See ref. [56], which argued for a boundary "c-theorem" using this boundary anomaly. In this work we focus on CFTs in even dimension, with an odd-dimensional boundary. In $d = 4$, using Wess-Zumino consistency, we identify two allowed boundary terms in the trace anomaly, but have nothing to add in $d \geq 6$.

To return to the a-type anomaly, the central observation is that the a dependent contribution to the trace anomaly (4.3) integrates to give a quantity proportional to the Euler characteristic for a manifold without boundary. The natural guess is then that in the presence of a boundary, one should add whatever boundary term is needed such that the integral continues to give a quantity proportional to the Euler characteristic. (Indeed we saw precisely this story play out in two dimensions in section 4.1.) The requisite boundary term is well known in the mathematics literature. See for example the review [68]. It is a Chern-Simons like term constructed from the Riemann and extrinsic curvatures. To write it down, we need some notation.

We start by introducing the orthonormal (co)frame one forms $e^A = e^A_\mu dx^\mu$, in terms of which the metric on M is $g_{\mu\nu} = \delta_{AB} e^A_\mu e^B_\nu$. Here and there, we also need their inverse E^μ_A, satisfying $E^\mu_A e^A_\nu = \delta^\mu_\nu$ and $E^\mu_A e^B_\mu = \delta^A_B$. From the e^A and the Levi-Civita connection $\Gamma^\mu{}_{\nu\rho}$, we construct the connection one-form $\omega^A{}_B$ via

$$\partial_\mu e^A_\nu - \Gamma^\rho{}_{\nu\mu} e^A_\rho + \omega^A{}_{B\mu} e^B_\nu = 0\,. \tag{4.54}$$

From this definition, it follows that $\omega^{AB} = -\omega^{BA}$ and the torsion one-form vanishes,

$$\mathrm{d}e^A + \omega^A{}_B \wedge e^B = 0\,. \tag{4.55}$$

Further, the curvature two-form built from $\omega^A{}_B$,

$$\mathcal{R}^A{}_B \equiv \mathrm{d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B = \frac{1}{2}\mathcal{R}^A{}_{B\mu\nu} dx^\mu \wedge dx^\nu \ , \tag{4.56}$$

is related to the Riemann curvature by

$$E^\mu_A \mathcal{R}^A{}_{B\rho\sigma} e^B_\nu = R^\mu{}_{\nu\rho\sigma} \,. \tag{4.57}$$

The curvature two-form satisfies the Bianchi identity

$$\mathrm{d}\mathcal{R}^A{}_B + \omega^A{}_C \wedge \mathcal{R}^{CB} - \mathcal{R}^A{}_C \wedge \omega^C{}_B = 0 \ . \tag{4.58}$$

The Euler form is then

$$\mathcal{E}_d \equiv \mathcal{R}^{A_1 A_2} \wedge \cdots \wedge \mathcal{R}^{A_{d-1}A_d} \epsilon_{A_1 \cdots A_d} \ . \tag{4.59}$$

where $\epsilon_{A_1\cdots A_d}$ is the totally antisymmetric Levi-Civita tensor in dimension d. The Euler form and Euler density are related in the obvious way $\mathcal{E}_d = E_d \,\mathrm{vol}(M)$, for $\mathrm{vol}(M)$ the volume form on M. In writing (4.59) we have normalized the Euler form so that its integral over an S^d is $d!\,\mathrm{Vol}(S^d)$.

To define the Chern-Simons like boundary term, it is convenient to define a connection one-form and curvature two-form that interpolate linearly between a reference one-form ω_0 and the actual one-form of interest ω:

$$\begin{aligned} \omega(t) &\equiv t\omega + (1-t)\omega_0 \,, \\ \mathcal{R}(t)^A{}_B &\equiv \mathrm{d}\omega(t)^A{}_B + \omega(t)^A{}_C \wedge \omega(t)^C{}_B \,. \end{aligned} \tag{4.60}$$

The boundary term is constructed from the $d-1$ form:

$$\mathcal{Q}_d \equiv \frac{d}{2}\int_0^1 \mathrm{d}t\,\dot\omega(t)^{A_1A_2} \wedge \mathcal{R}(t)^{A_3A_4} \wedge \cdots \wedge \mathcal{R}(t)^{A_{d-1}A_d} \epsilon_{A_1\cdots A_d} \,. \tag{4.61}$$

(The density Q_d is given by $\mathcal{Q}_d = Q_d \,\mathrm{vol}(\partial M)$.) If we also define

$$\mathcal{E}(t)_d \equiv \mathcal{R}(t)^{A_1A_2} \wedge \cdots \wedge \mathcal{R}(t)^{A_{d-1}A_d} \epsilon_{A_1\cdots A_d} \ , \tag{4.62}$$

then it follows, as we show below,

$$\mathcal{E}(1)_d - \mathcal{E}(0)_d = \mathrm{d}\mathcal{Q}_d \ . \tag{4.63}$$

The relevance of this construction to the Euler characteristic is that we can calculate the Euler characteristic for a manifold M with boundary by comparing it to a manifold M^0 with the same boundary and zero Euler characteristic. Because $\chi(A\times B) = \chi(A)\chi(B)$ and because χ vanishes in odd dimensions, one such zero characteristic manifold is a product manifold where both A and B are odd dimensional. In a patch near the boundary, we can always choose to express the metric in Gaussian normal coordinates,

$$g = \mathrm{d}r^2 + f(r,x)_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{4.64}$$

where the boundary is located at $r = r_0$. In this patch, we can choose a reference metric g_0 so that the patch is a product space,

$$g_0 = \mathrm{d}r^2 + f(r_0, x)_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \ . \tag{4.65}$$

Let ω_0 be the connection one-form associated with the metric g_0. By construction $\mathcal{E}_d(1) = \mathcal{E}_d$, and it follows from the local relation (4.63) that the Euler characteristic for a manifold with boundary is

$$\chi(M) = \frac{2}{d!\,\mathrm{Vol}(S^d)}\left(\int_M \mathcal{E}_d - \int_{\partial M} \mathcal{Q}_d\right) \ . \tag{4.66}$$

We have normalized the characteristic so that $\chi(S^d) = 2$.

On the boundary ∂M, we can give an explicit formula for $\dot{\omega}^{AB}$ in terms of the extrinsic curvature,

$$\dot{\omega}(t)^{AB} = \omega^{AB} - \omega_0^{AB} = \mathcal{K}^A n^B - \mathcal{K}^B n^A \,, \tag{4.67}$$

where we have defined the extrinsic curvature one-form $\mathcal{K}_\alpha \equiv K_{\alpha\beta}\mathrm{d}y^\beta$, and converted its index to a flat index through the e^A, metric, and embedding functions. Similarly, $n^A = e^A_\mu n^\mu$.

In analogy with the two dimensional variation (4.5), we therefore posit that the a-dependent piece of the Weyl anomaly is

$$\delta_\sigma W = (-1)^{d/2}\frac{4a}{d!\,\mathrm{Vol}(S^d)}\left(\int_M \mathcal{E}_d\delta\sigma - \int_{\partial M} \mathcal{Q}_d\delta\sigma\right) + \ldots \tag{4.68}$$

where the ellipsis denotes terms depending on c_i, the total divergence in the trace anomaly, and possibly other purely boundary contributions. We verify this claim in subsection 4.2.3 by showing that the anomaly (4.68) is Wess-Zumino consistent. With this variation in hand, we can integrate it in one of the same three ways we used in $d = 2$: guess work, using the integral (4.16), or dimensional regularization. The integral (4.16) gives the a dependent contribution to the effective anomaly action,

$$\mathcal{W}[g_{\mu\nu}, e^{-2\tau}g_{\mu\nu}] = (-1)^{d/2}\frac{4a}{d!\,\mathrm{Vol}(S^d)}\int_0^1 \mathrm{d}t\left\{\int_M \tau\mathcal{E}_d[g'] - \int_{\partial M}\tau\mathcal{Q}_d[g']\right\}\Bigg|_{g'_{\mu\nu}=e^{-2t\tau}g_{\mu\nu}} \,, \tag{4.69}$$

We also deduce $\mathcal{W}$ from dimensional regularization in subsection 4.2.5.

Let us next study the relation between $\mathcal{E}_d$ and $\mathcal{Q}_d$. The relation (4.63) is an example of a "transgression form" (see e.g. [69] for a modern summary of transgression forms). To prove it, consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)_d = \dot{\mathcal{R}}(t)^A{}_B \wedge \frac{\partial\mathcal{E}(t)_d}{\partial\mathcal{R}(t)^A{}_B} \,. \tag{4.70}$$

It is convenient to introduce an exterior covariant derivative D. It takes tensor-valued p-forms to tensor-valued $p+1$-forms. For example it acts on a matrix-valued p-form, $f^A{}_B$ as

$$\mathrm{D}f^A{}_B = \mathrm{d}f^A{}_B + \omega^A{}_C \wedge f^C{}_B - (-1)^p f^A{}_C \wedge \omega^C{}_B\,, \tag{4.71}$$

and correspondingly for (co)vector-valued forms. It has the Lifshitz property, e.g.

$$\mathrm{d}(f^{AB} \wedge g_{AB}) = \mathrm{D}(f^{AB} \wedge g_{AB}) = \mathrm{D}f^{AB} \wedge g_{AB} + (-1)^p f^{AB} \wedge \mathrm{D}g_{AB}\,. \tag{4.72}$$

Defining $\mathrm{D}(t)$, we then have

$$\mathrm{D}(t)\mathcal{R}(t)^A{}_B = 0\,, \qquad \dot{\mathcal{R}}(t)^A{}_B = \mathrm{D}(t)\dot{\omega}(t)^A{}_B\,. \tag{4.73}$$

The metric δ_{AB} and antisymmetric Levi-Civita tensor $\epsilon_{A_1\ldots A_d}$ are also constant under $\mathrm{D}(t)$, provided that we let the e^A depend on t so that $\omega(t)$ is associated with a metric $g(t)$. Consequently,

$$\mathrm{D}(t)\frac{\partial \mathcal{E}(t)_d}{\partial \mathcal{R}(t)^{AB}} = \frac{d}{2}\mathrm{D}(t)\left(\mathcal{R}(t)^{A_1A_2} \wedge \ldots \wedge \mathcal{R}(t)^{A_{d-3}A_{d-2}}\epsilon_{ABA_1\ldots A_d}\right) = 0\,, \tag{4.74}$$

and we can rewrite (4.70) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)_d &= \mathrm{d}\left(\dot{\omega}(t)^{AB} \wedge \frac{\partial \mathcal{E}(t)_d}{\partial \mathcal{R}(t)^{AB}}\right) \\ &= \mathrm{d}\left(\frac{d}{2}\dot{\omega}(t)^{A_1A_2} \wedge \mathcal{R}(t)^{A_3A_4} \wedge \ldots \wedge \mathcal{R}(t)^{A_{d-1}A_d}\varepsilon_{A_1\ldots A_d}\right)\,.\end{aligned} \tag{4.75}$$

Integrating this equality over $t \in [0,1]$ immediately yields (4.63).

4.2.2 An Explicit Expression For The Boundary Term

It will be expedient in the rest of this section to have an explicit expression for the boundary term $\int_{\partial M} \mathcal{Q}_d$, that is to perform the integral over t in (4.61). The final result is (4.81).

Before doing so, we will use that the pullback of $\mathcal{R}^{AB}$ to the boundary can be expressed in terms of the intrinsic and extrinsic curvatures of the boundary. The relations between $\mathcal{R}^{AB}$ and the boundary curvatures are known as the Gauss and Codazzi equations, and we discuss them in appendix 4.6.1.

Denoting the intrinsic Riemann curvature tensor of the boundary as $\mathring{R}^\alpha{}_{\beta\gamma\delta}$, we define the intrinsic curvature two-form

$$\mathring{\mathcal{R}}^\alpha{}_\beta \equiv \frac{1}{2}\mathring{R}^\alpha{}_{\beta\gamma\delta}\mathrm{d}y^\gamma \wedge \mathrm{d}y^\delta\,, \tag{4.76}$$

and thereby $\mathring{\mathcal{R}}^A{}_B$. Using the boundary covariant derivative $\mathring{\nabla}_\alpha$, we define a boundary exterior covariant derivative $\mathring{\nabla}$ just like D. The Gauss and Codazzi equations can then be summarized as

$$\mathcal{R}^A{}_B = \mathring{\mathcal{R}}^A{}_B - \mathcal{K}^A \wedge \mathcal{K}_B + n_B \mathring{\nabla}\mathcal{K}^A - n^A \mathring{\nabla}\mathcal{K}_B\,, \tag{4.77}$$

We can similarly decompose the pullback of $\mathcal{R}(t)$. On the boundary

$$\omega(t)^A{}_B = \omega^A{}_B + (t-1)(\mathcal{K}^A n_B - n^A \mathcal{K}_B)\ , \tag{4.78}$$

which implies that on the boundary

$$\begin{aligned}\mathcal{R}(t)^A{}_B &= \mathcal{R}^A{}_B + (t-1)\mathring{\nabla}(\mathcal{K}^A n_B - n^A\mathcal{K}_B) + (t-1)^2(\mathcal{K}^A n_C - n^A \mathcal{K}_C) \wedge (\mathcal{K}^C n_B - n^C \mathcal{K}_B) \\ &= \mathcal{R}^A{}_B - (t^2-1)\mathcal{K}^A \wedge \mathcal{K}_B + (t-1)\left(n_B\mathring{\nabla}\mathcal{K}^A - n^A \mathring{\nabla}\mathcal{K}_B\right)\ ,\end{aligned} \tag{4.79}$$

where we have used that $\mathring{\nabla} n^A = \mathcal{K}^A$. Putting this together with (4.77), we have

$$\mathcal{R}(t)^A{}_B = \mathring{\mathcal{R}}^A{}_B - t^2\mathcal{K}^A \wedge \mathcal{K}_B + t\left(n_B\mathring{\nabla}\mathcal{K}^A - n^A\mathring{\nabla}\mathcal{K}_B\right)\ . \tag{4.80}$$

Then on the boundary the definition of $\mathcal{Q}_d$ (4.61) becomes

$$\begin{aligned}\mathcal{Q}_d &= d\int_0^1 \mathrm{d}t\, n^{A_1}\mathcal{K}^{A_2} \wedge \left(\mathring{\mathcal{R}}^{A_3A_4} - t^2\mathcal{K}^{A_3}\wedge\mathcal{K}^{A_4}\right) \wedge \ldots \wedge \left(\mathring{\mathcal{R}}^{A_{d-1}A_d} - t^2\mathcal{K}^{A_{d-1}} \wedge \mathcal{K}^{A_d}\right)\epsilon_{A_1\ldots A_d} \\ &= d\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{(-1)^k}{2k+1}\mathring{\mathcal{R}}^{m-1-k} \wedge \mathcal{K}^{2k+1} n^A \epsilon_{A\ldots}\,,\end{aligned} \tag{4.81}$$

where we have defined $m \equiv \frac{d}{2}$ and in the last line we have suppressed the indices of the curvature forms, all of which are dotted into the epsilon tensor. We have also used that only one index of the epsilon tensor can be dotted into the normal vector n^A, and so the factors of $\mathring{\nabla}\mathcal{K}^A$ in $\mathcal{R}(t)$ never appear in $\mathcal{Q}_d$.

The integral representation of $\mathcal{Q}_d$ in the first line of (4.81) is not new. A similar expression appears in e.g. ref. [70].

For example, in four and six dimensions we have

$$\begin{aligned}\mathcal{Q}_4 &= 4n^A\mathcal{K}^B \wedge \left(\mathring{\mathcal{R}}^{CD} - \frac{1}{3}\mathcal{K}^C \wedge \mathcal{K}^D\right)\epsilon_{ABCD}\,, \\ \mathcal{Q}_6 &= 6n^A\mathcal{K}^B \wedge \left(\mathring{\mathcal{R}}^{CD} \wedge \mathring{\mathcal{R}}^{EF} - \frac{2}{3}\mathring{\mathcal{R}}^{CD} \wedge \mathcal{K}^E \wedge \mathcal{K}^F + \frac{1}{5}\mathcal{K}^B \wedge \mathcal{K}^C \wedge \mathcal{K}^D \wedge \mathcal{K}^E \wedge \mathcal{K}^F\right)\epsilon_{ABCDEF}\,.\end{aligned} \tag{4.82}$$

4.2.3 Wess-Zumino Consistency

We now verify that the posited term proportional to a in the Weyl anomaly (4.68) is Wess-Zumino consistent. In this setting, Wess-Zumino consistency requires that the anomaly satisfies

$$[\delta_{\sigma_1}, \delta_{\sigma_2}]W = 0\,. \tag{4.83}$$

Notating the anomalous variation proportional to a as

$$\delta_\sigma W_a = A\left(\int_M \delta\sigma\, \mathcal{E}_d - \int_{\partial M} \delta\sigma\, \mathcal{Q}_d\right), \qquad A \equiv (-1)^{d/2}\frac{4a}{d!\,\mathrm{Vol}(S^d)},$$

we consider

$$\delta_{\sigma_1}\delta_{\sigma_2} W_a = A\left(\int_M \delta\sigma_2 \delta_{\sigma_1}\mathcal{E}_d - \int_{\partial M} \delta\sigma_2\delta_{\sigma_1}\mathcal{Q}_d\right). \tag{4.84}$$

The variation of $\mathcal{E}_d$ is a total derivative,

$$\delta_\sigma \mathcal{E}_d = \mathrm{d}\left(\delta_\sigma\omega^{AB}\wedge\frac{\partial\mathcal{E}_d}{\partial\mathcal{R}^{AB}}\right), \tag{4.85}$$

with

$$\delta_\sigma\omega^{AB} = (e^A e^B_\mu - e^B e^A_\mu)\partial^\mu\delta\sigma\,. \tag{4.86}$$

It then follows that the bulk part of the second variation is

$$\begin{aligned}\delta_{\sigma_1}\delta_{\sigma_2} W_a &= 2dA\int_M e^{A_1}e^{A_2}_\mu\partial^\mu\delta\sigma_1\wedge\mathrm{d}\delta\sigma_2\wedge\mathcal{R}^{A_3A_4}\wedge\ldots\wedge\mathcal{R}^{A_{d-1}A_d}\epsilon_{A_1\ldots A_d} + (\text{boundary term})\,,\\ &= A\int_M d^dx\sqrt{g}\,\mathcal{X}^{\mu\nu}_d\partial_\mu\delta\sigma_1\partial_\nu\delta\sigma_2 + (\text{boundary term})\,,\end{aligned} \tag{4.87}$$

where we have defined

$$\mathcal{X}^{\mu\nu}_d \equiv \frac{d}{2^{d/2}}R_{\nu_1\nu_2\rho_1\rho_2}\ldots R_{\nu_{d-3}\nu_{d-2}\rho_{d-3}\rho_{d-2}}\epsilon^{\mu\rho\nu_1\ldots\nu_{d-2}}\epsilon^\nu{}_\rho{}^{\rho_1\ldots\rho_{d-2}}\,. \tag{4.88}$$

$\mathcal{X}^{\mu\nu}_d$ is symmetric, $\mathcal{X}^{\mu\nu}_d = \mathcal{X}^{\nu\mu}_d$, on account of $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$. The symmetry of $\mathcal{X}^{\mu\nu}_d$ together with the variation (4.87) imply

$$[\delta_{\sigma_1}, \delta_{\sigma_2}]W_a = (\text{boundary term})\,. \tag{4.89}$$

In other words, the bulk term in the a-anomaly is Wess-Zumino consistent. It suffices now to show that the boundary term also vanishes.

To proceed, we require the Weyl variations of the extrinsic and intrinsic curvatures. The variation of $K_{\alpha\beta}$ and so $\mathcal{K}^A$ is

$$\delta_\sigma K_{\alpha\beta} = \delta\sigma K_{\alpha\beta} + \gamma_{\alpha\beta}n^\mu\partial_\mu\delta\sigma\,, \qquad \delta_\sigma\mathcal{K}^A = e^A n^\mu\partial_\mu\delta\sigma = (\delta_\sigma\omega^A{}_B)n^B\,, \tag{4.90}$$

where e^A in the variation of $\mathcal{K}^A$ is pulled back to the boundary, while the variation of $\mathring{\mathcal{R}}^A{}_B$ is

$$\delta_\sigma\mathring{\mathcal{R}}^A{}_B = \mathring{\nabla}\delta_\sigma\mathring{\omega}^A{}_B\,, \tag{4.91}$$

for $\mathring{\omega}^A{}_B$ the connection one-form on the boundary. The variation of $\omega^A{}_B$ on the boundary is related to those of $\mathring{\omega}^A{}_B$ via

$$\delta_\sigma \omega^A{}_B = \delta_\sigma \mathring{\omega}^A{}_B + \left(n_B \delta_\sigma \omega^A{}_C - n^A \delta_\sigma \omega_{CB}\right) n^C . \tag{4.92}$$

Under a general variation of $\mathcal{K}^A$ and $\mathring{\mathcal{R}}^A{}_B$, $\mathcal{Q}_d$ in (4.81) varies as

$$\delta \mathcal{Q}_d = d \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^k \left\{ \delta\mathcal{K}^B \wedge \mathring{\mathcal{R}}^{CD} + \frac{m-1-k}{2k+1} \delta\mathring{\mathcal{R}}^{BC} \wedge \mathcal{K}^D \right\} \wedge \mathring{\mathcal{R}}^{m-2-k} \wedge \mathcal{K}^{2k} n^A \epsilon_{ABCD...} \tag{4.93}$$

Specializing to Weyl variations, this becomes

$$\begin{aligned} \frac{1}{d}\delta_\sigma \mathcal{Q}_d =& \delta_\sigma \omega^B{}_C n^C \mathcal{R}^{m-1} n^A \epsilon_{AB...} \\ & + \delta_\sigma \mathring{\omega}^{BC} \wedge \sum_{k=0}^{m-2} \binom{m-2}{k} (-1)^k (m-1) \mathring{\nabla}\mathcal{K}^D \wedge \mathring{\mathcal{R}}^{m-2-k} \wedge \mathcal{K}^{2k} n^A \epsilon_{ABCD...} \\ & + \mathrm{d} \left\{ \delta_\sigma \mathring{\omega}^{BC} \wedge \sum_{k=0}^{m-2} \binom{m-2}{k} (-1)^k \frac{m-1}{2k+1} \mathring{\mathcal{R}}^{m-2-k} \wedge \mathcal{K}^{2k+1} n^A \epsilon_{ABC...} \right\} , \end{aligned} \tag{4.94}$$

where we have used the Gauss equation in simplifying the $\delta_\sigma \mathcal{K}$ variation along with $\mathring{\nabla}\mathring{\mathcal{R}} = 0$ in simplifying the $\delta_\sigma \mathring{\mathcal{R}}$ variation. Using the Codazzi equation, $\mathcal{R}^A{}_B n^B = \mathring{\nabla}\mathcal{K}^A$, the second line combines with the first to give

$$\delta_\sigma \mathcal{Q}_d = \delta_\sigma \omega^{AB} \wedge \frac{\partial \mathcal{E}_d}{\partial \mathcal{R}^{AB}} + \mathrm{d}\left\{(m-1)\delta_\sigma \mathring{\omega}^{AB} \wedge (\mathcal{Q}_{d-2})_{AB}\right\} . \tag{4.95}$$

In writing the boundary term, we have defined the matrix-valued $(d-3)$-form $(\mathcal{Q}_{d-2})_{AB}$ to be

$$(\mathcal{Q}_{d-2})_{AB} \equiv d \sum_{k=0}^{m-2} \binom{m-2}{k} \frac{(-1)^k}{2k+1} \mathring{\mathcal{R}}^{m-2-k} \wedge \mathcal{K}^{2k+1} n^C \epsilon_{ABC...} \tag{4.96}$$

The reason for the name is the similarity with the explicit expression (4.81) for $\mathcal{Q}_d$: the sum (4.96) is identical to that in the expression for $\mathcal{Q}_d$, except it runs to $k = m-2$ rather than $k = m-1$.

Putting $\delta_\sigma \mathcal{Q}_d$ together with the variation of the Euler form (4.85), the boundary term in the variation of $\int_M \delta\sigma_2 \delta_{\sigma_1} \mathcal{E}_d$ cancels against the first half of the variation of $\mathcal{Q}_d$ in (4.95), so that

$$\begin{aligned} \delta_{\sigma_1}\delta_{\sigma_2} W_a &= A\left(\int_M d^d x \sqrt{g}\, \mathcal{X}^{\mu\nu} \partial_\mu \delta\sigma_1 \partial_\nu \delta\sigma_2 - 2(m-1) \int_{\partial M} e^A e^B_\alpha \partial^\alpha \delta\sigma_1 \wedge \mathrm{d}\delta\sigma_2 \wedge (\mathcal{Q}_{d-2})_{AB}\right) \\ &= A\left(\int_M d^d x \sqrt{g}\, \mathcal{X}^{\mu\nu} \partial_\mu \delta\sigma_1 \partial_\nu \delta\sigma_2 - \int_{\partial M} d^{d-1} y \sqrt{\gamma}\, \mathcal{Y}^{\alpha\beta} \partial_\alpha \delta\sigma_1 \partial_\beta \delta\sigma_2\right) , \end{aligned} \tag{4.97}$$

where $\mathcal{Y}^{\alpha\beta}$ is

$$\begin{aligned}\mathcal{Y}^{\alpha\beta} =& d\epsilon^{\alpha\gamma\gamma_1\dots\gamma_{d-3}}\epsilon^{\beta}{}_{\gamma}{}^{\delta_1\dots\delta_{d-3}}\sum_{k=0}^{m-2}\binom{m-2}{k}(-1)^k\frac{m-1}{(2k+1)2^{m-3-k}} \\ &\times \mathring{R}_{\gamma_1\gamma_2\delta_1\delta_2}\cdots\mathring{R}_{\gamma_{d-2k-5}\gamma_{d-2k-4}\delta_{d-2k-5}\delta_{d-2k-4}}K_{\gamma_{d-2k-3}\delta_{d-2k-3}}\cdots K_{\gamma_{d-3}\delta_{d-3}}\,.\end{aligned} \tag{4.98}$$

$\mathcal{Y}^{\alpha\beta}$ is symmetric owing to the symmetry of the boundary curvatures, $\mathring{R}_{\alpha\beta\gamma\delta} = \mathring{R}_{\gamma\delta\alpha\beta}$ and $K_{\alpha\beta} = K_{\beta\alpha}$. Then (4.97) yields

$$[\delta_{\sigma_1}, \delta_{\sigma_2}]W_a = 0\,, \tag{4.99}$$

which is what we sought to show.

4.2.4 A Complete Classification in $d = 4$ and Boundary Central Charges

The previous subsection was somewhat abstract. Let us see how the consistency works in $d = 4$. Along the way, we will also classify the potential boundary terms in the Weyl anomaly, finding two "boundary central charges." To our knowledge, one of these "central charges" was first noted in [71] and the other later in ref. [72].

In $d = 4$, $\mathcal{E}_4$ and $\mathcal{Q}_4$ are equivalent to the scalars

$$\begin{aligned}E_4 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,, \\ Q_4 &= 4\left(2\mathring{E}_{\alpha\beta}K^{\alpha\beta} + \frac{2}{3}\operatorname{tr}(K^3) - KK_{\alpha\beta}K^{\alpha\beta} + \frac{1}{3}K^3\right)\,,\end{aligned} \tag{4.100}$$

where $\mathring{E}_{\alpha\beta} = \mathring{R}_{\alpha\beta} - \frac{\mathring{R}}{2}\gamma_{\alpha\beta}$ is the boundary Einstein tensor, and the a-type term in the anomaly is

$$\delta_\sigma W_a = A\left(\int_M \mathrm{d}^4x\sqrt{g}\,\delta\sigma E_4 - \int_{\partial M}\mathrm{d}^3y\sqrt{\gamma}\,\delta\sigma Q_4\right)\,, \qquad A = \frac{a}{16\pi^2}\,. \tag{4.101}$$

The Weyl variations of E_4 and Q_4 are

$$\begin{aligned}&\delta_\sigma E_4 = -4\delta\sigma E_4 + 8\mathrm{D}_\mu\left(E^{\mu\nu}\partial_\nu\delta\sigma\right)\,, \\ &\delta_\sigma Q_4 = -3\delta\sigma Q_4 - 4\left\{R^{\alpha\beta}{}_{\alpha\beta}n^\mu\partial_\mu - 2\mathring{\nabla}_\alpha\left(K^{\alpha\beta} - K\gamma^{\alpha\beta}\right)\mathring{\nabla}_\beta\right\}\delta\sigma - 8\mathring{\nabla}_\alpha\left\{\left(K^{\alpha\beta} - K\gamma^{\alpha\beta}\right)\partial_\beta\delta\sigma\right\}\,,\end{aligned} \tag{4.102}$$

Using the Gauss and Codazzi equations (4.77), which here are

$$R_{\alpha\beta\gamma\delta} = \mathring{R}_{\alpha\beta\gamma\delta} - K_{\alpha\gamma}K_{\beta\delta} + K_{\alpha\delta}K_{\beta\gamma}\,, \qquad n^\mu R_{\mu\alpha\beta\gamma} = \mathring{\nabla}_\gamma K_{\alpha\beta} - \mathring{\nabla}_\beta K_{\alpha\gamma}\,, \tag{4.103}$$

we can rewrite the variation of Q_4 as

$$\delta_\sigma Q_4 = -3\delta\sigma Q_4 + 8n_\mu E^{\mu\nu}\partial_\nu\delta\sigma - 8\mathring{\nabla}_\alpha\left\{\left(K^{\alpha\beta} - K\gamma^{\alpha\beta}\right)\partial_\beta\delta\sigma\right\}\,. \tag{4.104}$$

The second variation of W_a is then

$$\delta_{\sigma_1}\delta_{\sigma_2}W_a = -8A\left(\int_M \mathrm{d}^4x\sqrt{g}\,E^{\mu\nu}(\partial_\mu\delta\sigma_1)(\partial_\nu\delta\sigma_2) + \int_{\partial M}\mathrm{d}^3y\sqrt{\gamma}\left(K^{\alpha\beta}-K\gamma^{\alpha\beta}\right)(\partial_\alpha\delta\sigma_1)(\partial_\beta\delta\sigma_2)\right), \quad (4.105)$$

which is manifestly symmetric under $\delta\sigma_1 \leftrightarrow \delta\sigma_2$, so that

$$[\delta_{\sigma_1},\delta_{\sigma_2}]W_a = 0\,. \quad (4.106)$$

In this instance, the tensors $\mathcal{X}^{\mu\nu}$ and $\mathcal{Y}^{\alpha\beta}$ are

$$\mathcal{X}^{\mu\nu} = -8E^{\mu\nu}\,, \qquad \mathcal{Y}^{\alpha\beta} = 8\left(K^{\alpha\beta}-K\gamma^{\alpha\beta}\right)\,. \quad (4.107)$$

So much for showing that the a-type anomaly is consistent. Are there any other boundary terms which may be allowed in the anomaly? This is essentially a cohomological question, which we answer in three steps:

1. Posit the most general boundary variation of W characterized by dimensionless coefficients.

2. Use the freedom to add local boundary counterterms to remove as many of these coefficients as possible.

3. Demand that the residual variation is Wess-Zumino consistent.

We perform this algorithm in Appendix 4.6.2. The final result is that the total Weyl anomaly for a $d=4$ CFT is

$$\delta_\sigma W = \frac{1}{16\pi^2}\int_M \mathrm{d}^4x\sqrt{g}\,\delta\sigma\left(aE_4 - cW^2_{\mu\nu\rho\sigma}\right) - \int_{\partial M}\mathrm{d}^3y\sqrt{\gamma}\,\delta\sigma\left(AQ_4 - b_1\mathrm{tr}\hat{K}^3 - b_2\gamma^{\alpha\gamma}\hat{K}^{\beta\delta}W_{\alpha\beta\gamma\delta}\right), \quad (4.108)$$

where $\hat{K}_{\alpha\beta}$ is the traceless part of the extrinsic curvature, $\hat{K}_{\alpha\beta} = K_{\alpha\beta} - \frac{K}{d-1}\gamma_{\alpha\beta}$, and $W_{\alpha\beta\gamma\delta}$ is the pullback of the Weyl tensor. The terms proportional to b_1 and b_2 are the additional type-B boundary terms in the anomaly. We refer to b_1 and b_2 as "boundary central charges," and they are formally analogous to c insofar as they multiply Weyl-covariant scalars. The purely extrinsic term proportional to b_1 first appeared in [71], and the second term proportional to b_2 later appeared in [72].

It is an interesting exercise to compute b_1 and b_2 for a conformally coupled scalar field. The simplest way to proceed is to look at existing heat kernel calculations for a scalar field in the presence of a boundary and then restrict to the conformally coupled case. The action for such a conformally coupled scalar is

$$S = \int_M \mathrm{d}^4x\sqrt{g}\left((\partial\phi)^2 + \frac{1}{6}R\phi^2\right) + \frac{1}{3}\int_{\partial M}\mathrm{d}^3y\sqrt{\gamma}K\phi^2\,. \quad (4.109)$$

Note that the last term ensures Weyl invariance. It is also necessary for coupling the theory to gravity.[6] By comparing this result with heat kernel calculations for a conformally coupled scalar field in the presence of a boundary, we can extract values for b_1 and b_2. There are two Weyl-invariant boundary conditions to consider, Dirichlet $\phi|_{\partial M} = 0$ (in which case the boundary term can be neglected) and the conformally-invariant Robin $(n^\mu \partial_\mu + \frac{1}{3}K)\phi|_{\partial M} = 0$. Comparing with for example (1.17) of [73] or the expressions for a_4 on p 5 of [74], we deduce that

$$b_1(\text{Robin}) = -\frac{1}{(4\pi)^2}\frac{2}{45} \ , \quad b_1(\text{Dirichlet}) = -\frac{1}{(4\pi)^2}\frac{2}{35} \ , \quad b_2 = \frac{1}{(4\pi)^2}\frac{1}{15} \ . \tag{4.110}$$

The value for b_1(Dirichlet) was computed before in eq. (19) of ref. [71], while b_1(Robin) can be found in eq. (55) of ref. [75]. The coefficient b_2 was computed in the Dirichlet case in eq. (15) of ref. [72]. (In our conventions, $a = 1/360$ and $c = 1/120$ for a $4d$ conformally coupled scalar.) As $|b_1(\text{Dirichlet})| > |b_1(\text{Robin})|$, and one can flow from the Robin theory to the Dirichlet theory by deforming the Robin theory by a "boundary mass" $\int \mathrm{d}^3 y \, m\phi^2$; it is tempting to speculate that b_1 satisfies a monotonicity property under boundary renormalization group flows, similar to the one conjectured for a by Cardy and now proven in $d = 4$ by ref. [14]. This conjecture is different from the "boundary F-theorem" conjectured in [76, 77, 78] for $d = 4$ boundary flows. We leave further analysis of these boundary central charges b_1 and b_2 for the future.

4.2.5 Dimensional Regularization

In the two dimensional case, we saw that an effective anomaly action could be constructed in dimensional regularization using a combination of the Einstein-Hilbert action and the Gibbons-Hawking surface term in $n = 2 + \epsilon$ dimensions. In the limit $\epsilon \to 0$, these objects sum together to give the Euler characteristic. The obvious guess, which we shall verify, is that to construct the anomaly action in d dimensions, we need to continue the Euler density along with the $\mathcal{Q}_d$ Chern-Simons like term to $n = d+\epsilon$ dimensions. In the mathematics community, such a dimensionally continued Euler density is called a Lipschiftz-Killing curvature, while in the physics community, these objects are used to construct actions for Lovelock gravities.

The mth Lipschitz-Killing curvature form in dimension n, $2m \leq n$, is:

$$\mathcal{E}_{n,m} \equiv \left(\bigwedge_{i=1}^{m} \mathcal{R}^{A_{2i-1}A_{2i}} \right) \wedge \left(\bigwedge_{i=2m+1}^{n} e^{A_i} \right) \epsilon_{A_1 \cdots A_n} \ , \tag{4.111}$$

where $\epsilon_{A_1 \cdots A_n}$ is the totally antisymmetric Levi-Civita tensor in dimension n. In $n = 2m$ dimensions, the Lipschitz-Killing form reduces to the Euler form, $\mathcal{E}_{2m,m} = \mathcal{E}_{2m}$. The analog

[6] If we are not interested in dynamical gravity, we could add an additional boundary term of the form $\phi(K + 3n^\mu \partial_\mu)\phi$ with arbitrary coefficient. This term preserves Weyl invariance. However, it does not modify the boundary conditions or the scalar functional determinant. Consequently the boundary central charges that we determine below do not depend on this term. See the appendix of [56] for a related discussion.

of the Gibbons-Hawking term we call $\mathcal{Q}_{n,m}$:

$$\mathcal{Q}_{n,m} \equiv m \int_0^1 \dot{\omega}(t)^{A_1 A_2} \wedge \left(\bigwedge_{i=2}^{m} \mathcal{R}(t)^{A_{2i-1}A_{2i}} \right) \wedge \left(\bigwedge_{i=2m+1}^{n} e^{A_i} \right) \epsilon_{A_1 \cdots A_n} \, \mathrm{d}t \; . \tag{4.112}$$

It is a $n-1$ degree Chern-Simons like form which is only defined on the boundary, which reduces to $\mathcal{Q}_d$ in $n = 2m$ dimensions.

The obvious guess for the effective action $\widetilde{W}[g_{\mu\nu}]$ in $n = d + \epsilon$ dimensions, i.e. the d dimensional analog of (4.17), is

$$\widetilde{W}[g_{\mu\nu}] = (-1)^m \frac{4a}{(n-2m)(2m)! \, \mathrm{Vol}(S^{2m})} \left(\int_M \mathcal{E}_{n,m} - \int_{\partial M} \mathcal{Q}_{n,m} \right) \; , \tag{4.113}$$

where $d = 2m$. The effective anomaly action is then just

$$\mathcal{W}[g_{\mu\nu}, e^{-2\tau} g_{\mu\nu}] = \lim_{n \to d} \left(\widetilde{W}[g_{\mu\nu}] - \widetilde{W}[e^{-2\tau} g_{\mu\nu}] \right) \; . \tag{4.114}$$

Note that this effective action only recovers the a dependent portion of the trace anomaly.

As in subsection 4.2.2, we can perform the integral over t in the definition of $\mathcal{Q}_{n,m}$ to deduce an explicit expression for $\mathcal{Q}_{n,m}$ in terms of the extrinsic and intrinsic curvatures of the boundary. The integration over t is identical to that performed in subsection 4.2.2, except now we have $n - 2m$ factors of e^A to account for. The final result is

$$\mathcal{Q}_{n,m} = 2m \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^k}{2k+1} \mathring{\mathcal{R}}^{m-1-k} \wedge \mathcal{K}^{2k+1} \wedge e^{n-2m} n^A \epsilon_{A\ldots} \, , \tag{4.115}$$

where for brevity we have suppressed the indices of the curvatures and factors of e^A, all of which are contracted with the remaining indices of the epsilon tensor.

Next we show that dimensional regularization (4.113) reproduces the a portion of the Weyl anomaly. Our approach is almost identical to the demonstration that the a-anomaly is Wess-Zumino consistent in subsection 4.2.3. We begin with the expressions (4.111) and (4.115) for $\mathcal{E}_{n,m}$ and $\mathcal{Q}_{n,m}$. We consider the Weyl variation of

$$\int_M \mathcal{E}_{n,m} - \int_{\partial M} \mathcal{Q}_{n,m} \, , \tag{4.116}$$

in n dimensions. We compute this variation in two steps. First we show that this difference does not depend on any variation of the connection one-form $\omega^A{}_B$ while keeping the e^A fixed.[7] Then the Weyl variation only arises from the Weyl variation of the e^A while keeping the $\omega^A{}_B$ fixed. This last variation is rather simple, as the e^A only appear through wedge products in $\mathcal{E}_{n,m}$ and $\mathcal{Q}_{n,m}$.

[7]This same computation shows that the Lovelock gravities have a well-defined variational principle for the metric $g_{\mu\nu}$ on a space with boundary (see ref. [79]).

Now consider a variation of the connection one-form $\omega^A{}_B$ whilst keeping the e^A and embedding of the boundary fixed. The bulk and boundary curvatures vary as

$$\delta_\omega \mathcal{R}^A{}_B = \mathrm{D}\delta\omega^A{}_B \,, \qquad \delta_\omega \mathring{\mathcal{R}}^A{}_B = \mathring{\nabla}\delta\mathring{\omega}^A{}_B \,, \qquad \delta_\omega \mathcal{K}^A = (\delta\omega^A{}_B)n^B \,, \tag{4.117}$$

where $\mathring{\omega}^A{}_B$ is the connection one-form on the boundary. The computation of this variation is virtually identical to that in subsection 4.2.3, as the only difference between $\mathcal{E}_{n,m}$ and $\mathcal{E}_d$, and $\mathcal{Q}_{n,m}$ and $\mathcal{Q}_d$, is an extra wedge product of $n-2m$ factors of the e^A. The analogues of (4.85) and (4.95) are

$$\begin{aligned} \delta_\omega \mathcal{E}_{n,m} &= \mathrm{d}\left(\delta\omega^{AB} \wedge \frac{\partial \mathcal{E}_{n,m}}{\partial \mathcal{R}^{AB}}\right) \,, \\ \delta_\omega \mathcal{Q}_{n,m} &= \delta\omega^{AB} \wedge \frac{\partial \mathcal{E}_{n,m}}{\partial \mathcal{R}^{AB}} + (\text{total deriative}) \,, \end{aligned} \tag{4.118}$$

so that

$$\delta_\omega \left(\mathcal{E}_{n,m} - \mathrm{d}\mathcal{Q}_{n,m}\right) = 0 \,, \tag{4.119}$$

as claimed.

Now consider a variation under which $\omega^A{}_B$ is fixed and the e^A vary as in an infinitesimal Weyl rescaling,

$$\delta_\sigma e^A = \delta\sigma e^A \,. \tag{4.120}$$

Then

$$\delta_\sigma \left(\mathcal{E}_{n,m} - \mathrm{d}\mathcal{Q}_{n,m}\right) = (n-2m)\delta\sigma \left(\mathcal{E}_{n,m} - \mathrm{d}\mathcal{Q}_{n,m}\right) \,, \tag{4.121}$$

so that the variation of the dimensionally regulated anomaly action $\widetilde{W}$ in (4.113) is

$$\delta_\sigma \widetilde{W} = (-1)^m \frac{4a}{(2m)!\,\mathrm{Vol}(S^{2m})} \left(\int_M \mathcal{E}_{n,m}\delta\sigma - \int_{\partial M} \mathcal{Q}_{n,m}\delta\sigma\right) \,. \tag{4.122}$$

In the $n \to 2m$ limit, this variation coincides with the a-anomaly (4.68).

4.3 Dilaton Effective Actions and Boundary Terms

In this section, we present the a contribution to the dilaton effective action in a spacetime with boundary in four and six dimensions. The $d=2$ dilaton effective action with a bounday term is given by (4.15). For $d>2$, the computation of boundary terms is more laborious. The details of a derivation using dimensional regularization are provided in appendix 4.6.3 in dimensions four and six. We save the general discussion of how the universal entanglement entropy arises from the boundary terms of these dilaton actions for the next section.

4.3.1 The Dilaton Effective Action in $d = 4$

The Euler density in $d = 4$ is given by

$$E_4 = \frac{1}{4}\delta^{\mu_1\cdots\mu_4}_{\nu_1\cdots\nu_4} R^{\nu_1\nu_2}{}_{\mu_1\mu_2} R^{\nu_3\nu_4}{}_{\mu_3\mu_4} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \,, \tag{4.123}$$

where $\delta^{\mu_1\cdots\mu_4}_{\nu_1\cdots\nu_4}$ is the fully antisymmetrized product of four Kronecker delta functions. The boundary term is

$$Q_4 = -4\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\; K^{\nu_1}_{\mu_1}\left(\frac{1}{2}R^{\nu_2\nu_3}{}_{\mu_2\mu_3} + \frac{2}{3}K^{\nu_2}_{\mu_2}K^{\nu_3}_{\mu_3}\right) = 4\left(2\mathring{E}_{\alpha\beta}K^{\alpha\beta} + \frac{2}{3}\mathrm{tr}(K^3) - KK_{\alpha\beta}K^{\alpha\beta} + \frac{1}{3}K^3\right) \tag{4.124}$$

Denote the Einstein tensor as

$$E^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\,. \tag{4.125}$$

In appendix 4.6.3, we find the dilaton effective action in $d = 4$ to be

$$\begin{aligned}\mathcal{W}[g_{\mu\nu}, e^{-2\tau}g_{\mu\nu}] \;=\; & \frac{a}{(4\pi)^2}\int_M \mathrm{d}^4x\sqrt{g}\left[\tau E_4 + 4E^{\mu\nu}(\partial_\mu\tau)(\partial_\nu\tau) + 8(\mathrm{D}_\mu\partial_\nu\tau)(\partial^\mu\tau)(\partial^\nu\tau) + 2(\partial\tau)^4\right] \\ & - \frac{a}{(4\pi)^2}\int_{\partial M}\mathrm{d}^3y\sqrt{\gamma}\left[\tau Q_4 + 4(K\gamma^{\alpha\beta} - K^{\alpha\beta})(\partial_\alpha\tau)(\partial_\beta\tau) + \frac{8}{3}\tau_n^3\right]\,,\end{aligned} \tag{4.126}$$

where $\tau_n = n^\mu\partial_\mu\tau$ is a normal derivative of the Weyl scale factor. The bulk term agrees with ref. [14, 55] while the boundary contribution is to our knowledge a new result.

4.3.2 The Dilaton Effective Action in $d = 6$

The Euler density in $d = 6$ is given by

$$E_6 = \frac{1}{8}\delta^{\mu_1\cdots\mu_6}_{\nu_1\cdots\nu_6} R^{\nu_1\nu_2}{}_{\mu_1\mu_2} R^{\nu_3\nu_4}{}_{\mu_3\mu_4} R^{\nu_5\nu_6}{}_{\mu_5\mu_6} \tag{4.127}$$

and the boundary term is

$$\begin{aligned}Q_6 = -\,6\delta^{\beta_1\cdots\beta_5}_{\alpha_1\cdots\alpha_5}K^{\alpha_1}_{\beta_1}\Bigg[&\left(\frac{1}{2}R^{\alpha_2\alpha_3}{}_{\beta_2\beta_3} + \frac{2}{3}K^{\alpha_2}_{\beta_2}K^{\alpha_3}_{\beta_3}\right)\left(\frac{1}{2}R^{\alpha_4\alpha_5}{}_{\beta_4\beta_5} + \frac{2}{3}K^{\alpha_4}_{\beta_5}K^{\alpha_4}_{\beta_5}\right) \\ & + \frac{4}{45}K^{\alpha_2}_{\beta_2}K^{\alpha_3}_{\beta_3}K^{\alpha_4}_{\beta_5}K^{\alpha_4}_{\beta_5}\Bigg]\,.\end{aligned} \tag{4.128}$$

To present the bulk dilaton action, we define

$$\begin{aligned}E^{(2)\mu\nu} &\equiv g^{\mu\nu}E_4 + 8R^\mu_\rho R^{\rho\nu} - 4R^{\mu\nu}R + 8R_{\rho\sigma}R^{\mu\rho\nu\sigma} - 4R^\mu{}_{\rho\sigma\tau}R^{\nu\rho\sigma\tau}\,, \\ C_{\mu\nu\rho\sigma} &\equiv R_{\mu\nu\rho\sigma} - g_{\mu\rho}R_{\nu\sigma} + g_{\mu\sigma}R_{\nu\rho}\,.\end{aligned} \tag{4.129}$$

In appendix 4.6.3, we use dimensional regularization to find the bulk dilaton action

$$
\begin{aligned}
&\mathcal{W}[g_{\mu\nu}, e^{-2\tau} g_{\mu\nu}]_{(\mathrm{Bulk})} = \\
&\quad \frac{a}{3(4\pi)^3} \int_M \mathrm{d}^6 x \sqrt{g} \left\{ -\tau E_6 + 3 E^{(2)}_{\mu\nu} \partial^\mu \tau \partial^\nu \tau + 16 C_{\mu\nu\rho\sigma} (\mathrm{D}^\mu \partial^\rho \tau)(\partial^\nu \tau)(\partial^\sigma \tau) \right. \\
&\qquad\qquad + 16 E_{\mu\nu} \left[(\partial^\mu \tau)(\partial^\rho \tau)(\mathrm{D}_\rho \partial^\nu \tau) - (\partial^\mu \tau)(\partial^\nu \tau) \Box \tau \right] - 6 R (\partial \tau)^4 \\
&\qquad\qquad \left. -24 (\partial\tau)^2 (\mathrm{D}\partial\tau)^2 + 24 (\partial\tau)^2 (\Box\tau)^2 - 36 (\Box\tau)(\partial\tau)^4 + 24 (\partial\tau)^6 \right\} .
\end{aligned}
\tag{4.130}
$$

This reproduces the bulk Wess-Zumino term first obtained in [15].

We have not been able to generate the boundary term in a general curved background. However, for a conformally flat geometry, we find

$$
\begin{aligned}
\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}] = &-\frac{a}{16\pi^3} \int_M \mathrm{d}^6 x \sqrt{g} \left\{ 2(\partial\tau)^2 (\partial_\mu \partial_\nu \tau)^2 - 2 (\partial\tau)^2 (\Box\tau)^2 + 3 \Box\tau (\partial\tau)^4 - 2(\partial\tau)^6 \right\} \\
&- \frac{a}{3(4\pi)^3} \int_{\partial M} \mathrm{d}^5 y \sqrt{\gamma} \Big[-\tau Q_6[\delta_{\mu\nu}] + 48 P^\alpha_\beta (\partial_\alpha \tau)(\partial^\beta \tau) + 3 Q_4[\delta_{\mu\nu}] (\mathring{\nabla}\tau)^2 \\
&\qquad + 48 K^{\alpha\beta} (\mathring{\Box}\tau)(\mathring{\nabla}_\alpha \partial_\beta \tau) + 24 K (\mathring{\nabla}_\alpha \partial_\beta \tau)^2 - 48 K_{\alpha\gamma} (\mathring{\nabla}^\beta \partial^\alpha \tau)(\mathring{\nabla}^\gamma \partial_\beta \tau) \\
&\qquad - 24 K (\mathring{\Box}\tau)^2 - 32 K (\mathring{\nabla}\tau)^2 \mathring{\Box}\tau - 16 K (\partial^\alpha \tau)(\partial^\beta \tau)(\mathring{\nabla}_\alpha \partial_\beta \tau) \\
&\qquad + 16 K_{\alpha\beta} (\partial^\alpha \tau)(\partial^\beta \tau) \mathring{\Box}\tau + 32 K_{\alpha\beta} (\mathring{\nabla}^\alpha \partial^\beta \tau)(\mathring{\nabla}\tau)^2 + 12 K \tau_n^4 \\
&\qquad + 12 K (\mathring{\nabla}\tau)^4 + 24 K (\mathring{\nabla}\tau)^2 \tau_n^2 + 48 (\mathring{\Box}\tau)(\mathring{\nabla}\tau)^2 (\tau_n) + 16 (\mathring{\Box}\tau)(\tau_n^3) \\
&\qquad - 24 (\mathring{\nabla}\tau)^2 \tau_n^3 - 36 \tau_n (\mathring{\nabla}\tau)^4 - \frac{36}{5} \tau_n^5 \Big] ,
\end{aligned}
\tag{4.131}
$$

where we have defined

$$
P^\alpha_\beta \equiv \left(K^2 - \mathrm{tr}(K^2) \right) K^\alpha_\beta - 2 K K^{\alpha\gamma} K_{\beta\gamma} + 2 K_{\gamma\delta} K^{\alpha\gamma} K^\delta_\beta . \tag{4.132}
$$

4.4 The Sphere Entanglement Entropy: General Result

We consider the entanglement entropy across a sphere with radius ℓ in flat space. The calculation is analogous to the discussion of the entanglement entropy for an interval in $d = 2$ in section 4.1.3. The information necessary to compute the entropy is contained in the causal development of the interior of the sphere, a ball of radius ℓ. We can then map that causal development to all of hyperbolic space cross the real line $R \times H^{d-1}$ using the change of variables

$$
\begin{aligned}
t &= \ell \frac{\sinh \tau/\ell}{\cosh u + \cosh \tau/\ell} , \\
r &= \ell \frac{\sinh u}{\cosh u + \cosh \tau/\ell} ,
\end{aligned}
\tag{4.133}
$$

where τ labels the new time, u is the radial coordinate in hyperbolic space while (t, r) are time and radius in polar coordinates in flat space. The line elements on flat space and $R \times H^{d-1}$ are related by a Weyl rescaling (see for example ref. [80])

$$\begin{aligned}\eta &= -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{d-2}^2 \ , \\ &= e^{2\sigma} \left[-\mathrm{d}\tau^2 + \ell^2 (\mathrm{d}u^2 + \sinh^2 u \, \mathrm{d}\Omega_{d-2}^2) \right] \ ,\end{aligned} \tag{4.134}$$

where $e^{-\sigma} = \cosh u + \cosh \tau/\ell$. We proceed by using the Euclidean version of this map, where τ_E is a periodic variable with period $2\pi\ell$ so that the theory is naturally at a temperature $T = \frac{1}{2\pi\ell}$, and the Euclidean geometry is conformal to $S^1 \times \mathbb{H}^{d-1}$. Note a difference here with the $d = 2$ case where the temperature was a free parameter.

The computation of the entanglement entropy across a sphere thus reduces to a computation of the thermodynamic entropy of the hyperbolic space $S_E = 2\pi\ell\langle H \rangle - W$ where $W \equiv -\ln \operatorname{tr} e^{-2\pi\ell H}$. As it did in $d = 2$, this computation in turn breaks down into three pieces, a computation of $\langle H \rangle$, a computation of the effective anomaly action $\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$ and a computation of a universal contribution to $\widetilde{W}[\delta_{\mu\nu}]$,

$$S_E = 2\pi\ell\langle H \rangle + \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}] - \widetilde{W}[\delta_{\mu\nu}] \, . \tag{4.135}$$

To compute $\langle H \rangle$, we shall not try to write down the Schwarzian derivative in arbitrary even d, but instead rely on an earlier closely related computation performed in ref. [1].

We have not been able to compute $\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$ in general d, but we shall argue based on computations in $d = 2$, 4 and 6 that it precisely cancels the contribution to S_E from $\langle H \rangle$. Finally, we compute $\widetilde{W}[\delta_{\mu\nu}]$ and show that the logarithmic contribution to it always reproduces the universal part of the sphere entanglement entropy.

4.4.1 Casimir Energy

The easy part of this computation is $\langle H \rangle$ because it has essentially been done in the chapter one, where the stress tensor in the vacuum on $R \times S^{d-1}$ in even d was computed, within the scheme where the the trace anomaly takes the form

$$\langle T^\mu{}_\mu \rangle = \sum_j c_j I_j - (-1)^{\frac{d}{2}} \frac{4a}{d! \, \mathrm{Vol}(S^d)} E_d \, , \tag{4.136}$$

i.e. in a scheme where local counterterms are used to remove the total divergence from the stress tensor trace. Within that scheme, the stress tensor is unambiguously determined by a to be

$$\langle T_0^0 \rangle = -\frac{4a}{(-\ell^2)^{d/2} d \, \mathrm{Vol}(S^d)} \ , \quad \langle T_j^i \rangle = \frac{4a}{(-\ell^2)^{d/2} d(d-1) \, \mathrm{Vol}(S^d)} \delta_j^i \ . \tag{4.137}$$

On $R \times H^{d-1}$ at the temperature $T = \frac{1}{2\pi\ell}$ it follows that

$$\langle T_0^0 \rangle = -\frac{4a}{d \, \ell^d \, \mathrm{Vol}(S^d)} \ , \quad \langle T_j^i \rangle = \frac{4a}{d(d-1)\ell^d \, \mathrm{Vol}(S^d)} \delta_j^i \ , \tag{4.138}$$

because the Riemann tensor is the opposite sign, and the result is constructed from the same product of $d/2$ Riemann tensors in each case. As the energy density is constant, the total energy is given by multiplying the energy density by the (divergent) volume of hyperbolic space, $\langle H \rangle = \langle T^{00} \rangle \, \mathrm{Vol}(H^{d-1})$. We need to isolate the logarithmic contribution to this volume

$$\mathrm{Vol}(H^{d-1}) = \ell^{d-1} \, \mathrm{Vol}(S^{d-2}) \int_0^{u_{\max}} \sinh^{d-2} u \, \mathrm{d}u \tag{4.139}$$

where our cut-off is

$$u_{\max} = -\ln \frac{\delta/\ell}{2 - \delta/\ell} \,. \tag{4.140}$$

We find that

$$\mathrm{Vol}(H^{d-1}) = \ldots + \frac{(-1)^{d/2}}{\pi} \ell^{d-1} \, \mathrm{Vol}(S^{d-1}) \ln \frac{\delta}{\ell} + \ldots \tag{4.141}$$

and hence that

$$2\pi\ell \langle H \rangle = \ldots + (-1)^{d/2} \frac{8a}{d} \frac{\mathrm{Vol}(S^{d-1})}{\mathrm{Vol}(S^d)} \ln \frac{\delta}{\ell} + \ldots \tag{4.142}$$

Like the stress tensor on $R \times S^{d-1}$, neither the stress tensor on $R \times H^{d-1}$ nor $\langle H \rangle$ is independent of the choice of scheme. For example, if one computes the partition function of a $d = 4$ conformal field theory in two different schemes in $d = 4$, their generating functionals may differ by the local counterterm

$$\xi \int d^4x \sqrt{g} R^2 \,, \tag{4.143}$$

where the coefficient ξ is real. Taking a metric variation of the counterterm, it is clear that the stress tensor on $R \times S^{d-1}$, or $\langle H \rangle$ on $R \times H^{d-1}$, depends on the choice of ξ. See refs. [1, 2, 81] for lengthier discussions of this issue. However, the dependence of W on ξ is linear in β. Thus while $\langle H \rangle$ depends on the choice of scheme, the result we obtain for the sphere entanglement entropy S_E does not.

In principle, we should also worry about boundary contributions to $\langle H \rangle$. We claim these contributions do not contribute to the logarithm. One way to compute them is to look at the metric variation of the boundary $Q_{n,m}$ term in $n = d + \epsilon$ dimensions. As we saw before, the variation of the metric through the spin connection will cancel against a bulk variation of $E_{n,m}$. The remaining variation comes only from the vielbeins, and cannot produce a logarithmic contribution.

4.4.2 Dilaton Effective Action

It is more involved to obtain $\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$. In $d = 2, 4$, and 6, we use the dilaton effective actions that we found in sections 4.1 and 4.3. We will see that logarithmic contributions from $\langle H \rangle$ and $\mathcal{W}$ cancel out, i.e. that

$$2\pi\ell\langle H\rangle + \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}] \tag{4.144}$$

has no logarithmic contribution. Thus, the entire entanglement entropy contribution comes from $\widetilde{W}[\delta_{\mu\nu}]$, which we will compute next.

In principle, we should be able to evaluate $\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$ for general even d and find the same cancelation of the logarithmic pieces. In practice, there is an issue of non-commuting limits in dimensional regularization which makes the calculation difficult. The correct order of limits is to take the metric to be completely general, take the $n \to d$ limit, and only then specialize to the metric of interest. To see that the other order of limits is problematic, consider the following example. If we first fix the metric $e^{-2\sigma}\delta_{\mu\nu}$ to be that of $S^1 \times H^{n-1}$ and then take the limit $n \to d$, we get a divergence that disappears in the other order of limits. Because $S^1 \times H^{n-1}$ contains an S^1 factor, the Euler characteristic, i.e. the leading $1/(n-d)$ singularity in $\widetilde{W}[e^{-2\sigma}\delta_{\mu\nu}]$, will vanish. In contrast, the leading $1/(n-d)$ singularity from the boundary contribution to $\widetilde{W}[\delta_{\mu\nu}]$ will not vanish. Thus $\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$ computed in this order will not even be finite.

We identify the conformal factor σ in the metric (4.134) with the dilaton τ of section 4.3 (not to be confused with hyperbolic time). For convenience, we divide up the bulk and boundary contributions to $\mathcal{W}$. We find the following results.

$d = 2$

The $d = 2$ case can be evaluated from the effective action (4.15). Denoting $\frac{c}{12} = a$ and recalling that an interval has two endpoints, we find the bulk contribution to $\mathcal{W}$ is

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]_{\text{Bulk}} = -\left(\frac{a}{2\pi}\right)\Big(2\pi u - 4\pi\ln(\sinh u)\Big)\,\text{Vol}(S^0) + ... \tag{4.145}$$

The boundary action contributes the following relevant divergence (the logarithmic divergence)

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]_{\text{Boundary}} = -\left(\frac{a}{2\pi}\right)\Big(4\pi u\Big)\,\text{Vol}(S^0) + \dots\,, \tag{4.146}$$

so that the logarithmic contribution to $\mathcal{W}$ is

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}] = -2au + \dots\,. \tag{4.147}$$

Using the expression (4.142) for $\langle H\rangle$, we see $2\pi\ell\langle H\rangle + \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$ has no logarithmic term.

$d = 4$

In $d = 4$, we find that the bulk and boundary terms in the expression (4.126) for $\mathcal{W}$ contribute the following logarithmically divergent terms

$$\begin{aligned} \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]_{\text{Bulk}} &= \frac{a}{(4\pi)^2}\Big(6\pi u - 16\pi \ln(\sinh u)\Big)\text{Vol}(S^2) + \dots\,, \\ \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]_{\text{Boundary}} &= \frac{a}{(4\pi)^2}\Big(16\pi u\Big)\text{Vol}(S^2) + \dots\,. \end{aligned} \tag{4.148}$$

$d = 6$

In $d = 6$, we find that the bulk and boundary terms in the expression (4.131) for $\mathcal{W}$ give

$$\begin{aligned} \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]_{\text{Bulk}} &= -\frac{a}{(4\pi)^3}\left(30\pi u - 96\pi \ln(\sinh u)\right)\text{Vol}(S^4) + \dots\,, \\ \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]_{\text{Boundary}} &= -\frac{a}{(4\pi)^3}(96\pi u)\text{Vol}(S^4) + \dots\,. \end{aligned} \tag{4.149}$$

In sum, using the dilaton effective action in $d = 2, 4, 6$, we confirm that there is no logarithmic contribution to $2\pi\ell\langle H\rangle + \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$, as advertised.

4.4.3 The Boundary Contribution to W in General Dimension

The last calculation to do is then an evaluation of the logarithmic contribution to $\widetilde{W}[\delta_{\mu\nu}]$ in general dimension. To keep the boundary parametrization simple, it is useful to work in the (τ, u) coordinate system. In that system, we have that the extrinsic curvature takes the form

$$K^\tau_\tau = -\frac{\sinh u}{\ell}\,, \qquad K^u_u = 0\,, \qquad K^j_i = \frac{1}{\ell}(\cosh\frac{\tau}{\ell}\coth u + \text{csch}\, u)\delta^i_j\,. \tag{4.150}$$

The bulk term in $\widetilde{W}$ vanishes identically in flat space, so it remains to evaluate the boundary term. Two useful integrals for evaluating that boundary term in flat space are, for even d,

$$\begin{aligned} \int_0^{2\pi} \frac{(1+\cosh u\cos t)^{d-2}}{(\cosh u + \cos t)^{d-1}}\text{d}t &= \frac{\pi}{\sinh u}\frac{(d-2)!}{2^{d-3}\left(\frac{d-2}{2}!\right)^2}\,, \\ \int_0^1 (1-s^2)^{d/2-1}\text{d}s &= \frac{\sqrt{\pi}\left(\frac{d-2}{2}\right)!}{2\left(\frac{d-1}{2}\right)!}\,. \end{aligned} \tag{4.151}$$

Starting with the expression (4.81) and using the Gauss equation to replace the non-zero $\mathring{\mathcal{R}}_{\alpha\beta\gamma\delta}$ with the vanishing $R_{\mu\nu\rho\sigma}$, the logarithmic contribution to the boundary term is

$$\int_{\partial M} \mathcal{Q}_{n,d/2} \;=\; \dots + \frac{2\pi(n-d)d!}{d-1}\text{Vol}(S^{d-2})\ln\frac{\delta}{\ell} + \dots\,. \tag{4.152}$$

Using that for even d,

$$\frac{\text{Vol}(S^{d-2})}{\text{Vol}(S^d)} = \frac{d-1}{2\pi}\,, \tag{4.153}$$

we then find the logarithmic contribution

$$-\widetilde{W}[\delta_{\mu\nu}] = \ldots + (-1)^{d/2} 4a \ln\frac{\delta}{\ell} + \ldots\,. \tag{4.154}$$

Using the expression (4.135) for S_E and that $2\pi\ell\langle H\rangle + \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$ has no logarithmic term, we indeed find that the universal term in the entanglement entropy S_E across a sphere is

$$S_E = \ldots + (-1)^{d/2} 4a \ln\frac{\delta}{\ell} + \ldots\,, \tag{4.155}$$

as claimed in ref. [40]. Often in these types of computations, knowing the value of a difference like $\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]$ is useful because there are symmetry reasons to believe that for the reference background $\widetilde{W}[\delta_{\mu\nu}]$ will vanish. Here, precisely because we had a boundary, $\widetilde{W}[\delta_{\mu\nu}]$ did not vanish. As a result, we needed an independent way of calculating $\widetilde{W}[\delta_{\mu\nu}]$, and in fact, when the dust settled, we saw that we only needed to calculate $\widetilde{W}[\delta_{\mu\nu}]$. Everything else canceled.

That $\widetilde{W}[\delta_{\mu\nu}]$ gives the right answer could perhaps have been anticipated. From ref. [49], it is known at least in four dimensions that the a dependent contribution to the entanglement entropy for a general entangling surface Σ is proportional to the Euler characteristic of that surface, $S_E \sim 2a\chi(\Sigma)\ln(\delta/\ell)$. The fact that $\widetilde{W}[\delta_{\mu\nu}]$ gives us the entanglement entropy in our case could be viewed as confirmation of ref. [49] in the case when Σ is a sphere. It is not too much of a stretch to imagine that in general even d, the a dependent part of the entanglement entropy will be $S_E \sim (-1)^{d/2} 2a\chi(\Sigma)(\ln\delta/\ell)$. Indeed, there are arguments to this effect in refs. [82, 83].

Before proceeding, we write down an expression for the thermal partition function $W_H = -\ln Z_H$ on H^{d-1} at temperature $T = 1/(2\pi\ell)$ whose logarithmic pieces agree with the results above

$$W_H = -a\frac{4\pi\ell}{(4\pi\ell^2)^{d/2}\left(\frac{d}{2}\right)!}\left[\Gamma(d) - 2^{d-1}\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)\right]\text{Vol}(H^{d-1}) + \ldots\,. \tag{4.156}$$

The first term is proportional to $\langle H\rangle$ and the second term gives the entanglement entropy. The quantity in brackets is A160481 in the Online Encyclopedia of Integer Sequences [84].

4.5 Discussion

We resolved the puzzle described in ref. [40]: the universal logarithmic term in the entanglement entropy (4.2) across a sphere in flat space (for a conformal theory) can be recovered by a Weyl transformation to hyperbolic space, provided one keeps careful track of boundary terms. One interesting consequence of our results is that the logarithmic term can be interpreted as a purely boundary effect. With the help of the conformal map to hyperbolic space cross a circle, focusing on the universal part, we identify the logarithmic contribution to the entanglement entropy S_E and the dimensionally regularized effective action $\widetilde{W}[\delta_{\mu\nu}]$:

$$S_{\rm E} \equiv -\operatorname{tr}(\rho_A \ln \rho_A) \sim -\widetilde{W}[\delta_{\mu\nu}] \ , \tag{4.157}$$

where $\widetilde{W}[\delta_{\mu\nu}]$ is given by eq. (4.113). $\widetilde{W}[\delta_{\mu\nu}]$ corresponds to a dimensionally continued Euler characteristic of the causal development of the interior of the sphere, a ball, which in turn receives contributions purely from the spherical boundary of the ball since the Riemann curvature and hence the Euler density vanish in flat space. The leading area law divergence in the entanglement entropy is also usually interpreted to be a boundary effect: entanglement entropy scales with the area of the boundary because in the ground state most of the entanglement is assumed to be local. But here we see that the subleading logarithmic divergence is also a boundary effect. Perhaps this result should have been anticipated since both divergences are regulated by a short distance cut-off δ, which one could think of as the distance between lattice points on either side of the boundary.

As we discussed in section 4.4, that $\widetilde{W}[\delta_{\mu\nu}]$ on its own gives the correct answer for the log term in the entanglement entropy across a sphere can be viewed as a special case of Solodukhin's result [49] using a squashed cone in $d = 4$ that the a contribution to the entanglement entropy across a general surface Σ can be written

$$S_E \sim 2a\chi(\Sigma) \ln(\delta/\ell) \ . \tag{4.158}$$

For non-spherical entangling surfaces, there will of course be other contributions to S_E, for example from the c_j central charges. While we are not aware of a derivation (refs. [82, 83] come close but ultimately only consider the sphere case), it seems reasonable that in general dimension, the only modification needed to make this formula correct in our conventions is a factor of $(-1)^{d/2}$.

In the process of resolving this puzzle, we produced a number of auxiliary results which are interesting in their own right. In two dimensions, where the trace anomaly is perhaps most powerful, we were able to use an effective anomaly action to reproduce three well-known results in conformal field theory, namely the Schwarzian derivative, the entanglement entropy of an interval, and also the Rényi entropies for the interval. Neither the effective anomaly action we use nor the results are new. However, we have not seen our form of the effective anomaly action used to derive these three results before.[8] Additionally, the story in two dimensions provides a simple warm-up example for the story in general dimension which we pursued next.

[8]See however ref. [64] for a similar calculation.

Between $d = 4$ and $d = 6$, our story is the most complete in $d = 4$. In four dimensions, we derived from general principles the most general Wess-Zumino consistent result for the trace anomaly on a manifold with a codimension one boundary, including two boundary central charges we denoted b_1 and b_2. It would be interesting to study b_1 and b_2 further (as well as their counter-parts in higher dimensions). What values[9] do they take for massless fermions? for a gauge field? for superconformal field theories? Might they be ordered under renormalization group flows, like the coefficient a?

Another pair of key results in this chapter are explicit formulae with boundary terms for the a contribution to the effective anomaly action in $d = 4$ and $d = 6$ dimensions. Previously, to our knowledge, only the bulk contribution had been worked out [14, 55, 15]. Unfortunately, in $d = 6$, we were only able to detail the boundary contribution to the action for a conformally flat metric. The conformally flat case was enough to study the entanglement entropy across a sphere. Nevertheless, it would be nice to write down the boundary contribution for a general metric.

We mostly adopted the dimensional regularization to construct $\mathcal{W}$. It would be interesting to construct $\mathcal{W}$ using the integral formula (4.69).

4.6 Appendix

4.6.1 Differential Geometry with a Boundary

Let M be a d-dimensional, orientable, Riemannian manifold with metric g with a boundary ∂M. We use x^μ to indicate coordinates on patches of M and y^α for coordinates on patches of ∂M. The boundary can be specified by means of the embedding functions $X^\mu(y^\alpha)$. These do not transform as tensors under reparameterizations in M, but their derivatives

$$f_\alpha{}^\mu \equiv \partial_\alpha X^\mu \,, \tag{4.159}$$

do. Rather, the f^μ_α transform as a vector under reparameterizations of the x^μ and as a one-form under reparameterizations of the y^α. The f^μ_α allow us to pull back covariant tensors on M to covariant tensors on ∂M. For instance, the metric g pulls back to the induced metric γ with components

$$\mathring{g}_{\alpha\beta}(y) = f_\alpha{}^\mu(y) f_\beta{}^\nu(y) g_{\mu\nu}(X(y)) \,. \tag{4.160}$$

We also define

$$f^\alpha{}_\mu \equiv g_{\mu\nu}\gamma^{\alpha\beta} f_\beta{}^\nu \,, \tag{4.161}$$

which satisfies

$$f^\alpha{}_\mu f_\beta{}^\mu = \delta^\alpha_\beta \,, \qquad f^\alpha{}_\mu f_\alpha{}^\nu \equiv h^\nu_\mu \,, \tag{4.162}$$

[9]The boundary central charges for fermions and gauge fields were recently computed in $d = 4$ in ref. [85].

with $h^{\mu\nu}$ a tangential projector. We can also define a unit-length, outward-pointing vector field n^μ after picking an orientation on ∂M via

$$n^\mu = \frac{1}{(d-1)!}\varepsilon^\mu{}_{\nu_1\ldots\nu_{d-1}}\varepsilon^{\alpha_1\ldots\alpha_{d-1}} f_{\alpha_1}{}^{\nu_1}\ldots f_{\alpha_{d-1}}{}^{\nu_{d-1}} \,. \tag{4.163}$$

The Covariant Derivative and the Second Fundamental Form

We use the Levi-Civita connection built from g to take derivatives D on M. From this connection we construct a connection on ∂M that allows us to take derivatives $\mathring{\nabla}$ of tensors on ∂M. $\mathring{\nabla}$ acts on e.g. a mixed tensor $\mathfrak{T}^\mu{}_\alpha$ via

$$\mathring{\nabla}_\alpha \mathfrak{T}^\mu{}_\beta = \partial_\alpha \mathfrak{T}^\mu{}_\beta + \Gamma^\mu{}_{\nu\alpha}\mathfrak{T}^\nu{}_\beta - \mathring{\Gamma}^\gamma{}_{\beta\alpha}\mathfrak{T}^\mu{}_\gamma \,, \tag{4.164}$$

with

$$\Gamma^\mu{}_{\nu\alpha} = \Gamma^\mu{}_{\nu\rho} f_\alpha{}^\rho \,, \qquad \mathring{\Gamma}^\alpha{}_{\beta\gamma} = f^\alpha{}_\mu \left(\partial_\gamma \delta^\mu_\nu + \Gamma^\mu{}_{\nu c}\right) f_\beta{}^\nu \,. \tag{4.165}$$

It is easy to show that $\mathring{\Gamma}^\alpha{}_{\beta\gamma}$ is the Levi-Civita connection constructed from the induced metric $\gamma_{\alpha\beta}$, and furthermore that the derivative satisfies

$$\mathring{\nabla}_\alpha g_{\mu\nu} = 0 \,, \qquad \mathring{\nabla}_\alpha \gamma_{\beta\gamma} = 0 \,. \tag{4.166}$$

There is a single tensor with one derivative that can be built from the data at hand, namely the second fundamental form $\mathrm{II}^\mu{}_{\alpha\beta}$,

$$\mathrm{II}^\mu{}_{\alpha\beta} \equiv \mathring{\nabla}_\alpha f_\beta{}^\mu \,. \tag{4.167}$$

One can show that

$$\mathrm{II}^\mu{}_{\alpha\beta} = \mathrm{II}^\mu{}_{\beta\alpha} \,, \qquad h_{\mu\nu}\mathrm{II}^\nu{}_{\alpha\beta} = 0 \,, \tag{4.168}$$

and the latter implies that

$$\mathrm{II}^\mu{}_{\alpha\beta} = -n^\mu K_{\alpha\beta} \,, \tag{4.169}$$

where $K_{\alpha\beta}$ is the extrinsic curvature of the boundary. From this and $n_\mu \mathring{\nabla}_\alpha n^\mu = 0$ we also find

$$\mathring{\nabla}_\alpha n_\mu = f^\beta{}_\mu K_{\alpha\beta} \,. \tag{4.170}$$

Let us relate this presentation to the more common one in terms of Gaussian normal coordinates. For some patch on M which includes a patch of ∂M, we choose coordinates so that g takes the form

$$g = \mathrm{d}r^2 + \hat{g}_{\alpha\beta}(r,y)\mathrm{d}y^\alpha \mathrm{d}y^\beta \,, \tag{4.171}$$

where the boundary is extended in the y^α at $r=0$. That is, the embedding functions are $f_\alpha{}^r = 0\,, f_\alpha{}^\beta = \delta^\beta_\alpha$, and consequently the induced metric is

$$\gamma_{\alpha\beta}(y) = \hat{g}_{\alpha\beta}(r=0,y) \,. \tag{4.172}$$

In this coordinate choice we have

$$n^r = 1\,, \qquad \mathrm{II}^r{}_{\alpha\beta} = \Gamma^r{}_{\alpha\beta} = -\frac{1}{2}\,\partial_r \hat{g}_{\alpha\beta}\big|_{y=0}\,. \tag{4.173}$$

Note that the trace of the extrinsic curvature, $K = \gamma^{\alpha\beta}K_{\alpha\beta}$ is

$$K = \frac{1}{2}\,\hat{g}^{\alpha\beta}\partial_r \hat{g}_{\alpha\beta}\big|_{r=0} = \left.\frac{\pounds_n\sqrt{\hat{g}}}{\sqrt{\hat{g}}}\right|_{r=0}\,, \tag{4.174}$$

with $\pounds_n$ the Lie derivative along n^μ, which coincides with a common formula used by physicists for the extrinsic curvature of a spacelike boundary.

Gauss and Codazzi

Consider the Levi-Civita connection one-form $\Gamma^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\mathrm{d}x^\rho$ and its curvature

$$R^\mu{}_\nu = \mathrm{d}\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu = \frac{1}{2}R^\mu{}_{\nu\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\,. \tag{4.175}$$

$R^\mu{}_{\nu\rho\sigma}$ is the Riemann curvature which can also be defined through the commutator of derivatives

$$[\mathrm{D}_\rho, \mathrm{D}_\sigma]\mathfrak{v}^\mu = R^\mu{}_{\nu\rho\sigma}\mathfrak{v}^\nu\,, \tag{4.176}$$

for $\mathfrak{v}^\mu$ a vector field. The pullback of $R^\mu{}_\nu$ to ∂M can be expressed in terms of the curvature $\mathring{R}^\mu{}_\nu$ of $\mathring{\Gamma}$ and the second fundamental form. The resulting expressions are the Gauss and Codazzi equations. They can be summarized as

$$\mathrm{P}[R^\mu{}_\nu] = \mathring{R}^\alpha{}_\beta f_\alpha{}^\mu f^\beta{}_\nu + \mathring{\nabla}\mathcal{M}^\mu{}_\nu - \mathcal{M}^\mu{}_\rho \wedge \mathcal{M}^\rho{}_\nu\,, \tag{4.177}$$

where $\mathring{\nabla}$ is the covariant exterior derivative and

$$\mathcal{M}^\mu{}_\nu = \mathrm{II}^\mu{}_\alpha f^\alpha{}_\nu - f_\alpha{}^\mu \mathrm{II}_\nu{}^\alpha\,, \qquad \mathrm{II}^\mu{}_\alpha \equiv \mathrm{II}^\mu{}_{\alpha\beta}\mathrm{d}y^\beta\,. \tag{4.178}$$

Alternatively, we can define

$$\tilde{\Gamma}^\mu{}_\nu = \Gamma^\mu{}_{\nu\alpha}\mathrm{d}y^\alpha - \mathcal{M}^\mu{}_\nu\,, \tag{4.179}$$

whose curvature satisfies

$$\tilde{R}^\mu{}_\nu = \mathring{R}^\alpha{}_\beta f_\alpha{}^\mu f^\beta{}_\nu\,. \tag{4.180}$$

In components, the Gauss and Codazzi equations read

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \mathring{R}_{\alpha\beta\gamma\delta} - K_{\alpha\gamma}K_{\beta\delta} + K_{\alpha\delta}K_{\beta\gamma}\,, \\ R_{\mu\alpha\beta\gamma}n^\mu &= -\mathring{\nabla}_\beta K_{\alpha\gamma} + \mathring{\nabla}_\gamma K_{\alpha\beta}\,, \end{aligned} \tag{4.181}$$

and we have used the embedding scalars to convert indices on the bulk Riemann tensor into indices on ∂M.

4.6.2 Wess-Zumino Consistency in $d = 4$

We now perform the algorithm described in Subsection 4.2.4, beginning with step 1. We need to parameterize the most general variation of W, which we denote as $\delta_\sigma W_b$. After some computation, we find that this variation contains sixteen independent terms[10]

$$\delta_\sigma W_b = \int_{\partial M} \mathrm{d}^3 y \sqrt{\gamma} \left\{ \sum_{I=1}^{8} b_I \mathcal{B}_I + \sum_{J=1}^{8} B_J \mathcal{D}_J \right\} \delta\sigma \,, \tag{4.182}$$

indexed by the eight b_I and eight B_J. (The coefficients b_I and B_J are used to denote boundary central charges.) We organize the terms in the following way. The eight $\mathcal{B}_I$ are three-derivative scalars. The eight $\mathcal{D}_J$ all involve derivatives of the Weyl variation $\delta\sigma$, and so we denote them with a calligraphic $\mathcal{D}$ to suggest a derivative. We distinguish the $\mathcal{B}_I$ and $\mathcal{D}_J$ for two reasons. First, the allowed three-derivative counterterms are given by the $\mathcal{B}_I$. Second, we will see shortly that those local counterterms redefine the coefficients of the $\mathcal{D}_J$.

In any case, the $\mathcal{B}_I$ are

$$\begin{aligned} &\mathcal{B}_1 = \mathring{R} K \,, \quad \mathcal{B}_2 = R K \,, \quad \mathcal{B}_3 = \mathring{R}_{\alpha\beta} K^{\alpha\beta} \,, \quad \mathcal{B}_4 = \operatorname{tr} K^3 \,, \\ &\mathcal{B}_5 = K^3 \,, \quad \mathcal{B}_6 = n^\mu \partial_\mu R \,, \quad \mathcal{B}_7 = \operatorname{tr} \hat{K}^3 \,, \quad \mathcal{B}_8 = W_{\alpha\beta\gamma\delta} \gamma^{\alpha\gamma} \hat{K}^{\beta\delta} \,, \end{aligned} \tag{4.183}$$

Here $W_{\alpha\beta\gamma\delta}$ is the pullback of the Weyl tensor to the boundary, and we have defined $\hat{K}$ to be the traceless part of the extrinsic curvature,

$$\hat{K}_{\alpha\beta} \equiv K_{\alpha\beta} - \frac{K}{d-1} \gamma_{\alpha\beta} \,, \tag{4.184}$$

which transforms covariantly under Weyl rescaling as $\hat{K}_{\alpha\beta} \to e^\sigma \hat{K}_{\alpha\beta}$. $\mathcal{B}_7$ and $\mathcal{B}_8$ are then manifestly covariant under Weyl rescaling. They are the only nonzero scalars that can be formed from either three factors of $\hat{K}$, or one factor of $\hat{K}$ and one of the Weyl tensor. They cannot be eliminated by the addition of a local counterterm and are trivially Wess-Zumino consistent, and so represent genuine boundary anomalies. The $\operatorname{tr}(\hat{K}^3)$ term first appeared in ref. [71], while the $W_{\alpha\beta\gamma\delta}\gamma^{\alpha\gamma}\hat{K}^{\beta\delta}$ term appeared later in ref. [72]. The $\mathcal{D}_J$ are

$$\begin{aligned} &\mathcal{D}_1 = \mathring{\Box} K \,, \quad \mathcal{D}_2 = \mathring{\nabla}_\alpha \mathring{\nabla}_\beta K^{\alpha\beta} \,, \quad \mathcal{D}_3 = \mathring{R} n^\mu \partial_\mu \,, \quad \mathcal{D}_4 = R n^\mu \partial_\mu \,. \\ &\mathcal{D}_5 = K_{\alpha\beta} K^{\alpha\beta} n^\mu \partial_\mu \,, \quad \mathcal{D}_6 = K^2 n^\mu \partial_\mu \,, \quad \mathcal{D}_7 = K n^\mu n^\nu \mathrm{D}_\mu \mathrm{D}_\nu \,, \quad \mathcal{D}_8 = n^\mu n^\nu n^\rho \mathrm{D}_\mu \mathrm{D}_\nu \mathrm{D}_\rho \,, \end{aligned} \tag{4.185}$$

Continuing with step 2, the most general local boundary counterterm is

$$W_{CT} = \int_{\partial M} \mathrm{d}^3 y \sqrt{\gamma} \sum_{I=1}^{6} d_I \mathcal{B}_I \,. \tag{4.186}$$

[10]In compiling the list of these sixteen terms, we have made extensive use of the Gauss and Codazzi equations (4.103). We also use that the action of $n^\mu \mathrm{D}_\mu$ is only well-defined on bulk tensors.

The d_I represent a choice of scheme. They can be adjusted to eliminate various coefficients in $\delta_\sigma W_b$. We would like to deduce which coefficients can be eliminated. This is an exercise in linear algebra. As $\sqrt{\gamma}\mathcal{B}_7$ and $\sqrt{\gamma}\mathcal{B}_8$ are invariant under Weyl rescalings, we do not include them in W_{CT}. The Weyl variation of W_{CT} may then be understood as a linear map $\Sigma : \mathbb{R}^6 \to \mathbb{R}^8$ which maps the $\{\mathcal{B}_I\}$ (for $I = 1, .., 6$) to the $\{\mathcal{D}_J\}$ as

$$\delta_\sigma \int_{\partial M} \mathrm{d}^3 y \sqrt{\gamma}\, \mathcal{B}_I = \int_{\partial M} \mathrm{d}^3 y \sqrt{\gamma} \sum_{J=1}^{8} \Sigma^J{}_I \mathcal{D}_J \,. \tag{4.187}$$

The number of $\mathcal{D}_J$ which can be eliminated is given by the dimension of the image of Σ, and the null vectors of Σ^t encode the linear combinations of the $\mathcal{D}_J$ which cannot be removed by a judicious choice of scheme.

A straightforward computation gives

$$\Sigma = \begin{pmatrix} -4 & -6 & -1 & 0 & 0 & 6 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & -3 \\ 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 3 \\ 0 & -6 & 0 & 0 & 9 & 3 \\ 0 & -6 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & -6 \end{pmatrix} \tag{4.188}$$

The map Σ is injective, so six of $\mathcal{D}_J$ can be eliminated. The null vectors of Σ^t are given by

$$\chi_1 = \begin{pmatrix} 3 & 1 & 4 & 0 & 0 & 0 & -3 & 4 \end{pmatrix} , \qquad \chi_2 = \begin{pmatrix} 0 & 0 & 0 & 6 & 0 & 0 & 3 & -2 \end{pmatrix} , \tag{4.189}$$

so the image of Σ is given by $\mathbb{R}^8$ modulo the $\mathbb{R}^2$ spanned by χ_1 and χ_2. In terms of the $\mathcal{D}_J$, the linear combinations

$$3\mathcal{D}_1 + \mathcal{D}_2 + 4\mathcal{D}_3 - 3\mathcal{D}_7 + 4\mathcal{D}_8 \,, \qquad 6\mathcal{D}_4 + 3\mathcal{D}_7 - 2\mathcal{D}_8 \,, \tag{4.190}$$

are never generated from the variation of W_{CT}. Said another way, the d_I can be adjusted to eliminate all of the $\mathcal{D}_J$ except for $\mathcal{D}_1$ and $\mathcal{D}_4$. So the most general boundary Weyl variation, having modded out by local counterterms, is

$$\delta_\sigma W_b = \int_{\partial M} \mathrm{d}^3 y \sqrt{\gamma} \left\{ \sum_{I=1}^{8} b_I \mathcal{B}_I + B_1 \mathring{\Box} K + B_4 R n^\mu \partial_\mu \right\} \delta\sigma \,. \tag{4.191}$$

Now we implement step 3, by computing the second Weyl variation. The second variations of $\mathcal{B}_1 \delta\sigma_2$ through $\mathcal{B}_8 \delta\sigma_2$ follow (almost) immediately from the $\delta_\sigma W_{CT}$ that we computed above. Let us then consider carefully the second Weyl variation of the terms proportional to B_1 and B_4. From these terms we get

$$\delta_{\sigma_1}\delta_{\sigma_2}W_b = \int \mathrm{d}^3y\sqrt{\gamma}\left\{B_1\left(3(n^\mu\partial_\mu\delta\sigma_1)(\mathring{\Box}\delta\sigma_2)+2K(\partial^\alpha\delta\sigma_1)(\partial_\alpha\delta\sigma_2)\right)\right.$$
$$\left. -6B_4(n^\mu\partial_\mu\delta\sigma_2)\left(\mathring{\Box}+n^\nu n^\rho \mathrm{D}_\nu \mathrm{D}_\rho + Kn^\nu\partial_\nu\right)\delta\sigma_1+\ldots\right\}, \quad (4.192)$$

where the ellipsis denotes terms that depend on b_1 through b_6. The only terms with a normal derivative of $\delta\sigma_2$ come from B_4. Given that fact, it is impossible to symmetrize under $\delta\sigma_1 \leftrightarrow \delta\sigma_2$ the term involving one normal derivative of $\delta\sigma_2$ and two normal derivatives of $\delta\sigma_1$. Thus Wess-Zumino consistency forces $B_4 = 0$.

It is slightly more involved to see that B_1 must vanish. First, observe that the $\mathcal{B}_6$ term is the only one which produces a second variation $\delta\sigma_2\mathcal{D}_8\delta\sigma_1$, which has three normal derivatives and is not symmetric under $\delta\sigma_1 \leftrightarrow \delta\sigma_2$ and so is not WZ consistent. So $b_6 = 0$. In fact, the same sort of reasoning tells us that $b_2 = b_4 = b_5 = 0$ and that b_3 is proportional to b_1 as $b_3 = -3b_1$. In terms of the remaining parameters b_1, B_1, the second Weyl variation is simply

$$\delta_{\sigma_1}\delta_{\sigma_2}W_b = \int \mathrm{d}^3y\sqrt{\gamma}\left\{3b_1\delta\sigma_2\hat{K}^{\alpha\beta}\mathring{\nabla}_\alpha\mathring{\nabla}_\beta\delta\sigma_1 + B_1\left(3(n^\mu\partial_\mu\delta\sigma_1)(\mathring{\Box}\delta\sigma_2)+2K(\partial^\alpha\delta\sigma_1)(\partial_\alpha\delta\sigma_2)\right)\right\}. \quad (4.193)$$

This expression is not symmetric under $\delta\sigma_1 \leftrightarrow \delta\sigma_2$ for any nonzero value of b_1 and B_1, and so WZ consistency enforces that they both vanish $b_1 = B_1 = 0$.

The only "boundary central charges" that survive are b_7 and b_8, and the boundary term in the anomaly is

$$\delta_\sigma W_b = \int_{\partial M}\mathrm{d}^3y\sqrt{\gamma}\left\{b_7\mathrm{tr}\hat{K}^3 + b_8\gamma^{\alpha\gamma}\hat{K}^{\beta\delta}W_{\alpha\beta\gamma\delta}\right\}. \quad (4.194)$$

Putting the pieces together, the total anomaly is given by (4.108) as advertised in Subsection 4.2.4. In the text, we relabel: $b_7 \to b_1$ and $b_8 \to b_2$.

4.6.3 Effective Action from Dimensional Regularization

In this appendix we consider the anomaly effective action $\mathcal{W}$ in even d dimensions as obtained from dimensional regularization via the expression (4.114), which we recall here

$$\mathcal{W}[g_{\mu\nu}, e^{-2\tau}g_{\mu\nu}] = A\lim_{n\to d}\frac{1}{n-d}\left\{\left(\int_M \mathcal{E}_{n,m} - \int_{\partial M}\mathcal{Q}_{n,m}\right) - \left(\int_M \hat{\mathcal{E}}_{n,m} - \int_{\partial M}\hat{\mathcal{Q}}_{n,m}\right)\right\}, \quad (4.195)$$

where $m = d/2$ and $A = (-1)^{d/2}4a/(d!\,\mathrm{Vol}(S^d))$. Here we obtain the explicit forms of $\mathcal{W}$ in $d = 4, 6$ including boundary terms. (In $d = 6$ the boundary action will be evaluated in a conformally flat geometry.) The bulk dilaton effective actions can be found in the literature; the boundary terms to our knowledge are new results.

We begin with the Lipschitz-Killing curvature $\mathcal{E}_{n,m}$ and the associated boundary term $\mathcal{Q}_{n,m}$ defined in (4.111) and (4.112) respectively. Denote the densities associated with these forms as $E_{n,m}$ and $Q_{n,m}$. The first step in evaluating the expression (4.114) for $\mathcal{W}$ is to deduce how $E_{n,m}$ and $Q_{n,m}$ change under Weyl rescalings. Starting with the metric $g_{\mu\nu}$ and performing a Weyl transformation to $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$, the transformed curvatures $\hat{E}_{n,m}$ and $\hat{Q}_{n,m}$ are

$$\begin{aligned} \sqrt{\hat{g}}\ \hat{E}_{n,m} &= \sqrt{g}\ e^{-(n-d)\tau} \left\{ E_d + \mathrm{D}_\mu J^\mu + (n-d)G + O(n-d)^2 \right\} \ . \\ \sqrt{\hat{\gamma}}\ \hat{Q}_{n,m} &= \sqrt{\gamma}\ e^{-(n-d)\tau} \left\{ Q_d + n_\mu J^\mu + \mathring{\nabla}_\alpha H^\alpha + (n-d)B + O(n-d)^2 \right\} \ , \end{aligned} \tag{4.196}$$

where it remains to determine J^μ, G, H^α, and B. Note that, in the $n \to d$ limit, (4.196) implies

$$\lim_{n\to d} \left(\int_M \hat{\mathcal{E}}_{n,m} - \int_{\partial M} \hat{\mathcal{Q}}_{n,m} \right) = \lim_{n\to d} \left(\int_M \mathcal{E}_{n,m} - \int_{\partial M} \mathcal{Q}_{n,m} \right) \ , \tag{4.197}$$

which is just a consequence of the fact that the Euler characteristic is a topological invariant and so is invariant under Weyl rescalings. This has the practical effect that the dimensionally regulated formula (4.114) for $\mathcal{W}$ is well-defined. From (4.196) we see that the integrand of (4.195) is

$$\begin{aligned} \sqrt{\hat{g}}\hat{E}_{n,m} - \sqrt{g}E_{n,m} &= \sqrt{g} \left\{ \mathrm{D}_\mu J^\mu - (n-d)\Big(\tau E_d - J^\mu \partial_\mu \tau - G + \mathrm{D}_\mu(\tau J^\mu) \Big) + O(n-d)^2 \right\} \ , \\ \sqrt{\hat{\gamma}}\hat{Q}_{n,m} - \sqrt{\gamma}Q_{n,m} &= \sqrt{\gamma} \left\{ n_\mu J^\mu + \mathring{\nabla}_\alpha H^\alpha - (n-d)\Big(\tau Q_d + \tau(n_\mu J^\mu + \mathring{\nabla}_\alpha H^\alpha) - B \Big) + O(n-d)^2 \right\} \ . \end{aligned}$$

In order to write $\mathcal{W}$ in as simple a way as possible, it will be useful to decompose G as

$$G = G_0 + \mathrm{D}_\mu K^\mu \,, \tag{4.198}$$

for some current K^μ. Putting the pieces together, we find that the anomaly action $\mathcal{W}$ is

$$\begin{aligned} \mathcal{W}[g_{\mu\nu}, e^{-2\tau} g_{\mu\nu}] = & A \Bigg(\int_M \mathrm{d}^d x \sqrt{g} \left\{ \tau E_d - J^\mu \partial_\mu \tau - G_0 \right\} \\ & - \int_{\partial M} \mathrm{d}^{d-1} y \sqrt{\gamma} \left\{ \tau Q_d - H^\alpha \partial_\alpha \tau - B + n^\mu K_\mu \right\} \Bigg) \ . \end{aligned} \tag{4.199}$$

Besides obtaining B and G defined in (4.196), we also need to determine J^μ, K^μ and H^α.

$d = 4$

To obtain the bulk action in $d = 4$, we find that J^μ is

$$J^\mu \ = -8 \left\{ E^{\mu\nu} \partial_\nu \tau + (D^\mu \partial_\nu \tau) \partial^\nu \tau + (\partial^\mu \tau)(\partial \tau)^2 - (\Box \tau) \partial^\mu \tau \right\} \ , \tag{4.200}$$

and we find it useful to split G into G_0 and K^μ as

$$\begin{aligned} K^\mu &= \frac{3}{2}J^\mu + 4E^{\mu\nu}\partial_\nu\tau\,, \\ G_0 &= 4E^{\mu\nu}(\partial_\mu\tau)(\partial_\nu\tau) - 8\Box\tau(\partial\tau)^2 + 6(\partial\tau)^4 \ . \end{aligned} \tag{4.201}$$

We find that the boundary data H^α and B are given by

$$\begin{aligned} H^\alpha &= 8\left\{\Big(K^{\alpha\beta} - \gamma^{\alpha\beta}K\Big)\partial_\beta\tau + \tau_n\partial^\alpha\tau\right\}, \\ B &= n^\mu K_\mu + 4\mathring{\nabla}_\alpha\left\{\partial_\beta\tau\left(K^{\alpha\beta} - \gamma^{\alpha\beta}K\right)\right\} - 4\left(K^{\alpha\beta} - \gamma^{\alpha\beta}K\right)(\partial_\alpha\tau)(\partial_\beta\tau) \\ &\quad - 8(\mathring{\nabla}\tau)^2\tau_n - \frac{8}{3}\tau_n^3\,, \end{aligned} \tag{4.202}$$

where we have denoted the normal derivative of τ as $\tau_n \equiv n^\mu\partial_\mu\tau$. Substituting these expressions into the general formula (4.199) for $\mathcal{W}$, we find the result (4.126) quoted in subsection 4.3.1.

$d = 6$

After some tedious computation, we find that the current J^μ in $d = 6$ for general $g_{\mu\nu}$ is given by

$$J^\mu_{(6d)} = J^\mu_1 + J^\mu_2 + J^\mu_3 + J^\mu_4 + J^\mu_5\ , \tag{4.203}$$

where J^μ_n contains n powers of τ, and

$$\begin{aligned} J^\mu_1 &= 6E^{(2)\mu}_\nu(\partial^\nu\tau)\ , \\ J^\mu_2 &= 48E^\mu_\nu\Big((\mathrm{D}_\rho\partial^\nu\tau)(\partial^\rho\tau) - (\partial^\nu\tau)\Box\tau\Big) + 48R^\mu{}_{\rho\nu\sigma}(\partial^\nu\tau)(\mathrm{D}^\sigma\partial^\rho\tau) \\ &\qquad +48R_{\nu\rho}\Big((\partial^\nu\tau)(\mathrm{D}^\rho\partial^\mu\tau) - (\mathrm{D}^\rho\partial^\nu\tau)(\partial^\mu\tau)\Big)\ , \\ J^\mu_3 &= 48E^\mu_\nu(\partial^\nu\tau)(\partial\tau)^2 + 48(\partial^\mu\tau)(\Box\tau)^2 - 96\Box\tau(\partial^\nu\tau)(\mathrm{D}_\nu\partial^\mu\tau) \\ &\qquad +96(\partial^\nu\tau)(\mathrm{D}_\rho\partial_\nu\tau)(\mathrm{D}^\rho\partial^\mu\tau) - 48(\mathrm{D}\partial\tau)^2(\partial^\mu\tau)\ , \\ J^\mu_4 &= -144(\partial\tau)^2\Box\tau(\partial^\mu\tau) + 144(\partial\tau)^2(\partial_\rho\tau)(\mathrm{D}^\rho\partial^\mu\tau)\ , \\ J^\mu_5 &= 144(\partial\tau)^4(\partial^\mu\tau)\ . \end{aligned} \tag{4.204}$$

The quantities $E^{(2)\mu\nu}$ and $C^{\mu\nu\rho\sigma}$ are defined in (4.129).

We have also computed G for a general metric $g_{\mu\nu}$. We split it into G_0 and K^μ so that the bulk part of the anomaly action $\mathcal{W}$ matches the expression obtained in ref. [15]. The resulting K^μ is

$$\begin{aligned} K^\mu &= \frac{11}{6}J^\mu - 5E^{(2)\mu\nu}\partial_\nu\tau + 16E^{\mu\nu}\Big((\partial_\nu\tau)\Box\tau - (\mathrm{D}^\rho\partial_\nu\tau)(\partial_\rho\tau)\Big) + 16C^\mu{}_{\nu\rho\sigma}(\mathrm{D}^\rho\partial^\nu\tau)(\partial^\sigma\tau) \\ &\qquad +48(\mathrm{D}^\mu\partial^\nu\tau)(\partial_\nu\tau)(\partial\tau)^2 + 72(\partial\tau)^4(\partial^\mu\tau) - 48(\partial\tau)^2\Box\tau(\partial^\mu\tau)\,, \end{aligned} \tag{4.205}$$

and the expression for G_0 is too lengthy to be worth writing here. It can be deduced by comparing the general expression for $\mathcal{W}$ given in (4.199) with the bulk part of the anomaly action in (4.130), using the formulae for J^μ and K^μ above. Similarly we decompose H^α into powers of τ as

$$H^\alpha = H_1^\alpha + H_2^\alpha + H_3^\alpha + H_4^\alpha \,. \tag{4.206}$$

The computation on the boundary becomes much more tedious. We have computed B in general but its expression is too lengthy to present here. We have not yet succeeded in finding the current H^α when for a general metric $g_{\mu\nu}$. When $\hat{g}_{\mu\nu}$ is conformally flat, $\hat{g}_{\mu\nu} = e^{-2\tau}\delta_{\mu\nu}$, we find

$$\begin{aligned}
H_1^\alpha &= 48P^\alpha_\beta\partial^\beta\tau + 6Q_4[\delta_{\mu\nu}]\partial^\alpha\tau \ , \\
H_2^\alpha &= 48K^\alpha_\beta(\partial^\beta\tau)\mathring{\Box}\tau - 48K^\alpha_\beta(\mathring{\nabla}_\gamma\partial^\beta\tau)(\partial^\gamma\tau) - 48K(\partial^\alpha\tau)\mathring{\Box}\tau \\
&\qquad + 48K^\beta_\gamma(\mathring{\nabla}^\gamma\partial_\beta\tau)(\partial^\alpha\tau) + 48K(\partial_\beta\tau)(\mathring{\nabla}^\alpha\partial^\beta\tau) - 48K^\beta_\gamma(\mathring{\nabla}^\alpha\partial_\beta\tau)(\partial^\gamma\tau) \ , \\
H_3^\alpha &= -48K^\alpha_\beta(\partial^\beta\tau)(\mathring{\nabla}\tau)^2 + 48K(\mathring{\nabla}\tau)^2(\partial^\alpha\tau) + 48K\tau_n^2(\partial^\alpha\tau) - 48\tau_n^2K^\alpha_\beta(\partial^\beta\tau) \\
&\qquad + 96\tau_n\mathring{\Box}\tau(\partial^\alpha\tau) - 96\tau_n(\mathring{\nabla}^\alpha\partial_\beta\tau)(\partial^\beta\tau) \ , \\
H_4^\alpha &= -144\tau_n(\mathring{\nabla}\tau)^2\partial^\alpha\tau - 48\tau_n^3(\partial^\alpha\tau)\,,
\end{aligned} \tag{4.207}$$

where we defined $P^{\alpha\beta}$ in (4.132). Using the expressions present above and the general expression for the boundary term of $\mathcal{W}$ in (4.199), we obtain the explicit form in $d = 6$ given in (4.131).

Chapter 5

Boundary Conformal Field Theory and a Boundary Central Charge

This chapter is an edited version of my publication [4], written in collaboration with Christopher Herzog.

Motivated by the boundary terms in the trace anomaly of the stress tensor, in this chapter we continue the investigation into the structure of boundary conformal field theory (bCFT) begun over thirty years ago [86, 87, 88].

A term in the trace anomaly of four dimensional CFTs mentioned is the square of the Weyl curvature with a coefficient conventionally called c. In flat space, the form of the two-point function of the stress tensor is fixed up to an overall normalization constant, a constant determined by c as well [89]. Less well known is what happens when there is a boundary. In curved space, one of the additional boundary localized terms in the trace of the stress tensor can be schematically written KW where W is the bulk Weyl curvature and K the extrinsic curvature of the boundary [3, 85, 90]. Let us call the coefficient of this term b_2. Ref. [85] observed that for free theories, b_2 and c were linearly related: $b_2 = 8c$ with our choice of normalization. A bottom up holographic approach to the problem suggests that for interacting theories, this relation may not always hold [91, 92, 93]. In this chapter, generalizing a method of Ref. [89] (see also [94]), we argue that b_2 is fixed instead by the near boundary limit of the stress tensor two-point function in the case where the two-point function is computed in a flat half space. For free theories, the bulk and boundary limits of the two-point function are related by a factor of two, and our proposal is then consistent with the $b_2 = 8c$ observation. More generally, we will find that interactions modify the relation between these limits.

To cross the logical chasm between b_2 and the stress tensor two-point function, our approach is to try to fill the chasm rather than just to build a bridge. With a view toward understanding the b_2 charge, we investigate bCFT more generally, in dimension $d > 2$, employing a variety of techniques from conformal block decompositions to Feynman diagrams. As a result, we find a number of auxiliary results which may have interest in their own right.

One such result is the observation that current and stress tensor two-point functions of free bCFTs have a universal structure. We consider stress tensor two-point functions for the scalar, fermion, and p-form in $2p+2$ dimensions as well as the current two-point functions for the scalar and fermion. Additionally, we describe their conformal block decompositions in detail. These calculations follow and generalize earlier work [87, 88, 95]. By conformal block decomposition, we are referring to a representation of the two-point functions as a sum over primary operators. In bCFT, there are two distinct such decompositions. Taking an operator in the bulk close to the boundary, we can re-express it as a sum over boundary primary fields, allowing for the boundary conformal block decomposition. Alternately, bringing two operators close together, we have the standard operator product expansion (OPE) where we can express the two operators as a sum over primary fields in the bulk, leading to the bulk conformal block decomposition. Our discussion of conformal blocks is in section 5.2.3 and appendix 5.7.1. Figure 5.1 represents the two types of conformal block decomposition in pictorial form.

We find generically for free theories that the two-point correlators can be described by a function of an invariant cross ratio v of the form $f(v) \sim 1 \pm v^{2\Delta}$, where Δ is a scaling dimension. Here, $v \to 1$ is the limit that the points get close to the boundary and $v \to 0$ is the coincident limit. (The behavior for free scalars is in general more complicated, but the limits $v \to 0$ and $v \to 1$ of $f(v)$ are the same as for the functions $1 \pm v^{2\Delta}$.) The 1 in $1 \pm v^{2\Delta}$ then corresponds to the two-point function in the absence of a boundary, and morally at least, we can think of the $v^{2\Delta}$ as the contribution of an image point on the other side of the boundary.

In the context of the b_2-charge, let us call the relevant cross-ratio function for the stress tensor $\alpha(v) \sim 1 + v^{2d}$. (Again, the function $\alpha(v)$ for a scalar is more complicated, but the limits $v \to 0$ and $v \to 1$ are the same.) In this case, we have the relation $\alpha(1) = 2\alpha(0)$. As we will see, c is proportional to the bulk limit $\alpha(0)$. It follows that there will be a corresponding relation between c and $\alpha(1)$ for free theories, which can be understood, given our proposed general relation (5.155) between $\alpha(1)$ and b_2, as the equality $b_2 = 8c$ in free theories.

What then happens for interacting theories? A canonical example of an interacting bCFT is the Wilson-Fisher fixed point, analyzed in either the ϵ [96, 87] or large N [88] expansion or more recently using boot strap ideas [95, 97]. Two choices of boundary conditions at the planar boundary are Dirichlet (ordinary) or Neumann (special). Indeed, one finds generically, in both the ϵ expansion and in the large N expansion, that $\alpha(1) \neq 2\alpha(0)$. In precisely the limit $d = 4$, the Wilson-Fisher theory however becomes free and the relation $\alpha(1) = 2\alpha(0)$ or equivalently $b_2 = 8c$ is recovered.

We would then like to search for an interacting bCFT in $d = 4$ dimensions that is tractable. Our strategy is to consider a free field in four dimensions coupled to a free field on the planar boundary in three dimensions through a classically marginal interaction that lives purely on the boundary. We consider in fact three different examples. Two of our examples turn out to be cousins of the Wilson-Fisher theory with a boundary, in the sense that, with appropriate fine tuning, they have a perturbative IR fixed point in the ϵ expansion, $d = 4-\epsilon$. The first example is a mixed dimensional Yukawa theory with a four dimensional scalar coupled to a three dimensional fermion. The second is a mixed dimensional scalar theory with coupled three and four dimensional scalar fields. At their perturbative interacting fixed points, both interacting theories give $\alpha(1) \neq 2\alpha(0)$ at leading order in perturbation theory. To our knowledge, neither theory has been examined in the literature. Given the interest in the Wilson-Fisher theory with a boundary, we suspect these cousins may deserve a more in depth analysis. Our calculations stop at one loop corrections to the propagators and interaction vertex. While these theories are free in the IR in $d = 4$ dimensions with $\epsilon = 0$, if we set $\epsilon = 1$ we may be able to learn some interesting data about fixed point theories in $d = 3$ with a two dimensional boundary. Unfortunately, neither of these interacting theories gives us an example of $b_2 \neq 8c$.

The third and perhaps most interesting example consists of a four dimensional photon coupled to a three dimensional massless fermion of charge g. The photon wave function is not renormalized at one or two loops [98, 99]. Indeed, a simple power counting argument suggests it is not perturbatively renormalized at all. A Ward identity then guarantees that the β function for the coupling g vanishes. Perturbatively, it follows that this mixed dimensional QED is exactly conformal for all values of g. The theory provides a controllable example where $\alpha(1) \neq 2\alpha(0)$ in exactly four dimensions. A leading order calculation in perturbation theory indeed demonstrates that $\alpha(1) \neq 2\alpha(0)$.

While we do not demonstrate the relation between $\alpha(1)$ and b_2 for mixed QED in particular, we do provide a general argument based on an effective anomaly action. The argument is similar in spirit to Osborn and Petkou's argument [89] relating c and $\alpha(0)$. The basic idea is the following. On the one hand, an effective anomaly action for the stress tensor will produce delta-function distributions that contribute to the stress tensor two-point function in the coincident and near boundary limits. As the effective anomaly action is constructed from the W^2 and KW curvature terms with coefficients c and b_2, these delta-function distributions will also have c and b_2 dependent coefficients. At the same time, the coincident limit of the stress tensor two-point function has UV divergences associated with similar delta-function distributions. Keeping track of boundary contributions, by matching the coefficients of these distributions, we obtain a constraint (5.155) relating b_2 and $\alpha(1)$.

The quantity $\alpha(1)$ is related to the coefficient of the two-point function of the displacement operator. In the presence of a boundary, the Ward identity for stress tensor conservation is modified to

$$\partial_\mu T^{\mu n} = D^n \delta(x_\perp) \, , \tag{5.1}$$

$$\partial_\mu T^{\mu A} = -\partial_B \hat{T}^{AB} \delta(x_\perp) \, , \tag{5.2}$$

where $\delta(x_\perp)$ is the Dirac delta function with support on the boundary, μ, ν are d dimensional indices, A, B are tangential indices and n is the normal direction. We can identify a scalar D^n displacement operator, sourced by perturbing the location of the boundary. Through a Gauss law pill box type argument, the operator D^n is equal to the boundary limit of T^{nn}. Moreover, the value $\alpha(1)$ is proportional to the contribution of D^n to the stress tensor two-point function in the boundary limit. A novel feature of all three boundary interacting theories, which distinguishes them from the Wilson-Fisher theory, is that in the perturbative limit, they have degrees of freedom that propagate on the boundary and an associated boundary stress tensor $\hat{T}^{AB}\delta(x_\perp)$. We expect that a classical non-zero $\hat{T}^{AB}$ generally exists in theories with boundary degrees of freedom that are coupled to bulk degrees of freedom. This boundary stress tensor is not conserved on its own, $\partial_B \hat{T}^{AB} \neq 0$, and conservation of energy and momentum in the full theory is guaranteed through an inflow mechanism involving the boundary limit of the normal-tangential component of the full stress tensor. We have

$$\text{Classical}: \quad T^{nn}|_{\text{bry}} = D^n \ , \quad T^{nA}|_{\text{bry}} = -\partial_B \hat{T}^{AB} \ . \tag{5.3}$$

While this story makes sense classically, renormalization effects alter the story non-perturbatively. Because $\hat{T}^{AB}$ is not conserved, its scaling dimension will shift upward from the unitarity bound at $\Delta = d-1$. It then no longer makes sense to separate out $\hat{T}^{AB}$ as a delta function-localized stress tensor; renormalization has "thickened" the degrees of freedom living on the boundary. Instead, one has just the bulk stress tensor $T^{\mu\nu}$, which is conserved, and whose conservation implies

$$\text{Operator}: \quad T^{nn}|_{\text{bry}} = D^n \ , \quad T^{nA}|_{\text{bry}} = 0 \ , \tag{5.4}$$

understood as an operator statement (at quantum level). Any insertion of $T^{nA}|_{\text{bry}}$ in a correlation function sets that correlation function to zero. In other words, there can be a localized, nonzero $T^{nA}|_{\text{bry}}$ classically, but quantum effects smear it out. This renormalization effect leads to subtleties with commuting the small coupling and near boundary limits in our perturbative calculations. For recent discussions of displacement operators, see [100, 101, 102, 103, 104, 105].[1]

Before moving to the details, it is worth remarking several features of this mixed dimensional QED theory. While its bCFT aspects have not to our knowledge been emphasized,

[1] As an application of the boundary conformal anomaly, in [9] we introduced a notation of reduction entropy (RE). We observed that the RE intriguingly reproduces the universal entanglement entropy upon a dimensional reduction, provided that $b_2 = 8c$ and a term $\langle T^{nn}\rangle$ is added in the RE. Interestingly, from the present chapter, we realize that $\langle T^{nn}\rangle$ in RE is the displacement operator. Moreover, since we find more generally that $b_2 \sim \alpha(1)$, the RE encodes the information about boundary conditions for interacting CFTs. The entanglement entropy, when computed by introducing a conical singularity, to our knowledge, however, does not seem to depend on boundary conditions. It would be interesting to revisit the calculations in ref. [9] in view of the results presented here.

the theory is closely related to models of graphene and has been studied over the years [106, 98, 99, 107, 108] in various contexts. Son's model [109] of graphene starts with charged, relativistic fermions that propagate in 2+1 dimensions with a speed $v_f < 1$ and their electric interactions with 4d photons. There is a β function for v_f with an IR fixed point at $v_f = 1$. Restoring the magnetic field and interactions at this IR fixed point, one finds precisely this mixed dimensional QED [98]. Similar statements about the non-renormalization of the coupling g can be found in the graphene literature (see e.g. [110]). This mixed QED was recently considered as a relativistic theory exhibiting fractional quantum Hall effect [111].

In the large N limit where one has many fermions, this QED-like theory can be mapped to three dimensional QED in a similar large N limit, with $g \sim 1/N$ [107]. Indeed, three dimensional QED is expected to flow to a conformal fixed point in the IR for sufficiently large N. This map thus replaces a discrete family of CFTs, indexed by N, with a continuous family of bCFTs, indexed by g. Such a map is reminiscent of AdS/CFT, with g playing the role of Newton's constant G_N. More recently, Hsiao and Son [112] conjectured that this mixed QED theory should have an exact S-duality. Such an S-duality has interesting phenomenological consequences. Using it, they calculate the conductivity at the self-dual point. Their calculation is in spirit quite similar to a calculation in an AdS/CFT context for the M2-brane theory [113].

An outline of this chapter is as follows. In section 5.1, we review the various boundary terms that appear in the trace anomaly of bCFTs. In section 5.2, we first review the general structure of the two-point functions in bCFT. Then, we discuss constraints on these two-point functions. We also give the boundary and bulk conformal block decompositions. Our decompositions for the current two-point function (5.84, 5.102–5.105, 5.107) have not yet been discussed in the literature to our knowledge. Nor have certain symmetry properties of the boundary blocks (5.109) and positivity properties of the current and stress tensor correlators (5.65, 5.67). In section 5.3, we give our argument relating $\alpha(1)$ to b_2-charge in 4d bCFTs. We also review how $\alpha(0)$ is related to the standard bulk c-charge. In section 5.4, we discuss two-point functions for free fields, including a conformal scalar, a Dirac fermion and gauge fields. In particular, the discussion of p-forms in $2p+2$-dimensions is to our knowledge new. Lastly, in section 5.5, we introduce our theories with classically marginal boundary interactions. In Appendix 5.7.1 we review how to derive the conformal blocks for scalar, vector, and tensor operators in the null cone formulation. Appendix 5.7.2 describes some curvature tensors and variation rules relevant to the discussion of the trace anomaly in sections 5.1 and 5.3. We discuss gauge fixing of the mixed QED in Appendix 5.7.3.

5.1 Boundary Conformal Anomalies

Considering a classically Weyl invariant theory embedded in a curved spacetime background, the counterterms added to regularize divergences give rise to the conformal (Weyl) anomaly, which is defined as a non-vanishing expectation of the trace of the stress tensor. The conformal anomaly in the absence of a boundary is well-known, in particular in $d = 2$ and $d = 4$ dimensions; see for instance [24, 114] for reviews. There is no conformal anomaly in odd dimensions in a compact spacetime. In the presence of a boundary, there are new Weyl anomalies localized on the boundary and their structure turns out to be rather rich. There are also new central charges defined as the coefficients of these boundary invariants. One expects that these boundary central charges can be used to characterize CFTs with a boundary or a defect, in a similar way that one characterizes CFTs without a boundary using the bulk central charges.

For an even dimensional CFT$_d$ with $d = 2n+2; n = 0, 1, 2, ...$, the Weyl anomaly can be written as

$$\langle T^{\mu}{}_{\mu}\rangle^{d=2n+2} = \frac{4}{d!\,\mathrm{Vol}(S^d)}\Big[\sum_i c_i \mathcal{I}_i + \delta(x_\perp)\sum_j b_j I_j - (-1)^{\frac{d}{2}} a_d \Big(E_d + \delta(x_\perp) E^{(\mathrm{bry})}\Big)\Big] \tag{5.5}$$

We normalize the Euler density E_d such that integrating E_d over an S^d yields $d!\mathrm{Vol}(\mathrm{S^d})$. We denote $E^{(\mathrm{bry})}$ as the boundary term of the Euler characteristic, which has a Chern-Simons-like structure [68, 79]. See the previous chapter or [3] for an extensive discussion. Notice that $E^{(\mathrm{bry})}$ is used to preserve the conformal invariance of the bulk Euler density when a boundary is present, so its coefficient is fixed by the bulk a-charge. We are here interested in a smooth and compact codimension-one boundary so we do not include any corner terms. The normalizations of local Weyl covariant terms, $\mathcal{I}_i$ and I_j, are defined here such that they simply have the same overall factor of the Euler anomaly. One can certainly adopt a different convention and rescale central charges a, c_i and b_j. The numbers of the local Weyl covariant terms vary depending on the dimensions. We emphasize that, since $\mathcal{I}_i$ and I_j are independently Weyl covariant, there are no constraints relating bulk charges c_i to b_j from an argument based solely on Weyl invariance of the integrated anomaly.

For an odd dimensional CFT$_d$ with $d = 2n+1$, $n = 1, 2, 3, \ldots$ there is no bulk Weyl anomaly. In the presence of a boundary, however, there can be boundary contributions. We write

$$\langle T^{\mu}{}_{\mu}\rangle^{d=2n+1} = \frac{2}{(d-1)!\,\mathrm{Vol}(S^{d-1})}\delta(x_\perp)\Big(\sum_i b_i I_i + (-1)^{\frac{(d+1)}{2}} a_d \mathring{E}_{d-1}\Big)\ , \tag{5.6}$$

where $\mathring{E}_{d-1}$ is the boundary Euler density defined on the $d-1$ dimensional boundary. The coefficient a_d with odd d is an a-type boundary charge. Similarly, I_i represents independent local Weyl covariant terms on the boundary.

An important boundary object is the traceless part of the extrinsic curvature defined as

$$\hat{K}_{AB} = K_{AB} - \frac{h_{AB}}{d-1}K\ , \tag{5.7}$$

where h_{AB} is the induced metric on the boundary. $\hat{K}_{AB}$ transforms covariantly under the Weyl transformation.

Note that we have dropped terms that depend on the regularization scheme in (5.5) and (5.6). For instance, the $\Box R$ anomaly in $d = 4$ CFTs can be removed by adding a finite counterterm R^2. It is worth mentioning that, from the previous chapter, Wess-Zumino consistency rules out the possibility of a boundary total derivative anomaly in $d = 4$ [3].

Let us consider explicit examples. In $d = 2$ one has

$$\langle T^\mu{}_\mu \rangle^{d=2} = \frac{a}{2\pi} \left(R + 2K\delta(x_\perp) \right) \ . \tag{5.8}$$

One can replace the anomaly coefficient a with the more common $d = 2$ central charge $c = 12a$. Note $c = 1$ for a free conformal scalar or a Dirac fermion. The $d = 2$ bCFTs have been a rich subject but since there is no new central charge, in this chapter we will not discuss $d = 2$ bCFTs. Interested readers may refer to [115] for relevant discussion of $d = 2$ bCFTs and their applications.

In $d = 3$ the anomaly contributes purely on the boundary. One has [66]

$$\langle T^\mu{}_\mu \rangle^{d=3} = \frac{\delta(x_\perp)}{4\pi} \left(a\mathring{R} + b \,\mathrm{tr}\, \hat{K}^2 \right) \ , \tag{5.9}$$

where $\mathrm{tr}\, \hat{K}^2 = \mathrm{tr}\, K^2 - \frac{1}{2}K^2$ and $\mathring{R}$ is the boundary Ricci scalar. Restricting to free fields of different spin s, the values of these charges are

$$a^{s=0} \ = \ -\frac{1}{96} \ \ (\mathrm{D}) \ , \quad a^{s=0} = \frac{1}{96} \ \ (\mathrm{R}) \ , \quad a^{s=\frac{1}{2}} = 0 \ , \tag{5.10}$$

and

$$b^{s=0} \ = \ \frac{1}{64} \ \ (\mathrm{D\ or\ R}) \ , \quad b^{s=\frac{1}{2}} = \frac{1}{32} \ , \tag{5.11}$$

where $(D)/(R)$ stands for Dirichlet/Robin boundary conditions. Neumann boundary conditions in general do not preserve conformal symmetry, but there is a particular choice of Robin boundary condition involving the extrinsic curvature which does. The quantity b for the scalar with Dirichlet and Robin boundary conditions was first computed to our knowledge by refs. [116] and [56] respectively. The complete table can be found in [117].

In $d = 4$ CFT, the conformal anomaly reads

$$\begin{aligned} \langle T^\mu{}_\mu \rangle^{d=4} \ &= \ \frac{1}{16\pi^2}\Big(cW^2_{\mu\nu\lambda\rho} - aE_4 \Big) \\ &\qquad + \frac{\delta(x_\perp)}{16\pi^2}\Big(aE_4^{(\mathrm{bry})} - b_1 \,\mathrm{tr}\, \hat{K}^3 - b_2 h^{\alpha\gamma}\hat{K}^{\beta\delta}W_{\alpha\beta\gamma\delta} \Big) \ , \end{aligned} \tag{5.12}$$

where

$$E_4 \ = \ \frac{1}{4}\delta^{\mu\nu\lambda\rho}_{\sigma\omega\eta\delta} R^{\sigma\omega}{}_{\mu\nu} R^{\eta\delta}{}_{\lambda\rho} \ , \tag{5.13}$$

$$E_4^{(\mathrm{bry})} \ = \ -4\delta^{ABC}_{DEF}\, K^D_A \left(\frac{1}{2} R^{EF}{}_{BC} + \frac{2}{3} K^E_B K^F_C \right) \ , \tag{5.14}$$

$$\mathrm{tr}\, \hat{K}^3 \ = \ \mathrm{tr}\, K^3 - K \,\mathrm{tr}\, K^2 + \frac{2}{9}K^3 \ , \tag{5.15}$$

$$h^{\alpha\gamma}\hat{K}^{\beta\delta}W_{\alpha\beta\gamma\delta} \ = \ R^\mu{}_{\nu\lambda\rho} K^\lambda_\mu n^\nu n^\rho - \frac{1}{2} R_{\mu\nu}(n^\mu n^\nu K + K^{\mu\nu}) + \frac{1}{6} KR \ , \tag{5.16}$$

with $\delta^{\mu\nu\lambda\rho}_{\sigma\omega\eta\delta}/\delta^{ABC}_{DEF}$ being the bulk/boundary generalized Dirac delta function, which evaluates to ± 1 or 0. Because of the tracelessness and symmetry of the Weyl tensor, one can write $h^{\alpha\gamma}\hat{K}^{\beta\delta}W_{\alpha\beta\gamma\delta} = -K^{AB}W_{nAnB}$. The coefficients b_1 and b_2 are new central charges. The values of these charges were computed for free theories. The bulk charges are independent of boundary conditions and are given by

$$a^{s=0} = \frac{1}{360}\ ,\ a^{s=\frac{1}{2}} = \frac{11}{360}\ ,\ a^{s=1} = \frac{31}{180}\ , \tag{5.17}$$

$$c^{s=0} = \frac{1}{120}\ ,\ c^{s=\frac{1}{2}} = \frac{1}{20}\ ,\ c^{s=1} = \frac{1}{10}\ , \tag{5.18}$$

(see e.g. [29]). The boundary charge b_1 of a scalar field depends on boundary conditions. One has

$$b_1^{s=0} = \frac{2}{35}\ (\mathrm{D})\ ,\ b_1^{s=0} = \frac{2}{45}\ (\mathrm{R})\ ,\ b_1^{s=\frac{1}{2}} = \frac{2}{7}\ (\mathrm{D\ or\ R})\ ,\ b_1^{s=1} = \frac{16}{35}\ (\mathrm{D\ or\ R})\ . \tag{5.19}$$

For scalar fields, these results were first obtained for Dirichlet boundary conditions by [71] and for Robin conditions by [75]. This list is duplicated from the more recent ref. [85] where standard heat kernel methods are employed. Finally, from free theories one finds

$$b_2 = 8c\ , \tag{5.20}$$

independent of boundary condition [85, 90]. (The result for b_2 for scalar fields with Dirichlet boundary conditions was computed first to our knowledge in [72].) It is one of the main motivations of this work to understand how general the relation (5.20) is.

The complete classification of conformal anomaly with boundary terms in five and six dimensions, to our knowledge, has not been given; see [90] for recent progress. Certainly, it is expected that the numbers of boundary Weyl invariants increase as one considers higher dimensional bCFTs.

5.2 Boundary Conformal Field Theory and Two-Point Functions

We would like to first review the general construction of conformal field theory two-point functions involving a scalar operator O, a conserved current J^μ, and a stress tensor $T^{\mu\nu}$ in the presence of a planar boundary. Much of our construction can be found in the literature, for example in refs. [87, 88, 95]. However, some details are to our knowledge new. We provide the conformal blocks for the current-current two-point functions (5.84, 5.102– 5.105, 5.107). We also remark on order of limits, positivity (5.65, 5.67) and some symmetry (5.109) properties more generally.

5.2.1 General Structure of Two-Point Functions

A conformal transformation g is a combination of a diffeomorphism $x^\mu \to x_g^\mu(x)$ and a local scale transformation $\delta_{\mu\nu} \to \Omega_g(x)^{-2}\delta_{\mu\nu}$ that preserves the usual flat metric $\delta_{\mu\nu}$ on $\mathbb{R}^d$. The group is isomorphic to $O(d+1,1)$ and is generated by rotations and translations, for which $\Omega = 1$, and spatial inversion $x^\mu \to x^\mu/x^2$, for which $\Omega = x^2$. In analogy to the rule for transforming the metric, given a tensor operator $O^{\mu_1\cdots\mu_s}$ of weight Δ, we can define an action of the conformal group

$$O^{\mu_1\cdots\mu_s}(x) \to \Omega_g^{\Delta+s}\left(\prod_{j=1}^{s}\frac{\partial {x_g}^{\mu_j}}{\partial x^{\nu_j}}\right) O^{\nu_1\cdots\nu_s}(x) \ . \tag{5.21}$$

In this language, J^μ and $T^{\mu\nu}$ have their usual engineering weights of $\Delta = d-1$ and d respectively. Notationally, it is useful to define the combination $(R_g)^\mu{}_\nu \equiv \Omega_g \frac{\partial {x_g}^\mu}{\partial x^\nu}$. Given the action of R_g on the metric, it is clearly an element of $O(d)$. In a coordinate system $x = (y, \mathbf{x})$, a planar boundary at $y = 0$ is kept invariant by only a $O(d,1)$ subgroup of the full conformal group, in particular, the subgroup generated by rotations and translations in the plane $y = 0$ along with inversion $x^\mu \to x^\mu/x^2$.

While in the absence of a boundary, one-point functions of quasi-primary operators vanish and two-point functions have a form fixed by conformal symmetry, the story is more complicated with a boundary. A quasi-primary scalar field O_Δ of dimension Δ can have an expectation value:

$$\langle O_\Delta(x)\rangle = \frac{a_\Delta}{(2y)^\Delta} \ . \tag{5.22}$$

The coefficients a_Δ play a role in the bulk conformal block decomposition of the two-point function, as we will see later. One-point functions for operators with spin are however forbidden by conformal invariance.

To some extent, the planar boundary functions like a mirror. In the context of two-point function calculations, in addition to the location $x = (y, \mathbf{x})$ and $x' = (y', \mathbf{x}')$ of the two operators, there are also mirror images at $(-y, \mathbf{x})$ and $(-y', \mathbf{x}')$. With four different locations in play, one can construct cross ratios that are invariant under the action of the conformal subgroup. Most of our results will be expressed in terms of the quantities

$$\xi = \frac{(x-x')^2}{4yy'} \ , \tag{5.23}$$

$$v^2 = \frac{(x-x')^2}{(x-x')^2 + 4yy'} = \frac{\xi}{\xi+1} \ . \tag{5.24}$$

Like four-point correlators in CFT without a boundary, the two-point correlators we consider can be characterized by a handful of functions of the cross ratios ξ or equivalently v. In the physical region, one has $0 \leq \xi \leq \infty$ and $0 \leq v \leq 1$. It will be useful to introduce also the differences

$$s \equiv x - x' \ , \quad \mathbf{s} \equiv \mathbf{x} - \mathbf{x}' \ . \tag{5.25}$$

Following ref. [88], we construct the two-point correlation functions out of weight zero tensors with nice bilocal transformation properties under $O(d,1)$. In addition to the metric $\delta_{\mu\nu}$, there are three:[2]

$$\begin{aligned} I_{\mu\nu}(x) &= \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \ , &(5.26)\\ X_\mu &= y\frac{v}{\xi}\partial_\mu \xi = v\left(\frac{2y}{s^2}s_\mu - n_\mu\right) \ , &(5.27)\\ X'_\mu &= y'\frac{v}{\xi}\partial'_\mu \xi = v\left(-\frac{2y'}{s^2}s_\mu - n_\mu\right) \ . &(5.28)\end{aligned}$$

The transformation rules are $X \to R_g(x)\cdot X$, $X' \to R_g(x')\cdot X'$, and the bilocal $I^{\mu\nu}(s) \to R_g(x)^\mu{}_\lambda R_g(x')^\nu{}_\sigma I^{\lambda\sigma}(s)$. One has $X'_\mu = I_{\mu\nu}(s)X^\nu$. In enforcing the tracelessness of the stress tensor, it will be useful to note that

$$X_\mu X^\mu = X'_\mu X'^\mu = 1 \ . \tag{5.29}$$

Two-Point Functions

We now tabulate the various two-point functions

$$\begin{aligned} \langle O_1(x)O_2(x')\rangle &= \frac{\xi^{-(\Delta_1+\Delta_2)/2}}{(2y)^{\Delta_1}(2y')^{\Delta_2}}G_{O_1O_2}(v) \ , &(5.30)\\ \langle J_\mu(x)O(x')\rangle &= \frac{\xi^{1-d}}{(2y)^{d-1}(2y')^{\Delta}}X_\mu f_{JO}(v) \ , &(5.31)\\ \langle T_{\mu\nu}(x)O(x')\rangle &= \frac{\xi^{-d}}{(2y)^{d}(2y')^{\Delta}}\alpha_{\mu\nu} f_{TO}(v) \ , &(5.32)\end{aligned}$$

$$\begin{aligned} \langle J_\mu(x)J_\nu(x')\rangle &= \frac{\xi^{1-d}}{(2y)^{d-1}(2y')^{d-1}}\Big(I_{\mu\nu}(s)P(v) + X_\mu X'_\nu Q(v)\Big) \ , &(5.33)\\ \langle T_{\mu\nu}(x)V_\lambda(x')\rangle &= \frac{\xi^{-d}}{(2y)^{d}(2y')^{\Delta}}\Big[\left(I_{\mu\lambda}(s)X_\nu + I_{\nu\lambda}(s)X_\mu - \frac{2}{d}g_{\mu\nu}X'_\lambda\right) f_{TV}(v) \\ &\qquad + \alpha_{\mu\nu}X'_\lambda\, g_{TV}(v)\Big] \ , &(5.34)\\ \langle T_{\mu\nu}(x)T_{\lambda\sigma}(x')\rangle &= \frac{\xi^{-d}}{(2y)^{d}(2y')^{d}}\Big[\alpha_{\mu\nu}\alpha'_{\sigma\rho}A(v) + \beta_{\mu\nu,\sigma\rho}B(v) + I_{\mu\nu,\sigma\rho}(s)C(v)\Big] \ , &(5.35)\end{aligned}$$

[2] In this section we follow the notation in [87, 88] where the normal vector is inward-pointing. In following sections we will adopt instead an outward-pointing normal vector.

where Δ_1/Δ_2 is the scaling dimension of O_1/O_2 and

$$\alpha_{\mu\nu} = \left(X_\mu X_\nu - \frac{1}{d}\delta_{\mu\nu}\right) , \quad \alpha'_{\mu\nu} = \left(X'_\mu X'_\nu - \frac{1}{d}\delta_{\mu\nu}\right) , \tag{5.36}$$

$$\begin{aligned}\beta_{\mu\nu,\sigma\rho} &= \Big(X_\mu X'_\sigma I_{\nu\rho}(s) + X_\nu X'_\sigma I_{\mu\rho}(s) + X_\mu X'_\rho I_{\nu\sigma}(s) + X_\nu X'_\rho I_{\mu\sigma}(s) \\ &\quad -\frac{4}{d}\delta_{\sigma\rho}X_\mu X_\nu - \frac{4}{d}\delta_{\mu\nu}X'_\sigma X'_\rho + \frac{4}{d^2}\delta_{\mu\nu}\delta_{\sigma\rho}\Big) ,\end{aligned} \tag{5.37}$$

$$I_{\mu\nu,\sigma\rho}(s) = \frac{1}{2}\Big(I_{\mu\sigma}(s)I_{\nu\rho}(s) + I_{\mu\rho}(s)I_{\nu\sigma}(s)\Big) - \frac{1}{d}\delta_{\mu\nu}\delta_{\sigma\rho} . \tag{5.38}$$

In writing the tensor structures on the right hand side, we have enforced tracelessness $T^\mu_\mu = 0$. However, we have not yet made use of the conservation conditions $\partial_\mu J^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$.[3] The conservation conditions fix (5.31) and (5.32) up to constants c_{JO} and c_{TO}:

$$f_{JO} = c_{JO}v^{d-1} , \quad f_{TO} = c_{TO}v^d . \tag{5.39}$$

The mixed correlator $\langle T^{\mu\nu}(x)V^\lambda(x')\rangle$ is fixed up to two constants, $c^\pm_{TV}$:

$$f_{TV} = c^+_{TV}v^{d+1} + c^-_{TV}v^{d-1} , \tag{5.40}$$

$$g_{TV} = -(d+2)c^+_{TV}v^{d+1} + (d-2)c^-_{TV}v^{d-1} . \tag{5.41}$$

If we further insist that the vector $V^\mu = J^\mu$ is a conserved current, such that $\Delta = d-1$, then the correlator is fixed up to one undetermined number, $c^\pm_{TV} = c_{TJ}$.

The $\langle J^\mu(x)J^\nu(x')\rangle$ and $\langle T^{\mu\nu}(x)T^{\lambda\sigma}(x')\rangle$ correlation functions on the other hand are fixed up to a single function by conservation. The differential equations are

$$v\partial_v(P+Q) = (d-1)Q , \tag{5.42}$$

$$(v\partial_v - d)(C+2B) = -\frac{2}{d}(A+4B) - dC , \tag{5.43}$$

$$(v\partial_v - d)((d-1)A + 2(d-2)B) = 2A - 2(d^2-4)B . \tag{5.44}$$

This indeterminancy stands in contrast to two (and three) point functions without a boundary, where conformal invariance uniquely fixes their form up to constants.

In the coincidental or bulk limit $v \to 0$, the operators are much closer together than they are to the boundary, and we expect to recover the usual conformal field theory results in the absence of a boundary. We thus apply the boundary conditions $A(0) = B(0) = Q(0) = 0$. The asymptotic values $C(0)$ and $P(0)$ are then fixed by the corresponding stress tensor and current two-point functions in the absence of a boundary; we adopt the standard notation, $C(0) = C_T$ and $P(0) = C_J$. The observables C_T and C_J play important roles when analyzing CFTs. In particular, for a free $d = 4$ conformal field theory of N_s scalars, N_f Dirac fermions, and N_v vectors, one has [89]

$$C_T = \frac{1}{4\pi^4}\left(\frac{4}{3}N_s + 8N_f + 16N_v\right) . \tag{5.45}$$

[3]While these conservation conditions may be altered by boundary terms involving displacement operators, away from the boundary they are strictly satisfied.

By unitarity (or reflection positivity), $C_T > 0$.[4] A trivial theory has $C_T = 0$. Similarly, we require that $G_{O_1 O_2}(0) = \kappa \delta_{\Delta_1 \Delta_2}$ for some constant $\kappa > 0$.

The decomposition of the two-point functions into $A(v)$, $B(v)$, $C(v)$, $P(v)$, and $Q(v)$ was governed largely by a sense of naturalness with respect to the choice of tensors X^μ and $I^{\mu\nu}$ rather than by some guiding physical principal. Indeed, an alternate decomposition was already suggested in the earlier paper ref. [87]. While uglier from the point of view of the tensors X_μ and $I_{\mu\nu}$, it is nevertheless in many senses a much nicer basis. This alternate decomposition, discussed below, is more natural from the point of view of reflection positivity. It also diagonalizes the contribution of the displacement operators in the boundary conformal block decomposition.

This basis adopts the following linear combinations:

$$\alpha(v) = \frac{d-1}{d^2}\left[(d-1)(A+4B)+dC\right] , \tag{5.46}$$

$$\gamma(v) = -B - \frac{1}{2}C , \tag{5.47}$$

$$\epsilon(v) = \frac{1}{2}C . \tag{5.48}$$

Ref. [87] motivated these combinations by restricting x and x' to lie on a line perpendicular to the boundary, taking $\mathbf{x} = \mathbf{x}' = 0$:

$$\lim_{\mathbf{x}=\mathbf{x}'\to 0} \langle T_{\mu\nu}(x) T_{\sigma\rho}(x')\rangle = \frac{A_{\mu\nu\sigma\rho}}{s^{2d}} . \tag{5.49}$$

In this case, one finds

$$A_{nnnn} = \alpha(v) , \tag{5.50}$$

$$A_{ABnn} = A_{nnAB} = -\frac{1}{d-1}\alpha(v)\delta_{AB} , \tag{5.51}$$

$$A_{AnBn} = \gamma(v)\delta_{AB} , \tag{5.52}$$

$$A_{ABCD} = \epsilon(v)(\delta_{AC}\delta_{BD} + \delta_{AD}\delta_{BC}) - \frac{1}{d-1}\left(2\epsilon(v) - \frac{\alpha(v)}{d-1}\right)\delta_{AB}\delta_{CD} . \tag{5.53}$$

Recall that the coincidental limit corresponds to $v = 0$ and the boundary limit to $v = 1$, where in this perpendicular geometry $v = \frac{|y-y'|}{y+y'}$. Relating these new linear combination to $C(0)$, for a non-trival unitary conformal field theory, we have

$$\alpha(0) = \frac{d-1}{d}C_T > 0 , \quad \gamma(0) = -\epsilon(0) = -\frac{1}{2}C_T < 0 . \tag{5.54}$$

One can play exactly the same game with the current:

$$\lim_{\mathbf{x}=\mathbf{x}'\to 0} \langle J_\mu(x) J_\nu(x')\rangle = \frac{A_{\mu\nu}}{s^{2(d-1)}} , \tag{5.55}$$

[4] See [118] for the discussion of C_T in non-unitary CFTs with four-and six-derivative kinetic terms.

where

$$A_{nn} = \pi(v) = P(v) + Q(v) \ , \tag{5.56}$$

$$A_{AB} = \rho(v)\delta_{AB} = P(v)\delta_{AB} \ . \tag{5.57}$$

Comments on Order of Limits

There are subtleties when considering various limits of the objects v, X_μ and X'_μ. We define the coincidental (or bulk) limit to be $s \to 0$ with y, $y' \neq 0$. In this limit, $v \to 0$ and

$$\lim_{s\to 0} X_\mu = 0 = \lim_{s\to 0} X'_\mu \ . \tag{5.58}$$

We define the boundary limit to be $y \to 0$ and $y' \to 0$ with $s \neq 0$. In this limit, we find instead that $v \to 1$ and

$$\lim_{y,y'\to 0} X_\mu = -n_\mu = \lim_{y,y'\to 0} X'_\nu \ . \tag{5.59}$$

We see that if one imposes the coincidental limit after the boundary limit has been imposed, the result is different from (5.58).

In the special case where both x and x' lie on a perpendicular to the boundary, depending on the sign of $y - y'$, one instead finds

$$\lim_{\mathbf{s}=0} X_\mu = -\lim_{\mathbf{s}=0} X'_\mu = n_\mu \quad (y > y' \neq 0) \ , \tag{5.60}$$

$$\lim_{\mathbf{s}=0} X_\mu = -\lim_{\mathbf{s}=0} X'_\mu = -n_\mu \quad (y < y' \neq 0) \ . \tag{5.61}$$

The following quantity is then independent of the relative magnitudes of y and y',

$$\lim_{\mathbf{s}=0} X_\mu X'_\nu = -n_\mu n_\nu \ . \tag{5.62}$$

A confusing aspect about this third case is that having taken this collinear limit, if we then further take a boundary $y \to 0$ or a coincident $y \to y'$ limit, the answer does not agree with either (5.58) or (5.59). In the near boundary limit, one finds that $A_{nAnB} = -\gamma\delta_{AB}$ while restricting the insertions to a line perpendicular to the boundary, one finds instead $A_{nAnB} = \gamma\delta_{AB}$. In general, when comparing physical quantities, one will have to fix an order of limits to avoid the sign ambiguity. In this case, however, due to our previous arguments, we expect that $\gamma(1) = 0$ generically under conformal boundary conditions.

5.2.2 Reflection Positivity and Bounds

Unitarity in Lorentzian quantum field theory is equivalent to the reflection positivity in quantum field theory with Euclidean signature. To apply reflection positivity, let us consider the case where the coordinates

$$x = (y, z, \mathbf{0}) \ , \ \ x' = (y, -z, \mathbf{0}) \ , \ \ s_\mu = (0, 2z, \mathbf{0}) \ , \tag{5.63}$$

lie in a plane located at a non-zero y, parallel to the boundary. Denoting this plane as $\mathcal{P}$, we introduce a reflection operator $\Theta_{\mathcal{P}}$ such that the reflection with respect to $\mathcal{P}$ gives $\Theta_{\mathcal{P}}(x) = x'$. The square of $\Theta_{\mathcal{P}}$ is the identity operator. Acting on a tensor field, $\Theta_{\mathcal{P}}(F_{\mu_1\cdots\mu_n}(x))$, $\Theta_{\mathcal{P}}$ will flip the overall sign if there are an odd number of 2 (z-direction) indices. The statement of reflection positivity for a tensor operator is that

$$\langle F_{\mu_1\cdots\mu_n}(x)\Theta_{\mathcal{P}}(F_{\nu_1\cdots\nu_n}(x))\rangle \ , \tag{5.64}$$

treated as a $d^n \times d^n$ matrix, has non-negative eigenvalues. (Note this reflection operator acts on just one of the points; when it acts on the difference it gives $\Theta_{\mathcal{P}}(s) = 0$.) In our particular choice of frame (5.63), $\Theta_{\mathcal{P}}(I_{\mu\nu}(s)) = \delta_{\mu\nu}$ and $\Theta_{\mathcal{P}}(X'_\mu) = X_\mu$. Making these substitutions in the current and stress tensor correlators (5.33) and (5.35), we can deduce eigenvectors and corresponding eigenvalues.

For the current two-point function, X_μ is an eigenvector with eigenvalue proportional to π while $\delta_{\mu 3}$ is an eigenvector with eigenvalue proportional to ρ, with positive coefficients of proportionality. (Instead of 3, we could have chosen any index not corresponding to the y and z directions.) Thus we conclude that

$$\pi(v) \geq 0 \ , \quad \rho(v) \geq 0 \ , \tag{5.65}$$

for all values of v, $0 \leq v \leq 1$. For the stress tensor, $\alpha_{\mu\nu}$, $X_{(\mu}\delta_{\nu)3}$, and $\delta_{3(\mu}\delta_{\nu)4}$ are eigenvectors with eigenvalues proportional to α, $-\gamma$, and ϵ, demonstrating the positivity that[5]

$$\alpha(v) \geq 0 \ , \quad -\gamma(v) \geq 0 \ , \quad \epsilon(v) \geq 0 \ . \tag{5.67}$$

With these positivity constraints in hand, one can deduce a couple of monotonicity properties from the conservation relations, re-expressed in terms of π, ρ, α, γ, and ϵ:

$$\begin{aligned}(v\partial_v - (d-1))\pi &= -(d-1)\rho \leq 0 \ , &(5.68)\\ (v\partial_v - d)\alpha &= 2(d-1)\gamma \leq 0 \ , &(5.69)\\ (v\partial_v - d)\gamma &= \frac{d}{(d-1)^2}\alpha + \frac{(d-2)(d+1)}{d-1}\epsilon \geq 0 \ . &(5.70)\end{aligned}$$

The last two inequalities further imply $(v\partial_v - d)^2\alpha \geq 0$. While these inequalities provide some interesting bounds for all values of v, they unfortunately do not lead to a strong constraint on the relative magnitudes of the two end points of α, $\alpha(1)$ and $\alpha(0)$, a constraint, as we will see, that could be interesting in relating the boundary charge b_2 in (5.132) to the usual central charge c in $d = 4$ CFTs.

[5] For instance, the eigen-equation for $\alpha_{\mu\nu}$ is

$$\langle T_{\mu\nu}(x)\Theta_P(T_{\lambda\sigma}(x))\rangle\alpha^{\lambda\sigma} = \frac{d}{d-1}\frac{\alpha(v)}{s^{2d}}\alpha_{\mu\nu} \ . \tag{5.66}$$

Using current conservation and our new basis of cross-ratio functions, we can write the stress tensor and current two-point functions in yet a third way, eliminating ρ, γ, and ϵ in favor of derivatives of π and α. This third way will be useful when we demonstrate the relationship between $\alpha(1)$ and the boundary central charge b_2. We write

$$\langle J_\mu(x) J_\nu(x')\rangle = \frac{1}{s^{2d-2}}\left(\pi(v) I_{\mu\nu}(s) - \frac{v\partial_v \pi}{d-1}\hat{I}_{\mu\nu}(s)\right) , \tag{5.71}$$

$$\begin{aligned}\langle T_{\mu\nu}(x) T_{\rho\sigma}(x')\rangle &= \frac{1}{s^{2d}}\Big[\alpha \frac{d}{d-1} I_{\mu\nu,\rho\sigma}(s) + v^2\partial_v^2\alpha \frac{\hat{I}_{\mu\nu,\rho\sigma}}{(d-2)(d+1)} \\ &\quad - v\partial_v\alpha\left(\frac{\hat{\beta}_{\mu\nu,\rho\sigma}}{2(d-1)} + \frac{(2d-1)\hat{I}_{\mu\nu,\rho\sigma}}{(d-2)(d+1)}\right)\Big] ,\end{aligned} \tag{5.72}$$

where we have defined some new tensorial objects in terms of the old ones:

$$\hat{I}_{\mu\nu}(s) \equiv I_{\mu\nu}(s) - X_\mu X'_\nu , \tag{5.73}$$

$$\hat{I}_{\mu\nu,\rho\sigma}(s) \equiv I_{\mu\nu,\rho\sigma}(s) - \frac{d}{d-1}\alpha_{\mu\nu}\alpha'_{\rho\sigma} - \frac{1}{2}\hat{\beta}_{\mu\nu,\rho\sigma} , \tag{5.74}$$

$$\hat{\beta}_{\mu\nu,\rho\sigma} \equiv \beta_{\mu\nu,\rho\sigma} - 4\alpha_{\mu\nu}\alpha'_{\rho\sigma} . \tag{5.75}$$

One nice feature of the hatted tensors is their orthogonality to the X_μ and X'_ρ tensors. In particular

$$X_\mu \hat{I}^{\mu\rho} = 0 = \hat{I}^{\mu\rho} X'_\rho , \tag{5.76}$$

$$X_\mu \hat{I}^{\mu\nu,\rho\sigma} = 0 = \hat{I}^{\mu\nu,\rho\sigma} X'_\rho , \tag{5.77}$$

$$X_\mu X_\nu \hat{\beta}^{\mu\nu,\rho\sigma} = 0 = \hat{\beta}^{\mu\nu,\rho\sigma} X'_\rho X'_\sigma . \tag{5.78}$$

In the near boundary limit, $v \to 1$, since X_μ, $X'_\mu \to -n_\mu$, only the tangential components $\hat{I}_{AB}$ and $\hat{I}_{AB,CD}$ of $\hat{I}_{\mu\nu}$ and $\hat{I}_{\mu\nu,\rho\sigma}$ are nonzero. In fact, in this limit, these tensors may be thought of as the $d-1$ dimensional versions of the original tensors $I_{\mu\nu}$ and $I_{\mu\nu,\rho\sigma}$. For $\hat{\beta}_{\mu\nu,\rho\sigma}$, only the mixed components $\hat{\beta}_{(nA),(nB)}$ survive in a near boundary limit.

5.2.3 Conformal Block Decomposition

Like four point functions in CFT without a boundary, the two-point functions $\langle O_1(x) O_2(x')\rangle$, $\langle J^\mu(x) J^\nu(x')\rangle$, and $\langle T^{\mu\nu}(x) T^{\lambda\sigma}(x')\rangle$ admit conformal block decompositions. We distinguish two such decompositions: the bulk decomposition in which the two operators get close to each other and the boundary decomposition in which the two operators get close to the boundary (or equivalently their images). Our next task is to study the structure of these decompositions. For simplicity, in what follows, we will restrict to the case that the dimensions of O_1 and O_2 are equal and take $\Delta_1 = \Delta_2 = \eta$.

Bulk Decomposition

Recall that in the presence of a boundary the one-point functions for operators with spin violate conformal symmetry. As a result, the bulk conformal block decomposition will involve only a sum over scalar operators with coefficients proportional to the a_Δ.

Allowing for an arbitrary normalization κ of the two-point function, the bulk OPE for two identical scalar operators can be written as

$$O_\eta(x)O_\eta(x') = \frac{\kappa}{s^{2\eta}} + \sum_{\Delta\neq 0} \lambda_\Delta B(x-x', \partial_{x'}) O_\Delta(x') \quad , \quad \lambda_\Delta \in \mathbb{R} \ , \tag{5.79}$$

where the sum is over primary fields. The bulk differential operator $B(x-x', \partial_{x'})$ is fixed by bulk conformal invariance and produces the sum over descendants. As the OPE (5.79) reflects the local nature of the CFT, this OPE is unchanged when a boundary is present. The bulk channel conformal block decomposition is given by taking the expectation value of (5.79) using (5.22) and then matching the result with (5.30). We write

$$G_{O_\eta O_\eta}(v) \;\; = \;\; \kappa + \sum_{\Delta\neq 0} a_\Delta \lambda_\Delta G_{\text{bulk}}(\Delta, v) \ , \tag{5.80}$$

where we have pulled out the leading bulk identity block contribution.[6] There are analogous expressions for the functions $P(v)$, $Q(v)$, $A(v)$, $B(v)$, and $C(v)$ out of which we constructed $\langle J^\mu(x) J^\nu(x')\rangle$ and $\langle T^{\mu\nu}(x) T^{\lambda\sigma}(x')\rangle$. We can write for example

$$Q(v) \;\; = \;\; \sum_{\Delta\neq 0} a_\Delta \lambda_\Delta Q_{\text{bulk}}(\Delta, v) \ , \tag{5.81}$$

$$A(v) \;\; = \;\; \sum_{\Delta\neq 0} a_\Delta \lambda_\Delta A_{\text{bulk}}(\Delta, v) \ , \tag{5.82}$$

where $G_{\text{bulk}}(\Delta, v)$, $Q_{\text{bulk}}(\Delta, v)$, and $A_{\text{bulk}}(\Delta, v)$ have a very similar form:

$$G_{\text{bulk}}(\Delta, v) \;\; = \;\; \xi^{\frac{\Delta}{2}} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}, 1-\frac{d}{2}+\Delta; -\xi\right) \ , \tag{5.83}$$

$$Q_{\text{bulk}}(\Delta, v) \;\; = \;\; \xi^{\frac{\Delta}{2}} {}_2F_1\left(1+\frac{\Delta}{2}, 1+\frac{\Delta}{2}, 1-\frac{d}{2}+\Delta; -\xi\right)(1+\xi) \ , \tag{5.84}$$

$$A_{\text{bulk}}(\Delta, v) \;\; = \;\; \xi^{\frac{\Delta}{2}} {}_2F_1\left(2+\frac{\Delta}{2}, 2+\frac{\Delta}{2}, 1-\frac{d}{2}+\Delta; -\xi\right)(1+\xi)^2 \ . \tag{5.85}$$

Indeed, one is tempted to define a general form for which each of these functions is a special case:

$$G^{(s)}_{\text{bulk}}(\Delta, v) \;\; = \;\; \xi^{\frac{\Delta}{2}} {}_2F_1\left(s+\frac{\Delta}{2}, s+\frac{\Delta}{2}, 1-\frac{d}{2}+\Delta, -\xi\right)(1+\xi)^s \ . \tag{5.86}$$

[6] For two-point functions of scalar operators of different dimension, $\Delta_1 \neq \Delta_2$, G_{bulk} will depend on Δ_1 and Δ_2. We refer to the literature [88, 95] for the more general expression, but suppress it here as we are interested in the simpler case.

The remaining functions $B(v)$, $C(v)$, and $P(v)$ can be straightforwardly constructed from the conservation equations (5.42)-(5.44), and can be represented as sums of hypergeometric functions. Note that the bulk identity block does not contribute to $Q(v)$, $A(v)$, and $B(v)$, but it does to $C(v)$ and $P(v)$. We review the derivation of these conformal block decompositions using the null cone formalism in Appendix 5.7.1.

Boundary Decomposition

In the presence of a boundary, a bulk scalar operator O_η of dimension η can be expressed as a sum over boundary operators denoted as $\mathring{O}_\Delta(\mathbf{x})$. We write

$$O_\eta(x) = \frac{a_O}{(2y)^\eta} + \sum_{\Delta \neq 0} \tilde{\mu}_\Delta \mathring{B}(y, \mathring{\partial}) \mathring{O}_\Delta(\mathbf{x}) \quad , \quad \tilde{\mu}_\Delta \in \mathbb{R}, \tag{5.87}$$

where the sum is over boundary primary fields. Boundary conformal invariance fixes the operator $\mathring{B}(y, \mathring{\partial})$. The two-point function of two identical boundary operators is normalized to be

$$\langle \mathring{O}_\Delta(\mathbf{x}) \mathring{O}_\Delta(\mathbf{x}) \rangle = \frac{\kappa_{d-1}}{\mathbf{s}^{2\Delta}} \ , \tag{5.88}$$

where κ_{d-1} is a constant. The one-point function of the boundary operator vanishes. Reflection positivity guarantees the positivity of these boundary two-point functions for unitary theories. The boundary channel conformal block decomposition is given by squaring (5.87), taking the expectation value using (5.88), and then matching the result with (5.30). We write

$$G_{OO}(v) = \xi^\eta \left[a_O^2 + \sum_{\Delta \neq 0} \mu_\Delta^2 G_{\text{bry}}(\Delta, v) \right] \ , \tag{5.89}$$

where [119]

$$G_{\text{bry}}(\Delta, v) \quad = \quad \xi^{-\Delta} {}_2F_1 \left(\Delta, 1 - \frac{d}{2} + \Delta, 2 - d + 2\Delta, -\frac{1}{\xi} \right) \ . \tag{5.90}$$

To remove the η dependence from the conformal block, it is useful to include an explicit factor of ξ^η in the decomposition (5.89). We have made a redefinition $\mu_\Delta^2 = \tilde{\mu}_\Delta^2 \kappa_{d-1}$ to allow for more generally normalized two-point functions. Reflection positivity applied to the boundary two-point functions along with the fact that $\tilde{\mu}_\Delta \in \mathbb{R}$ guarantees the coefficients μ_Δ^2 in the boundary expansion are non-negative. There was no such constraint on the bulk conformal block decomposition.

For a field of spin s, there is an extra subtlety that the sum, by angular momentum conservation, can involve boundary fields of spin s' up to and including s. For a conserved current, we need to consider boundary fields of spin $s' = 0$ and 1, while for the stress tensor we will need $s' = 0$, 1, and 2 boundary fields. Fortunately, because of the restricted form of the $\langle J_\mu(x)O(x')\rangle$, $\langle T_{\mu\nu}(x)O(x')\rangle$, and $\langle T_{\mu\nu}(x)V_\lambda(x')\rangle$ correlation functions, the sum over fields with spin strictly less than s is restricted, and the situation simplifies somewhat. Consider first $\langle J_\mu(x)O(x')\rangle$ in the boundary limit, which vanishes for $\Delta < d-1$ and blows up for $\Delta > d-1$. We interpret the divergence to mean that the corresponding coefficient c_{JO} must vanish when $\Delta > d-1$. It follows that in the boundary conformal block expansion of $\langle J_\mu(x)J_\nu(x')\rangle$, the only scalar field that contributes will have $\Delta = d-1$. An analogous argument in the stress tensor case implies that only scalar fields and vectors of dimension $\Delta = d$ can contribute in the boundary conformal block expansion.

These restrictions on the boundary conformal block expansion are reflected in the possible near boundary behaviors of the functions π, ρ, α, γ, and ϵ allowed by the current conservation equations (5.68)–(5.70). From the definitions of π (5.56) and ρ (5.57), ρ corresponds to vector exchange on the boundary and π to scalar exchange. If we exchange a boundary vector $\hat{V}^A$ of dimension $d-2+\delta_V$, where δ_V is an anomalous dimension, the near boundary behavior for ρ is $\rho \sim (1-v)^{-1+\delta_V}$, which can be deduced from the boundary conformal block expressions given in this section and the current conservation equations. The unitary bound implies $\delta_V > 0$, and there is a descendant scalar operator $\partial_A \hat{V}^A$ of dimension $d-1+\delta_V$. (A boundary vector operator at the unitarity bound $d-2$ would be conserved, $\partial_A \hat{J}^A = 0$.) Correspondingly, eq. (5.68) enforces the near boundary behavior $\pi \sim (1-v)^{\delta_V}$. The exception to this rule is when $\delta_V = 1$. Then the conservation equations allow ρ and π to have independent order one contributions near the boundary, corresponding to the possibility of having both vector and scalar primaries of dimension $d-1$.

The story is similar for the stress tensor with ϵ representing spin two exchange, γ spin one exchange, and α scalar exchange. The generic case is boundary exchange of a spin two operator $\hat{S}_{AB}$ of dimension $d-1+\delta_S$ with $\delta_S > 0$. The near boundary behaviors of the stress tensor correlation function are then $\epsilon \sim (1-v)^{-1+\delta_S}$, $\gamma \sim (1-v)^{\delta_S}$, and $\alpha \sim (1-v)^{1+\delta_S}$ where the scaling of γ and α is consonant with the existence of descendants of the form $\partial_A \hat{S}^{AB}$ and $\partial_A \partial_B \hat{S}^{AB}$. Again, there is one exception to this story, when $\delta_S = 1$. In this case, the conservation equations allow α, γ, and ϵ to have independent order one contributions near the boundary, corresponding to scalar, vector, and spin two exchange of dimension d.

The scalar of $\Delta = d$ plays a special role in bCFT. It is often called the displacement operator. The presence of a boundary affects the conservation of the stress tensor, $\partial_\mu T^{\mu n}(x) = D^n(x)\delta(y)$, where D^n is a scalar operator of $\Delta = d$. The scalar displacement operator D^n is generally present in boundary and defect CFTs.

For interesting reasons, discussed in what follows, a vector of dimension $\Delta = d$ and scalar of dimension $\Delta = d-1$ are generically absent from the conformal block decompositions of these two-point functions. In the case of the current two-point function, a natural candidate for a scalar of dimension $\Delta = d-1$ is the boundary limit of J^n. If there are no degrees of freedom on the boundary, then J^n must vanish as a boundary condition or the corresponding charge is not conserved. If there are charged degrees of freedom on the boundary characterized by a boundary current $\hat{J}^A$, then current conservation implies $J^n|_{\text{bry}} = -\partial_A \hat{J}^A$ and the total charge is conserved by an inflow effect. From the point of view of the conformal field theory living on the boundary, the current $\hat{J}^A$ is no longer conserved, and $J^n|_{\text{bry}}$ becomes a descendant of $\hat{J}^A$. Because conservation on the boundary is lost, the scaling dimension of $\hat{J}^A$ must shift upward from $d-2$ by a positive amount δ_J. Correspondingly, the scaling dimension of J^n shifts upwards by δ_J from $d-1$, and it will appear in the conformal block decomposition not as a primary but as a descendant of $\hat{J}^A$. We thus expect generically that a scalar primary of $\Delta = d-1$ is absent from the boundary conformal block expansion of the current-current two-point function.

The story for a vector of dimension $\Delta = d$ is similar. A natural candidate for such an operator is the boundary limit of T^{nA}. In the free models we consider, the boundary conditions force this quantity to vanish. The interacting models we introduce in section 5.5 have extra degrees of freedom that propagate on the boundary and an associated boundary stress tensor $\hat{T}^{AB}$. By conservation of the full stress tensor, the boundary limit of T^{nA} is equal to the descendant operator $\partial_A \hat{T}^{AB}$, neither of which will necessarily vanish classically. The scaling dimension of $\hat{T}^{AB}$ must shift upward from $d-1$ by a positive amount δ_T. The boundary operator corresponding to $T^{nA}|_{\text{bry}}$ now enters the boundary conformal block decomposition not as a vector primary but as a descendant of the spin two field $\hat{T}^{AB}$. We expect generically that a vector of $\Delta = d$ is absent from the boundary conformal block expansion of $\langle T_{\mu\nu}(x) T_{\lambda\sigma}(x')\rangle$.

We will nevertheless keep these vectors and scalars in our boundary conformal block decomposition. The reason is that for these interacting models, we only perform leading order perturbative calculations. At this leading order, we cannot see the shift in dimension of T^{nA} and J^n, and it is useful to continue to treat them as primary fields.

The boundary block expansions for $\langle J^\mu(x) J^\nu(x')\rangle$ and $\langle T^{\mu\nu}(x) T^{\lambda\sigma}(x')\rangle$ have the forms

$$\pi(v) = \xi^{d-1} \left(\mu_{(0)}^2 \pi_{\text{bry}}^{(0)}(v) + \sum_{\Delta \geq d-2} \mu_\Delta^2 \pi_{\text{bry}}^{(1)}(\Delta, v) \right) , \tag{5.91}$$

$$\alpha(v) = \xi^{d} \left(\mu_{(0)}^2 \alpha_{\text{bry}}^{(0)}(v) + \mu_{(1)}^2 \alpha_{\text{bry}}^{(1)}(v) + \sum_{\Delta \geq d-1} \mu_\Delta^2 \alpha_{\text{bry}}^{(2)}(\Delta, v) \right) , \tag{5.92}$$

where the indices (0), (1) and (2) denote the spins. One has similar expressions for the other functions $\rho(v)$, $\gamma(v)$, and $\epsilon(v)$.

In this basis, we find the following blocks[7]

$$\alpha^{(0)}_{\rm bry}(v) = \frac{1}{4(d-1)}(v^{-1}-v)^d(d(v^{-1}+v)^2-4) \ , \tag{5.93}$$

$$\gamma^{(0)}_{\rm bry}(v) = -\frac{d}{4(d-1)^2}(v^{-1}-v)^d(v^{-2}-v^2) \ , \tag{5.94}$$

$$\epsilon^{(0)}_{\rm bry}(v) = \frac{d}{4(d-1)^2(d+1)}(v^{-1}-v)^d(v^{-2}-v^2)^2 \ . \tag{5.95}$$

In the boundary limit $\xi \to \infty$, the combinations $\xi^d \gamma^{(0)}_{\rm bry}$ and $\xi^d \epsilon^{(0)}_{\rm bry}$ vanish while $\xi^d \alpha^{(0)}_{\rm bry} \to 1$. In this basis, the contribution of the displacement operator D^n to the boundary block expansion is encoded purely by $\alpha^{(0)}_{\rm bry}$.

Similarly, for the spin one exchange, we find

$$\alpha^{(1)}_{\rm bry}(v) = \frac{d-1}{d}(v^{-1}-v)^d(v^{-2}-v^2) \ , \tag{5.96}$$

$$\gamma^{(1)}_{\rm bry}(v) = -\frac{1}{2}(v^{-1}-v)^d(v^{-2}+v^2) \ , \tag{5.97}$$

$$\epsilon^{(1)}_{\rm bry}(v) = \frac{1}{2(d+1)}(v^{-1}-v)^d(v^{-2}-v^2) \ , \tag{5.98}$$

where now $\xi^d \gamma^{(1)}_{\rm bry} \to -1$ in the boundary limit while the other two vanish. For spin two exchange with weight $\Delta = d$, we have

$$\alpha^{(2)}_{\rm bry}(d,v) = (v^{-1}-v)^d(v^{-1}-v)^2 \ , \tag{5.99}$$

$$\gamma^{(2)}_{\rm bry}(d,v) = -\frac{1}{d-1}(v^{-1}-v)^d(v^{-2}-v^2) \ , \tag{5.100}$$

$$\epsilon^{(2)}_{\rm bry}(d,v) = \frac{1}{(d^2-1)(d-2)}(v^{-1}-v)^d(d(v^{-1}+v)^2-2(v^{-2}+v^2)) \ , \tag{5.101}$$

where now $\xi^d \epsilon^{(2)}_{\rm bry} \to 4/(d+1)(d-2)$ and the other two vanish. We have shifted the normalization convention here relative to (5.93) and (5.97) so that we may write the higher dimensional blocks (5.106) for $\alpha^{(2)}_{\rm bry}(\Delta, v)$ in a simpler and uniform way.

Playing similar games with the current, we find

$$\pi^{(0)}_{\rm bry}(v) = \frac{1}{2}(v^{-1}-v)^{d-1}(v^{-1}+v) \ , \tag{5.102}$$

$$\rho^{(0)}_{\rm bry}(v) = \frac{1}{2(d-1)}(v^{-1}-v)^d \ , \tag{5.103}$$

and

$$\pi^{(1)}_{\rm bry}(d-1,v) = (v^{-1}-v)^d \ , \tag{5.104}$$

$$\rho^{(1)}_{\rm bry}(d-1,v) = \frac{1}{d-1}(v^{-1}-v)^{d-1}(v^{-1}+v) \ . \tag{5.105}$$

[7] The results in the basis of $A(v), B(v), C(v)$ are given in [95].

For higher dimension operators, we have

$$\alpha^{(2)}_{\rm bry}(\Delta, v) = \xi^{-\Delta-2}\, {}_2F_1\left(2+\Delta, 1-\frac{d}{2}+\Delta, 2-d+2\Delta; -\frac{1}{\xi}\right) , \tag{5.106}$$

$$\pi^{(1)}_{\rm bry}(\Delta, v) = \xi^{-\Delta-1}\, {}_2F_1\left(1+\Delta, 1-\frac{d}{2}+\Delta, 2-d+2\Delta; -\frac{1}{\xi}\right) . \tag{5.107}$$

The remaining functions $\gamma^{(2)}_{\rm bry}(\Delta, v)$, $\epsilon^{(2)}_{\rm bry}(\Delta, v)$ and $\rho^{(1)}_{\rm bry}(\Delta, v)$ have a more cumbersome form but can be straightforwardly derived from the conservation equations (5.42)-(5.44). Evidently, $G_{\rm bry}(\Delta, v)$, $\pi^{(1)}_{\rm bry}(\Delta, v)$, and $\alpha^{(2)}_{\rm bry}(\Delta, v)$ all are special cases of the general form

$$\xi^{-\Delta-s}\, {}_2F_1\left(s+\Delta, 1-\frac{d}{2}+\Delta, 2-d+2\Delta; -\frac{1}{\xi}\right) . \tag{5.108}$$

We have written all of these blocks to make a symmetry under $v \to v^{-1}$ apparent. The transformation $v \to v^{-1}$ or equivalently $\xi \to -1-\xi$ corresponds to a reflection $y' \to -y'$ keeping y fixed. Under such a partial reflection, the blocks are eigenvectors with eigenvalue ± 1 for integer Δ:

$$f^{(s)}_{\rm bry}(\Delta, \frac{1}{v}) = (-1)^{\Delta+s+\sigma} f^{(s)}_{\rm bry}(\Delta, v) . \tag{5.109}$$

The shift σ is one for $\rho^{(s)}_{\rm bry}$ and $\gamma^{(s)}_{\rm bry}$ and zero otherwise. For the higher dimensional exchanged operators, this reflection property relies on a hypergeometric identity

$${}_2F_1(a,b,c;z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right) , \tag{5.110}$$

in the special case where $c = 2b$.

5.2.4 Crossing Relations

A crossing relation for boundary conformal field theory is the statement that two-point functions can be expressed either as a sum over boundary conformal blocks or as a sum over bulk conformal blocks. (See figure 5.1. The left/right plot represents the bulk/boundary channel.)

The field theories we consider are either free or have some weak interactions that are constrained to live on the boundary. The solutions to crossing for the current and stress tensor correlation functions are remarkably universal for the family of theories we consider. Roughly speaking, they all involve a decomposition of a function of an invariant cross ratio of the form

$$G(v) = 1 + \chi v^\eta . \tag{5.111}$$

The parameter χ will depend on the boundary conditions. Roughly, one can think of this expression in terms of the method of images, where the 1 reproduces the answer in the

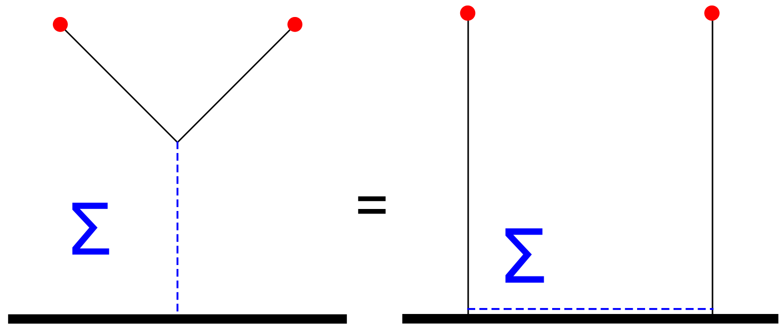

Figure 5.1: Crossing symmetry for two-point functions in bCFTs.

coincident/bulk limit, in the absence of a boundary, and the v^η represents the correlation between points and their images on the other side of the boundary. In the bulk channel, 1 is the identity block and v^η will generically involve a sum over a tower of fields. In the boundary channel, we first decompose $G(v) = \frac{1}{2}(1+\chi)(1+v^\eta) + \frac{1}{2}(1-\chi)(1-v^\eta)$ into eigenfunctions of the reflection operator $v \to 1/v$ and then find infinite sums of boundary blocks that reproduce $1 \pm v^\eta$. The two-point function may not be precisely of the form $1+\chi v^\eta$, but the discrepancy can always be accounted for by adjusting the coefficients of a few blocks of low, e.g. $\Delta = d-1$ or d, conformal dimension.

A number of the blocks have a very simple form. In the bulk, we find

$$G_{\text{bulk}}(d-2, v) = v^{d-2} \ , \quad Q_{\text{bulk}}(d, v) = v^d \ , \quad A_{\text{bulk}}(d+2, v) = v^{d+2} \ . \tag{5.112}$$

In the boundary, we already saw that the blocks of dimension $d-1$ for $\langle J^\mu(x) J^\nu(x')\rangle$ and of dimension d for $\langle T^{\mu\nu}(x) T^{\lambda\sigma}(x')\rangle$ have a polynomial form. However, we neglected to point out that for the scalar two-point functions, the boundary blocks of dimension $\frac{d-2}{2} + n$ where n is a non-negative integer also have a simple polynomial form. The polynomial like expressions satisfy the recursion relation

$$\begin{aligned} G_{\text{bry}}\Big(\frac{d-2}{2}+n, v\Big) &= \frac{4(2n-1)}{(2n-d)}\Big[(1+2\xi) G_{\text{bry}}\left(\frac{d-2}{2}+n-1, v\right) \\ &\quad + \frac{4\xi(\xi+1)}{(d-4+2n)} \partial_\xi G_{\text{bry}}\left(\frac{d-2}{2}+n-1, v\right)\Big] \ . \end{aligned} \tag{5.113}$$

The first two values are

$$\xi^{\frac{d-2}{2}} G_{\text{bry}}\left(\frac{d-2}{2}, v\right) = \frac{1}{2}(1+v^{d-2}) \ , \tag{5.114}$$

$$\xi^{\frac{d-2}{2}} G_{\text{bry}}\left(\frac{d}{2}, v\right) = \frac{2}{d-2}(1-v^{d-2}) \ . \tag{5.115}$$

These two particular cases are degenerate in fact: they satisfy the same differential equation (see Appendix 5.7.1). We have imposed boundary conditions that are consistent with the recursion relation (5.113) and the reflection symmetry (5.109).

These simple expressions for the conformal blocks motivate the following remarkably simple relation:

$$\xi^{\frac{d-2}{2}}\left[\frac{1+\chi}{2}G_{\rm bry}\left(\frac{d-2}{2},v\right)+\frac{1-\chi}{2}\frac{d-2}{2}G_{\rm bry}\left(\frac{d}{2},v\right)\right]=1+\chi G_{\rm bulk}(d-2,v)\ . \quad (5.116)$$

(For $\chi=\pm 1$, this relation is pointed out in [95].) In the next section, we will compute the two-point function for a free scalar field of dimension $\Delta=\frac{d-2}{2}$. We find a free scalar takes advantage of precisely such a crossing relation (5.116). Moreover, the case $\chi=1$ corresponds to Neumann boundary conditions, in which case the contribution from a boundary operator $\partial_n\phi$ of dimension $\Delta=\frac{d}{2}$ is absent. Correspondingly, the case $\chi=-1$ is Dirichlet boundary conditions, and the boundary operator ϕ itself is absent. An absent or trivial boundary is the case $\chi=0$. The contribution from the bulk comes simply from the identity operator and the composite operator ϕ^2. By adding an interaction on the boundary, we will be able to move perturbatively away from the limiting cases $\chi=\pm 1$. However, positivity of the boundary decomposition (5.89) implies the bounds:

$$-1\leq\chi\leq 1\ . \quad (5.117)$$

Given these bounds, one might interpret that $\chi=\pm 1$ correspond to "corners" in the bootstrap program.

More generally, for a function of the form $G_{OO}(v)=a_O^2\xi^\Delta+1\pm v^{2\Delta}$, the boundary and bulk decompositions will involve a sum over infinite numbers of operators. Here ξ^Δ corresponds to the boundary identity block and the 1 to the bulk identity block. With a little bit of guess work, one can deduce a general form of these series expansions. (For a more rigorous derivation, one can use the α-space formalism [120, 121].) One has the boundary decompositions

$$\frac{\xi^{-\Delta}}{2}\left(1+v^{2\Delta}\right)=\sum_{n\in 2\mathbb{Z}^*}\mu_n^2 G_{\rm bry}(\Delta+n,v)\ , \quad (5.118)$$

$$\frac{\xi^{-\Delta}}{2}\left(1-v^{2\Delta}\right)=\sum_{n\in 2\mathbb{Z}^*+1}\mu_n^2 G_{\rm bry}(\Delta+n,v)\ , \quad (5.119)$$

where $\mathbb{Z}^*$ denotes a non-negative integer and the coefficients are

$$\mu_n^2 \;=\; \frac{2^{d-2\Delta-2n}\sqrt{\pi}\,\Gamma(n+2\Delta-d+1)\Gamma(n+\Delta)}{\Gamma(\Delta)\Gamma\left(n+\Delta-\frac{d-1}{2}\right)\Gamma(n+1)\Gamma\left(\Delta+1-\frac{d}{2}\right)}\ , \quad (5.120)$$

where $\mu_0^2=1$. In contrast, for the bulk decomposition, the boundary identity block decomposes into bulk conformal blocks

$$\xi^\Delta \;=\; \sum_{n=0}^{\infty}\frac{[(\Delta)_n]^2}{n!\left(2\Delta-\frac{d}{2}+n\right)_n}G_{\rm bulk}(2\Delta+2n,v)\ . \quad (5.121)$$

One also has the bulk decomposition

$$v^{2\Delta} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\Delta)_n \left(\Delta - \frac{d}{2} + 1\right)_n}{\left(2\Delta + n - \frac{d}{2}\right)_n} G_{\rm bulk}(2\Delta + 2n, v) \ . \tag{5.122}$$

There are similar decompositions for the $\langle J^\mu(x) J^\nu(x')\rangle$ and $\langle T^{\mu\nu}(x) T^{\lambda\sigma}(x')\rangle$ correlation functions. For the current, we need to give the decomposition of

$$Q(v) = 2\chi^2 v^{2d-2} \ , \quad \pi(v) = 1 - \chi^2 v^{2d-2} \ , \tag{5.123}$$

and for the stress tensor, we need to give the decomposition of

$$A(v) = \frac{4d}{d-1} \chi^2 v^{2d} \ , \quad \alpha(v) = 1 + \chi^2 v^{2d} \ . \tag{5.124}$$

(For free theories, $\chi^2 = 1$.) Using the relations

$$\frac{1}{2}(1 + v^{2d-2}) \ = \ \xi^{d-1} \left(\pi^{(0)}_{\rm bry}(v) + \sum_{n \in 2\mathbb{Z}^* + 1} \mu_n^2 \pi^{(1)}_{\rm bry}(d - 1 + n, v) \right) \ , \tag{5.125}$$

$$\frac{1}{2}(1 - v^{2d-2}) \ = \ \xi^{d-1} \sum_{n \in 2\mathbb{Z}^*} \mu_n^2 \pi^{(1)}_{\rm bry}(d - 1 + n, v) \ , \tag{5.126}$$

where

$$\mu_n^2 = \frac{2^{1-d-2n} \sqrt{\pi}\, \Gamma(d + n - 2) \Gamma(d + n)}{\Gamma(d-2) \Gamma\left(\frac{d}{2}\right) \Gamma(n+2) \Gamma\left(\frac{d-1}{2} + n\right)} \ , \tag{5.127}$$

and $\mu_0^2 = (d-1)/2$, we can find a decomposition similar in spirit to the lhs of (5.116). Similarly, for the stress tensor

$$\frac{1}{2}(1 + v^{2d}) \ = \ \xi^d \left(\alpha^{(0)}_{\rm bry}(v) + \sum_{n \in 2\mathbb{Z}^*} \mu_n^2 \alpha^{(2)}_{\rm bry}(d + n, v) \right) \ , \tag{5.128}$$

$$\frac{1}{2}(1 - v^{2d}) \ = \ \xi^d \left(\frac{d^2}{4(d-1)} \alpha^{(1)}_{\rm bry}(v) + \sum_{n \in 2\mathbb{Z}^* + 1} \mu_n^2 \alpha^{(2)}_{\rm bry}(d + n, v) \right) \ , \tag{5.129}$$

where

$$\mu_n^2 = \frac{2^{-d-2n} \sqrt{\pi}\, \Gamma(d + n - 1) \Gamma(d + n + 2)}{\Gamma(d) \Gamma\left(\frac{d}{2} - 1\right) \Gamma(n+3) \Gamma\left(\frac{d+1}{2} + n\right)} \ , \tag{5.130}$$

where $\mu_0^2 = (d-2)d(d+1)/8(d-1)$. Finally, there are also corresponding bulk decompositions for which there is no obvious positivity constraint. We can write decompositions for the scalar, conserved current, and stress tensor two-point functions in a unified form:

$$v^{2\Delta} \ = \ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\Delta + s)_n \left(\Delta + 1 - \frac{d}{2} - s\right)_n}{\left(2\Delta + n - \frac{d}{2}\right)_n} G^{(s)}_{\rm bulk}(2\Delta + 2n, v) \ . \tag{5.131}$$

Similar decompositions of $1 \pm v^{2\Delta}$ were discussed in the appendices of ref. [95]. As a result, many of the formulae here are not entirely new. We have made an attempt to present them in a way that stresses their symmetry properties under $v \to 1/v$ and also stresses the important role played by the decomposition of $1 \pm v^{2\Delta}$ in free theories – for scalar, vector, and tensor operators.

5.3 A Boundary Central Charge

Consider $d = 4$ CFTs in curved space with a smooth codimension one boundary $\partial\mathcal{M}$. The conformal anomaly is given by

$$\langle T^\mu_\mu \rangle = \frac{1}{16\pi^2}\Big(cW^2_{\mu\nu\lambda\rho} - aE_4\Big) \\ + \frac{\delta(y)}{16\pi^2}\Big(aE_4^{(\text{bry})} - b_1 \operatorname{tr}\hat{K}^3 - b_2 h^{AB}\hat{K}^{CD}W_{ACBD}\Big) \ . \tag{5.132}$$

We construct a projector onto the boundary metric $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ with n_μ being a unit, outward normal vector to $\partial\mathcal{M}$; E_4 is the $d = 4$ Euler density, $W_{\mu\nu\lambda\rho}$ is the Weyl tensor and $\hat{K}_{AB} = K_{AB} - \frac{K}{3}h_{AB}$ is the traceless part of the extrinsic curvature.

The energy-momentum (stress) tensor in the Euclidean signature is defined by

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g}}\frac{\delta W}{g^{\mu\nu}(x)} \ , \tag{5.133}$$

where W is the generating functional for connected Green's functions. The two-point function in flat space is

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(x')\rangle \ = \lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\left((-2)^2\frac{\delta^2}{\delta g^{\sigma\rho}(x')\delta g^{\mu\nu}(x)}W\right) \ . \tag{5.134}$$

We will denote $\widetilde{W}$ as the anomalous part of W. Note in general there can be Weyl invariant contributions to correlation functions. The theory is assumed to be regulated in a diffeomorphism-invariant way.

We will adopt the dimensional regulation and will be interested in the mass scale, μ, dependence in the correlation functions. The a-anomaly is topological so it does not produce any μ dependence. The b_1-charge does not contribute to the two-point function in the flat limit, since $K^3 \sim \mathcal{O}(g_{\mu\nu})^3$. One will be able to extract b_1 from a study of three-point functions in the presence of a boundary– we discuss them in the next chapter. We here only consider the c and b_2 anomalies. The relevant pieces of the anomaly effective action are

$$\widetilde{W}^{(c)} = \frac{c}{16\pi^2}\frac{\mu^\epsilon}{\epsilon}\int_{\mathcal{M}} W^2_{\mu\nu\lambda\rho} \ , \quad \widetilde{W}^{(b_2)} = \frac{b_2}{16\pi^2}\frac{\mu^\epsilon}{\epsilon}\int_{\partial\mathcal{M}} K^{AB}W_{nAnB} \ . \tag{5.135}$$

These pieces should allow us to compute anomalous contributions to stress tensor correlation functions in the coincident limit.[8]

We will perform the metric variation twice on the anomaly action to obtain anomalous contributions to the two-point function of the stress tensor. We work in Gaussian normal coordinates. While we do not impose that $\delta g_{\mu\nu} = 0$ on the boundary, we do keep $\delta g_{nA} = 0$. In the flat boundary limit (see Appendix 5.7.2),

$$\lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \delta K_{AB} = \frac{1}{2}\partial_n \delta g_{AB} \ . \tag{5.136}$$

Note the δg_{nn} contribution vanishes in the flat limit in the transformed extrinsic curvature. The transformed Weyl tensor can be written as

$$\lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \delta W_{\mu\sigma\rho\nu} = -2P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta \delta g^{\alpha\beta} \ , \tag{5.137}$$

where $P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}$, defined in (5.303), is a projector that shares the same symmetries as the Weyl tensor:

$$P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} = P_{\alpha\gamma\delta\beta,\mu\sigma\rho\nu} \ , \tag{5.138}$$

$$P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}P_{\mu\sigma\rho\nu,\eta\chi\epsilon\omega} = P_{\alpha\gamma\delta\beta,\eta\chi\epsilon\omega} \ . \tag{5.139}$$

It will be convenient to define the following fourth order differential operator using the projector:

$$P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\partial^\sigma\partial^\rho\partial^\gamma\partial^\delta = \frac{(d-3)}{4(d-2)}\Delta^T_{\mu\nu\alpha\beta} \ . \tag{5.140}$$

Some additional properties of this tensor along with its definition can be found in Appendix 5.7.2.

It is useful first to recall the story [89] without a boundary. The argument that gives a relation between c and $\alpha(0)$ will also work with a boundary, provided we arrange for the variation $\delta g_{\mu\nu}$ to vanish as we approach the boundary, eliminating any boundary terms that may arise through integration by parts. We then have, in the bulk limit, that

$$\lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \delta^2\Big(\lim_{v\to 0}\widetilde{W}^{(c)}\Big) = \frac{c}{4\pi^2}\frac{\mu^\epsilon}{\epsilon}\int_{\mathcal{M}} P_{\alpha\gamma\delta\beta,\eta\chi\phi\psi}(\delta g^{\eta\psi})(\partial^\phi\partial^\chi\partial^\gamma\partial^\delta \delta g^{\alpha\beta}) \ . \tag{5.141}$$

From the definition of the stress tensor as a variation with respect to the metric, one infers the scale dependent contribution:

$$\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x')T_{\alpha\beta}(x'')\rangle^{(c)} = \frac{c}{4\pi^2}\Delta^T_{\mu\nu\alpha\beta}\delta^4(x'-x'') \ . \tag{5.142}$$

[8] We remark that the $\Box R$ anomaly in $d = 4$ does not affect the scale dependent contribution to the two-point function, since the corresponding effective action, R^2, is finite.

The general form of the two-point function without a boundary (or with a boundary but in the bulk limit) is given by

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(x')\rangle = C_T\frac{I_{\mu\nu,\sigma\rho}}{s^8} = \frac{C_T}{320}\Delta^T_{\mu\nu\sigma\rho}\frac{1}{s^4} , \tag{5.143}$$

where we have used (5.140) in $d = 4$. We next regularize the UV divergence in the two-point function in $d = 4$ by taking [122]

$$\mathcal{R}\frac{1}{x^4} = -\partial^2\left(\frac{\ln\mu^2x^2}{4x^2}\right) , \tag{5.144}$$

from which we obtain

$$\mu\frac{\partial}{\partial\mu}\Big(\mathcal{R}\frac{1}{x^4}\Big) = 2\pi^2\delta^4(x) , \tag{5.145}$$

and hence

$$\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\rangle = C_T\frac{\pi^2}{160}\Delta^T_{\mu\nu\sigma\rho}\delta^4(x) . \tag{5.146}$$

Matching (5.146) with (5.142), one identifies

$$c = \frac{\pi^4}{40}C_T , \tag{5.147}$$

where $C_T = C(0) = \frac{4}{3}\alpha(0)$.

Now let us consider the variation of the boundary term in the trace anomaly. Given the variation rules, the b_2-anomaly action gives

$$\lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\delta^2\widetilde{W}^{(b_2)} = \frac{b_2}{16\pi^2}\frac{\mu^\epsilon}{\epsilon}\int_{\partial\mathcal{M}}(\partial_n\delta g^{AB})(P_{AnBn,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\delta g^{\alpha\beta}) . \tag{5.148}$$

Thus,

$$\mu\frac{\partial}{\partial\mu}\langle T_{AB}(x')T_{\alpha\beta}(x'')\rangle^{(b_2)} = \frac{b_2}{2\pi^2}\partial_y\delta(y-y')P_{AnBn,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\delta^4(x'-x'')|_{y\to0} . \tag{5.149}$$

However, it is peculiar that such a boundary term should be present at all. By simple power counting, we do not expect a pure boundary, log divergent contribution to the stress tensor two-point function. The corresponding momentum space correlator has odd mass dimension, $4+4-3=5$, which naively should not involve a logarithmic divergence. More convincing, perhaps, is the flip in sign of this term under reflection $y\to-y$. As we saw in the boundary conformal block decomposition of the stress tensor, under reflection the $ABCD$ and $ABnn$ components of the two-point function restricted to the boundary should be even. Although these two arguments fall short of a rigorous proof, it seems natural for such a pure boundary log divergence to cancel against something else.

Consider whether this boundary term (5.149) may cancel against boundary terms we dropped in calculating (5.141). There is an immediate subtlety associated with the noncommutativity of the boundary and coincident limits. The boundary term (5.149) exists in a strict boundary limit, while the calculation (5.141), which reproduces the anomalous part of the $\frac{1}{s^8} I_{\mu\nu,\rho\sigma}$ tensor structure, was performed in the coincident limit. As we see from the two-point function (5.72), the coefficient $\alpha(v)$ of the $I_{\mu\nu,\rho\sigma}$ structure varies as v changes from the coincident limit 0 to the boundary limit 1.

We posit the existence of an effective action which computes correlation functions of the stress tensor. Almost everywhere, the scale dependent part of this action is $\widetilde{W}^{(c)}$. However, if we introduce a small distance ϵ to separate the stress tensor insertions, in a very thin layer of thickness less than ϵ along the boundary, we should replace the constant c in $\widetilde{W}^{(c)}$ with a generally different constant c_{bry}. The idea is that c_{bry} will give us both the freedom to reproduce the scale dependence of the $\alpha(1) I_{\mu\nu,\rho\sigma}$ contribution to the two-point function (5.72) and to cancel the offensive boundary term (5.149). In contrast, the terms in the expression (5.72) proportional to $\partial_v \alpha$ and $\partial_v^2 \alpha$ give vanishing contribution to the $nnnn$ and $nnAB$ components of the two-point function. The term proportional to $\hat{\beta}_{\mu\nu\rho\sigma}$ in (5.72) near the boundary only has $nAnB$ contributions. Because of this index incompatibility, it seems unlikely that terms in an effective action that would produce this index structure would also lead to a cancellation of the boundary term (5.149). Unfortunately, we cannot offer a rigorous proof.

Keeping the surface terms, by varying the metric such that $\delta g_{\mu\nu}$ is nonzero close to the boundary, the near-boundary limit of the c-anomaly action gives

$$\begin{aligned}
\lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \delta^2\Big(\lim_{v\to 1} \widetilde{W}^{(c_{\text{bry}})}\Big) &= \frac{c_{\text{bry}}}{4\pi^2}\frac{\mu^\epsilon}{\epsilon}\int_{\mathcal{M}} P_{\alpha\gamma\delta\beta,\eta\chi\phi\psi}(\delta g^{\eta\psi})(\partial^\phi\partial^\chi\partial^\gamma\partial^\delta \delta g^{\alpha\beta}) \\
&\quad + \frac{c_{\text{bry}}}{4\pi^2}\frac{\mu^\epsilon}{\epsilon}\int_{\partial\mathcal{M}} P_{\alpha\gamma\delta\beta,\eta n\phi\psi}(\partial^\phi \delta g^{\eta\psi})(\partial^\gamma\partial^\delta \delta g^{\alpha\beta}) \\
&\quad - \frac{c_{\text{bry}}}{4\pi^2}\frac{\mu^\epsilon}{\epsilon}\int_{\partial\mathcal{M}} P_{\alpha\gamma\delta\beta,\eta\chi n\psi}(\delta g^{\eta\psi})(\partial^\chi\partial^\gamma\partial^\delta \delta g^{\alpha\beta}) \ , \qquad (5.150)
\end{aligned}$$

where we have performed integration by parts near the boundary. Consequently, we find for the scale dependence of the two-point function in the near boundary limit that[9]

$$\begin{aligned}
\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x')T_{\alpha\beta}(x'')\rangle^{(c)} &= \frac{c_{\text{bry}}}{4\pi^2}\Delta^T_{\mu\nu\alpha\beta}\delta^4(x'-x'') \\
&\quad - \frac{2c_{\text{bry}}}{\pi^2}\partial_y\delta(y-y')P_{\mu n\nu n,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\delta^4(x'-x'')|_{y\to 0} \\
&\quad - \frac{2c_{\text{bry}}}{\pi^2}\delta(y-y')P_{\mu nA\nu,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\partial^A\delta^4(x'-x'')|_{y\to 0} \\
&\quad - \frac{2c_{\text{bry}}}{\pi^2}\delta(y-y')P_{\mu\phi n\nu,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\partial^\phi\delta^4(x'-x'')|_{y\to 0} \ . \qquad (5.151)
\end{aligned}$$

[9]The two-point functions presented in this section generalize the results given in [8], which has assumed a certain boundary condition on boundary geometry that removes normal derivatives acting on the metric variations [90].

Next observe, through a direct computation, that

$$\lim_{y\to 0} P_{\mu n A\nu,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\partial^A\frac{1}{x^4} = \lim_{y\to 0} P_{\mu\phi n\nu,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\partial^\phi\frac{1}{x^4} = 0 \ . \tag{5.152}$$

This implies, after adopting the regularized expression (5.145), the last two lines of (5.151) do not contribute.[10] The second line of (5.151) suggests to evaluate

$$\lim_{y\to 0} P_{\mu n\nu n,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\frac{1}{x^4} \ , \tag{5.153}$$

which turns out to be non-zero. However, this second line has precisely the right form to cancel the earlier boundary contribution we found from varying the b_2 anomaly (5.149). As explained above, we will eliminate this problematic boundary term by requiring a cancellation between b_2 and c-contributions:

$$b_2 = 4c_{\rm bry} \ . \tag{5.154}$$

On the other hand, to reproduce the near boundary structure of the stress tensor two-point function, $\alpha(1)I_{\mu\nu,\rho\sigma}$, we must have that $c_{\rm bry} = \pi^4\alpha(1)/30$. Thus, we conclude that

$$b_2 = \frac{2\pi^4}{15}\alpha(1) \ . \tag{5.155}$$

With the relation (5.155), we can achieve a better understanding of the previously conjectured equality (5.20) (i.e $b_2 = 8c$), and discuss how general it is. Observe first that the relation (5.20) is true only when $\alpha(1) = 2\alpha(0)$. (Recall in general one has $c = \frac{\pi^4}{30}\alpha(0)$.) We will find that $\alpha(1) = 2\alpha(0)$ indeed holds for a large class of free CFTs in the following sections. However, in the 4d mixed dimensional QED theory which we discuss in section 5.5, the boundary value $\alpha(1)$ depends on the coupling, while the bulk theory is the standard Maxwell theory with an unchanged value of c or $\alpha(0)$. In other words, the mixed dimensional QED can provide a counterexample to the relation (5.20).

5.4 Free Fields and Universality

In this section, we consider three families of free conformal field theories: a conformally coupled massless scalar in d dimensions, a massless fermion in d dimensions and an abelian p-form in $2p+2$ dimensions. We will see that the corresponding two-point functions take a remarkably universal form. They correspond to special cases of the crossing relations we found in section 5.2 with the parameter $\chi = \pm 1$. The parameter χ can be promoted to a matrix, with $\chi^2 = \mathbb{1}$, an identity. To construct CFTs with more general eigenvalues of χ^2 away from unity, we will include boundary interactions in the next section.

[10] If we also turn on δg_{nA} in the Gaussian normal coordinates when varying the b_2 action, restoring the last term of (5.311) in the flat limit, we find the additional contributions to the two-point function do not have a scale dependence.

5.4.1 Free Scalar

We start with the classical Minkowski action for a conformally coupled scalar in d-dimensions with a possibly curved codimension-one boundary term:

$$I = -\int_{\mathcal{M}} \frac{1}{2}\Big((\partial\phi)^2 + \frac{(d-2)}{4(d-1)}R\phi^2\Big) - \frac{(d-2)}{4(d-1)}\int_{\partial\mathcal{M}} K\phi^2 \ , \tag{5.156}$$

where R is the Ricci scalar and K is the trace of the extrinsic curvature. The surface term is required by Weyl invariance. Restricting to flat space with a planar boundary at $y = 0$, the usual improved stress tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{4}\frac{1}{d-1}\Big((d-2)\partial_\mu\partial_\nu + \delta_{\mu\nu}\partial^2\Big)\phi^2 - \frac{(d-2)}{4(d-1)}\delta(y)h_{\mu\nu}(\partial_n\phi^2) \ , \tag{5.157}$$

with n_μ an outward pointing unit normal vector to the boundary. While in the bulk, the stress tensor is traceless (on shell), the boundary term requires either Dirichlet $\phi = 0$ or Neumann $\partial_n\phi = 0$ boundary conditions to preserve the tracelessness.

Let us consider a more general case with a vector of scalar fields, i.e $\phi \to \phi^a$. (We will suppress the index a in what follows.) Then, we can introduce two complementary projectors $\Pi_\pm$ such that $\Pi_+ + \Pi_- = \mathbb{1}$ and $\Pi_\pm^2 = \Pi_\pm$. The generalized boundary conditions are then[11]

$$\partial_n(\Pi_+\phi)|_{y=0} = 0 \ , \quad \Pi_-\phi|_{y=0} = 0 \ . \tag{5.158}$$

For a single scalar, one can only have either $\Pi_+ = 1, \Pi_- = 0$ or $\Pi_+ = 0, \Pi_- = 1$. For the scalar, the nA component of the stress tensor is

$$T_{nA} = \frac{d}{2(d-1)}(\partial_n\phi)(\partial_A\phi) - \frac{(d-2)}{2(d-1)}\phi\,\partial_A\partial_n\phi \ . \tag{5.159}$$

The boundary conditions (5.158) force that T_{nA} vanishes at $y = 0$.

It is perhaps useful to discuss the case of a transparent boundary. We have fields ϕ_R and ϕ_L on each side of the boundary. Given the second order equation of motion, the boundary conditions are continuity of the field $\phi_R = \phi_L$ and its derivative $\partial_n\phi_R = \partial_n\phi_L$. We can use the folding trick to convert this interface CFT into a bCFT by replacing the ϕ_R fields with their mirror images $\tilde\phi_R$ on the left hand side. We still have continuity of the fields as a boundary condition $\tilde\phi_R = \phi_L$, but having reflected the normal direction, continuity of the derivative is replaced with $\partial_n\tilde\phi_R = -\partial_n\phi_L$. In terms of the projectors (5.158), we have

$$\Pi_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \ , \quad \phi = \begin{pmatrix} \tilde\phi_R \\ \phi_L \end{pmatrix} \ . \tag{5.160}$$

[11]These boundary conditions are sometimes called mixed in the literature; for instance, see section 5.3 in [123].

As the fields $\tilde{\phi}_R$ and ϕ_L do not interact, it is straightforward to go back to the unfolded theory. One slightly tricky point relates to composite operators like the stress tensor. In the original theory, there is no reason for a classical T_{nA} to vanish at the boundary. However, in the folded theory (or bCFT), by our previous argument, we saw the T_{nA} does vanish classically. In this case there are really two, separately conserved stress tensors, one associated with $\tilde{\phi}_R$ and one associated with ϕ_L. The statement that T_{nA} vanishes classically in the bCFT is really the statement that T_{nA} computed from the $\tilde{\phi}_R$ fields cancels T_{nA} computed from the ϕ_L fields at the boundary. More generally, a nonzero classical T_{nA} in a bCFT corresponds to a discontinuity in T_{nA} for the interface theory. From the pill box argument mentioned before, this situation corresponds to non-conservation of the boundary stress tensor $\partial_B \hat{T}^{AB}$. (As mentioned before, we expect quantum effects to restore the condition $T_{nA} = 0$ on the boundary for general bCFTs.)

We note in passing that the component T^{nn} of the scalar field will in general not vanish on the boundary. Indeed, as discussed before, it corresponds to the displacement operator which is generally present in bCFTs.

The two-point function for the elementary fields ϕ can be constructed using the method of images:

$$\langle \phi(y)\phi(y')\rangle = \frac{\kappa}{s^{d-2}}(\mathbb{1} + \chi v^{d-2}) \ , \tag{5.161}$$

where we denote

$$\kappa = \frac{1}{(d-2)\mathrm{Vol}(S_{d-1})} \ , \quad \mathrm{Vol}(S_{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \ . \tag{5.162}$$

Applying the boundary conditions (5.158), one finds that

$$\chi = \Pi_+ - \Pi_- \ . \tag{5.163}$$

From the properties of the projectors, $\chi^2 = \mathbb{1}$. The eigenvalues of χ must be ± 1, $+1$ for Neumann boundary conditions and -1 for Dirichlet. The relevant cross-ratio function (5.30) is then $G_{\phi\phi}(v) = \mathbb{1} + \chi v^{d-2}$. In section 5.2, we saw that this particular $G_{\phi\phi}(v)$ admitted the decomposition (5.116) into a pair of bulk and a pair of boundary blocks. In fact, because of the restriction on the eigenvalues of χ, we only require a single boundary block, of dimension $\frac{d-2}{2}$ for Neumann boundary conditions or dimension $\frac{d}{2}$ for Dirichlet. We will see in the next section how to move away from eigenvalues ± 1 perturbatively by adding a boundary interaction.

Next we consider $\langle \phi^2(x)\phi^2(x')\rangle$. There is a new element here because ϕ^2 has a nontrivial one-point function

$$\langle \phi^2(y)\rangle = \frac{\kappa\,\mathrm{tr}(\chi)}{(2y)^{d-2}} \ . \tag{5.164}$$

For N scalars, one finds the following cross-ratio function for the two-point correlator:

$$G_{\phi^2\phi^2}(v) = 2\kappa^2 \operatorname{tr}(\mathbb{1} + \chi v^{d-2})^2 + \kappa^2 \operatorname{tr}(\chi)^2 \xi^{d-2} \ . \tag{5.165}$$

This function $G_{\phi^2\phi^2}(v)$ is straightforward to decompose into boundary and bulk blocks, using the results of section 5.2. For the boundary decomposition, the last term on the rhs of (5.165), proportional to ξ^{d-2}, is the boundary identity block. We may decompose $1 + v^{2(d-2)}$ using the infinite sum (5.118). The piece proportional to $2\operatorname{tr}(\chi)v^{d-2}$ can be expressed using $v^{d-2} = \xi^{d-2} G_{\text{bry}}(d-2, v)$. One may worry that this term comes with a negative coefficient when $\operatorname{tr}(\chi) < 0$, violating reflection positivity. In fact, in the infinite sum (5.118), the block $G_{\text{bry}}(d-2, v)$ has coefficient one, which, in the case of Dirichlet boundary conditions, precisely cancels the $G_{\text{bry}}(d-2, v)$ reproduced from $-v^{d-2}$. Indeed, for Dirichlet boundary conditions, the boundary ϕ^2 operator is absent. There is no issue for Neumann boundary conditions since all the coefficients are manifestly positive. The bulk decomposition is similarly straightforward. The "one" in (5.165) is the bulk identity block. The term proportional to v^{d-2} can be expressed again as a single block, this time in the bulk, $G_{\text{bulk}}(d-2, v) = v^{d-2}$. The pieces proportional to ξ^{d-2} and $v^{2(d-2)}$ decompose into bulk blocks using (5.121) and (5.122).

For the stress tensor two-point function, using Wick's theorem one obtains

$$\alpha(v) = (d-2)^2\kappa^2 \left(\operatorname{tr}(\mathbb{1}) + \operatorname{tr}(\chi^2)v^{2d} + \operatorname{tr}(\chi)\frac{d(d-2)(d+1)}{4(d-1)} v^{d-2}(1-v^2)^2 \right) , \tag{5.166}$$

$$\begin{aligned} A(v) &= \frac{d(d-2)^2\kappa^2}{4(d-1)^2} \Big(\operatorname{tr}(\chi) v^d \left(-2d(d^2-4) + d(d-2)^2 v^{-2} + (d^2-4)(d+4)v^2 \right) \\ &\qquad + 16(d-1)\operatorname{tr}(\chi^2)v^{2d} \Big) \ . \end{aligned} \tag{5.167}$$

Setting $\chi = \pm 1$ we recover the results computed in [87, 88] for a single scalar under Dirichlet or Neumann boundary condition. In the boundary decomposition, looking at $\alpha(v)$, we recognize the $v^{d-2}(1-v^2)^2$ piece as a contribution from $\alpha^{(2)}_{\text{bry}}(d, v)$, with a sign depending on the boundary conditions. Then, decomposing $1 + v^{2d}$ using (5.128), we see that the coefficient of the $\alpha^{(2)}_{\text{bry}}(d, v)$ is precisely of the right magnitude to cancel out the possibly negative contribution from $v^{d-2}(1-v^2)^2$, consistent with the absence of a $(\partial_A\phi)(\partial_B\phi)$ type boundary operator for Dirichlet boundary conditions. Regarding the bulk decomposition, we can write $\alpha^{(2)}_{\text{bry}}(d, v)$ as a linear combination of $\alpha_{\text{bulk}}(d-2, v)$, $\alpha_{\text{bulk}}(d, v)$, and $\alpha_{\text{bulk}}(d+2, v)$, all of which are polynomials in $v^{d\pm2}$ and v^d, giving a trivial solution of the crossing equations.

Let us also consider a complexified scalar $\phi = \phi_1 + i\phi_2$, or equivalently a pair of real scalars to define a conserved current. We have

$$J_\mu = \frac{i}{2}\left[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right] = -\phi_1\partial_\mu\phi_2 + \phi_2\partial_\mu\phi_1 \ . \tag{5.168}$$

We introduce real projectors, $\Pi_\pm^\dagger = \Pi_\pm$, acting on the complexified combinations, $\partial_n(\Pi_+\phi) = 0$ and $\Pi_-\phi = 0$. With these boundary conditions, the current is conserved at the boundary,

$J_n = 0$. Changing the $\phi(x)$ to $\phi^*(x)$ in (5.161) and using Wick's Theorem, one finds

$$\begin{aligned} Q(v) &= \frac{(d-2)\kappa^2}{2}\left(\operatorname{tr}(\chi)v^{d-2}((d-2)-dv^2) - 2\operatorname{tr}(\chi^2)v^{2d-2}\right) \ , &(5.169)\\ \pi(v) &= \frac{(d-2)\kappa^2}{2}\left(\operatorname{tr}(\mathbb{1}) + (d-1)\operatorname{tr}(\chi)v^{d-2}(1-v^2) - \operatorname{tr}(\chi^2)v^{2d-2}\right) \ . &(5.170)\end{aligned}$$

Looking at $\pi(v)$, we recognize $(d-1)v^{d-2}(1-v^2)$ as a contribution from $\pi^{(1)}_{\text{bry}}(d-1,v)$. The $1-v^{2d-2}$ dependence of $\pi(v)$ decomposes into boundary blocks according to (5.126). Similar to the $\langle\phi^2(x)\phi^2(x')\rangle$ case we analyzed above, one might again be worried that the contribution from $\pi^{(1)}_{\text{bry}}(d-1,v)$ is negative, violating reflection positivity. However, for Dirichlet boundary conditions, the contributions from $1-v^{2d-2}$ and $(d-1)v^{d-2}(1-v^2)$ precisely cancel, consistent with the absence of a $\phi\partial_A\phi$ type boundary operator. It turns out that $\pi^{(1)}_{\text{bry}}(d-1,v)$ and $\pi_{\text{bulk}}(d-2,v)$ are proportional, giving a trivial solution of the crossing equations. Indeed, looking at $Q(v)$ we recognize $v^{d-2}((d-2)-dv^2)$ as a contribution from $Q_{\text{bulk}}(d-2,v)$. Similar to what we found for the $\langle\phi^2(x)\phi^2(x')\rangle$ correlation function, looking now at the $1-v^{2d-2}$ dependence of $\pi(v)$, we recognize the one as the bulk identity block and decompose the v^{2d-2} using (5.131).

5.4.2 Free Fermion

The Minkowski action for Dirac fermions in curved space is

$$I = \frac{i}{2}\int_{\mathcal{M}}\left(\bar\psi\gamma_\mu D^\mu\psi - (D^\mu\bar\psi)\gamma_\mu\psi\right) \ , \tag{5.171}$$

where, as usual, the covariant derivative contains the spin connection and the bar is defined by $\bar\psi = \psi^\dagger\gamma^0$. The scaling dimension of the fermion ψ is $\Delta = \frac{1}{2}(d-1)$. The action is conformally invariant without any boundary term needed. Using a Minkowski tensor with mostly plus signature the Clifford algebra is given by $\{\gamma_\mu,\gamma_\nu\} = -2\eta_{\mu\nu}$. In the flat space, the current and stress tensor in terms of the spinor field ψ are

$$\begin{aligned} J_\mu &= \bar\psi\gamma_\mu\psi \ , &(5.172)\\ T_{\mu\nu} &= \frac{i}{2}\left((\partial_{(\mu}\bar\psi)\gamma_{\nu)}\psi - \bar\psi\gamma_{(\mu}\partial_{\nu)}\psi\right) \ . &(5.173)\end{aligned}$$

We symmetrize the indices with strength one, such that

$$T_{nn} = \frac{i}{2}\left((\partial_n\bar\psi)\gamma_n\psi - \bar\psi\gamma_n\partial_n\psi\right) \ . \tag{5.174}$$

Following [124, 87], we define the following hermitian projectors Π_+ and Π_-:

$$\Pi_\pm = \frac{1}{2}(\mathbb{1}\pm\chi) \ , \tag{5.175}$$

with the parameter $\chi = \Pi_+ - \Pi_-$ for the fermion theory acting on the Clifford algebra such that

$$\chi\gamma_n = -\gamma_n\bar{\chi} \ , \quad \chi\gamma_A = \gamma_A\bar{\chi} \ , \quad \chi^2 = \bar{\chi}^2 = \mathbb{1} \ , \tag{5.176}$$

where $\bar{\chi} = \gamma^0\chi^\dagger\gamma^0$. Since the action only has first-order derivatives we only need boundary conditions imposed on half of the spinor components. We consider boundary conditions $\Pi_-\psi = 0$ and its conjugate $\bar{\psi}\Pi_- = 0$. In terms of χ, they become

$$(\mathbb{1} - \chi)\psi|_{\partial\mathcal{M}} = 0 \ , \quad \bar{\psi}(\mathbb{1} - \bar{\chi})|_{\partial\mathcal{M}} = 0 \ . \tag{5.177}$$

As a consequence, from the equation of motion one can deduce a related but not independent Neumann boundary condition $\partial_n(\Pi_+\psi) = 0$. A physical interpretation of these boundary conditions is that they make sure J_n and T_{nA} vanish on the boundary. The two-point function of the spinor field is then

$$\langle\psi(x)\bar{\psi}(x')\rangle = -\kappa_f\left(\frac{i\gamma\cdot(x-x')}{|x-x'|^d} + \chi\frac{i\gamma\cdot(\bar{x}-x')}{|\bar{x}-x'|^d}\right) \ , \tag{5.178}$$

where $\bar{x} = (-x_1, \mathbf{x}) \equiv (-y, \mathbf{x})$. The parameter χ enters naturally in the fermion theory with a boundary. We consider a typical choice of normalization of the two-point function $\kappa_f = (d-2)\kappa = 1/\operatorname{Vol}(S^{d-1})$.

A straightforward application of Wick's theorem then allows us to calculate the $\langle J_\mu(x)J_\nu(x')\rangle$ and $\langle T_{\mu\nu}(x)T_{\lambda\sigma}(x')\rangle$ correlators. In fact, as we have seen, it is enough to work out just the components with all normal indices. The remaining components can then be calculated using the conservation relations. One finds

$$\pi(v) = \kappa_f^2\operatorname{tr}_\gamma(\mathbb{1})\left(1 - \operatorname{tr}(\chi^2)v^{d-1}\right) \ , \tag{5.179}$$

$$\alpha(v) = \frac{1}{2}(d-1)\kappa_f^2\operatorname{tr}_\gamma(\mathbb{1})\left(1 + \operatorname{tr}(\chi^2)v^{2d}\right) \ , \tag{5.180}$$

where the value of $\operatorname{tr}_\gamma(\mathbb{1})$ depends on the particular Clifford algebra we choose. Essentially the same result for $\alpha(v)$ can be found in ref. [87]; for Dirac fermions, it is common in the literature to take $\operatorname{tr}_\gamma(\mathbb{1}) = 2^{\lfloor d/2\rfloor}$.

The same conformal block decompositions that we worked out for the scalar apply to the free fermions as well. Observe that, $(d-2)\operatorname{tr}_\gamma(\mathbb{1})$ scalars, half of which have Dirichlet and half of which have Neumann boundary conditions, produce the same $\langle J_\mu(x)J_\nu(x')\rangle$ two-point function as the spinor. Similarly, $\frac{d-1}{2}\operatorname{tr}_\gamma(\mathbb{1})$ scalars, again split evenly between Neumann and Dirichlet boundary conditions, produce the same stress tensor two-point function as our spinor field.

5.4.3 Free p-Form Gauge Fields

Now we consider an abelian p-form in d dimensions in the presence of a planar, codimension one boundary. The Minkowski action is

$$I = -\frac{1}{2(p+1)!}\int_{\mathcal{M}} \mathrm{d}^d x\, H_{\mu_1\cdots\mu_{p+1}} H^{\mu_1\cdots\mu_{p+1}} \ , \tag{5.181}$$

where $H_{\mu_1\cdots\mu_{p+1}} = D_{\mu_1} B_{\mu_2\cdots\mu_{p+1}} \pm$ cylic permutations; D_μ is the standard covariant derivative. The action in $d = 2(p+1)$ is conformally invariant without any boundary term neeeded. Important special cases are a Maxwell field in four dimensions and a 2-form in six dimensions. We will again work in a flat half-space with coordinate system $x_\mu = (y, \mathbf{x})$ with a boundary at $y = 0$. In ref. [125], the authors computed two- and three-point functions of the stress tensor in the absence of a boundary. Here we will generalize their two-point calculations to include a planar boundary. The stress tensor in flat space is given by

$$T_{\mu\nu} = \frac{1}{p!} H_{\mu\mu_1\cdots\mu_p} H_\nu{}^{\mu_1\cdots\mu_p} - \frac{1}{2(p+1)!}\delta_{\mu\nu} H_{\mu_1\cdots\mu_{p+1}} H^{\mu_1\cdots\mu_{p+1}} \ . \tag{5.182}$$

This stress tensor is traceless only when $d = 2p+2$.

We fix a generalization of Feynman gauge by adding $\frac{1}{2(p-1)!}(\partial_\mu B^{\mu\nu_1\cdots\nu_{p-1}})^2$ to the action.[12] The two-point function of the B-field is then

$$\langle B_{\mu_1\cdots\mu_p}(x) B^{\nu_1\cdots\nu_p}(x')\rangle = \kappa \delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p}\left(\frac{1}{(x-x')^{d-2}} + \chi \frac{1}{((\mathbf{x}-\mathbf{x}')^2 + (y+y')^2)^{(d-2)/2}}\right) . \tag{5.183}$$

The choice of χ is based on the presence or absence of a normal index.[13] There are two possible choices of boundary conditions, generalizing the "absolute" and "relative" boundary conditions of the Maxwell field $F_{\mu\nu}$ [123]. The Neumann-like or "absolute" choice corresponds to setting the normal component of the field strength to zero $H_{nA_1\cdots A_p} = 0$ and leads to the two conditions $\partial_n B_{A_1\cdots A_p} = 0$ and $B_{nA_2\cdots A_p} = 0$. The Dirichlet-like or "relative" choice means $B_{A_1\cdots A_p} = 0$ which, along with the gauge fixing condition $\partial_\mu B^{\mu\mu_2\cdots\mu_p} = 0$, leads to the additional constraint $\partial_n B^{nA_2\cdots A_p} = 0$. To keep things general, we set $\chi = \chi_\perp$ when one of the indices of B is the normal index and $\chi = \chi_\parallel$ otherwise.

Conformal covariance suggests that the two-point function of H with itself can be written

[12] We remark that there are additional subtleties in p-form theories that are worthy of further consideration. First, the gauge fixing process breaks conformal invariance. An ameliorating factor is that the ghost and gauge fixing sectors to a large extent decouple from the rest of the theory. For example, the two-point function of $\partial \cdot B$ and $H = dB$ vanishes in general. Second, the ghosts required in the gauge fixing process require further ghost degrees of freedom, so-called "ghosts for ghosts" (see e.g. [126, 127]).

[13] The parameter χ is a c-number for gauge fields, not a matrix.

in the form

$$\begin{aligned}\langle H_{\mu_1\cdots\mu_{p+1}}(x)H_{\nu_1\cdots\nu_{p+1}}(x')\rangle &= \frac{1}{s^d}\sum_{g,h\in\Sigma^{p+1}}(-1)^{g+h}\Big(a(v)\prod_{i=1}^{p+1}I_{g(\mu_i)\,h(\nu_i)}(s) \\ &\quad +b(v)X_{g(\mu_{p+1})}X'_{h(\nu_{p+1})}\prod_{i=1}^{p}I_{g(\mu_i)\,h(\nu_i)}(s)\Big)\ , \qquad (5.184)\end{aligned}$$

where Σ^p is the permutation group of p elements. The objects $I_{\mu\nu}$, X_μ and X'_ν were defined in section 5.2.

To fix $a(v)$ and $b(v)$ in (5.184), we don't need to calculate all components of the two-point function. Let us focus on the diagonal components. In fact, we can further restrict to the perpendicular geometry where $\mathbf{s}=0$. From (5.183), we find

$$\langle H_{2\cdots p+2}(x)H^{2\cdots p+2}(x')\rangle = \frac{\kappa(d-2)}{s^d}(p+1)(1+\chi_\parallel v^d)\ , \qquad (5.185)$$

$$\langle H_{1\cdots p+1}(x)H^{1\cdots p+1}(x')\rangle = \frac{\kappa(d-2)}{s^d}\Big(p+1-d+(p\chi_\perp+(d-1)\chi_\parallel)v^d\Big)\ . \qquad (5.186)$$

We then compare these expressions with (5.184) in the same limit,

$$\langle H_{2\cdots p+2}(x)H^{2\cdots p+2}(x')\rangle = \frac{(p+1)!}{s^d}a\ , \qquad (5.187)$$

$$\langle H_{1\cdots p+1}(x)H^{1\cdots p+1}(x')\rangle = -\frac{p!}{s^d}\Big((p+1)a+b\Big)\ . \qquad (5.188)$$

Solving for $a(v)$ and $b(v)$ yields

$$a(v) = \frac{(d-2)\kappa}{p!}(1+\chi_\parallel v^d)\ , \qquad (5.189)$$

$$\begin{aligned}b(v) &= \frac{(d-2)\kappa}{p!}\Big(d-2(p+1)-(\chi_\parallel(d+p)+\chi_\perp p)v^d\Big) \\ &= -\frac{(d-2)\kappa}{p!}\left(\chi_\parallel(d+p)+\chi_\perp p\right)v^d\ , \qquad (5.190)\end{aligned}$$

where we have set $d=2(p+1)$ to have a traceless stress tensor. In the absolute and relative cases where $\chi_\parallel=-\chi_\perp=\pm1$, we find the simpler

$$a(v) = \frac{(d-2)\kappa}{p!}(1\pm v^d)\ , \qquad (5.191)$$

$$b(v) = \frac{(d-2)\kappa}{p!}\Big(d-2(p+1)\mp dv^d\Big) = \mp d\frac{(d-2)\kappa}{p!}v^d\ . \qquad (5.192)$$

To pin down the form of the stress tensor, we need the following three two-point functions:

$$\langle T_{nn}(x)T_{nn}(y)\rangle = \frac{(p!)^2}{2s^{2d}}\left(\binom{d-1}{p}((p+1)a+b)^2+\binom{d-1}{p+1}(p+1)^2a^2\right)\ , \qquad (5.193)$$

$$\langle T_{n2}(x)T_{n2}(y)\rangle = -\frac{(p!)^2}{s^{2d}}\binom{d-2}{p}((p+1)a+b)(p+1)a\ , \qquad (5.194)$$

$$\langle T_{23}(x)T_{23}(y)\rangle = \frac{(p!)^2}{s^{2d}}\left(\binom{d-3}{p-1}((p+1)a+b)^2+\binom{d-3}{p}(p+1)^2a^2\right)\ . \qquad (5.195)$$

Away from $d = 2p + 2$, the calculation becomes inconsistent because the stress tensor is no longer traceless and there should be additional structures that need to be matched to fix the complete form of the stress tensor two-point function. For $d = 2p + 2$, we find

$$\begin{aligned} A(v) &= 2(2p)!b^2 \\ &= \frac{2(d-2)^2\kappa^2(2p)!}{(p!)^2}\left(\chi_\parallel(d+p) + \chi_\perp p\right)^2 v^{2d} , \end{aligned} \tag{5.196}$$

$$\begin{aligned} B(v) &= -\frac{1}{2}(2p)!b^2 \\ &= -\frac{(d-2)^2\kappa^2(2p)!}{2(p!)^2}\left(\chi_\parallel(d+p) + \chi_\perp p\right)^2 v^{2d} , \end{aligned} \tag{5.197}$$

$$\begin{aligned} C(v) &= (2p)!\left(2(p+1)^2a^2 + 2ab(p+1) + b^2\right) \\ &= \frac{(d-2)^2\kappa^2(2p)!}{2(p!)^2}\big[d^2 - (d-2)dv^d(\chi_\parallel + \chi_\perp) + \\ &\quad + \frac{1}{2}\Big((4 + d(5d-8))\chi_\parallel^2 + 4(d-2)(d-1)\chi_\parallel\chi_\perp + (d-2)^2\chi_\perp^2\Big)v^{2d}\big] . \end{aligned} \tag{5.198}$$

Note that in the bulk limit $v \to 0$, this result agrees with [125], as it should. Restricting to the absolute and relative boundary conditions where $\chi_\parallel = -\chi_\perp$, we find that

$$\begin{aligned} \alpha(v) &= \frac{d-1}{d}C(v) \\ &= \frac{d(d-1)(d-2)^2\kappa^2(2p)!}{2(p!)^2}(1 + \chi^2 v^{2d}) . \end{aligned} \tag{5.199}$$

Observe that, $\frac{(2p+2)!}{2(p!)^2}$ scalars, split evenly between Neumann and Dirichlet boundary conditions, reproduce the same stress tensor as this p-form with either absolute or relative boundary conditions. This equivalence means that the conformal block decomposition for the p-form is the same as that for the scalar.

From (5.199), the 4d U(1) gauge field has the following values:

$$\alpha(0) = \frac{3}{\pi^4} , \quad \alpha(1) = \frac{6}{\pi^4} . \tag{5.200}$$

From the bulk relations (5.147) and (5.54), we indeed recover the bulk c-charge given in (5.18). From the relation (5.155), we get $b_2 = \frac{4}{5}$, which is consistent with the heat kernel computation of the gauge field [85]. Indeed, the free theories considered in this section all have the relation $\alpha(1) = 2\alpha(0)$, which implies that $b_2 = 8c$ as we mentioned earlier. In the next section, we will see how the story changes when interactions are introduced on the boundary.

5.5 Models with Boundary Interactions

The free theories we studied generically have a current two-point function characterized by a $\pi(v) \sim 1 - v^{2d-2}$ and stress tensor two-point function characterized by an $\alpha(v) \sim$

$1 + v^{2d}$.[14] Since we saw generally that $\chi^2 = \mathbb{1}$, there was as a result no way to modify the coefficients of v^{2d-2} and v^{2d} in $\pi(v)$ and $\alpha(v)$ (respectively) relative to the bulk identity block contribution. On the other hand, we saw in the boundary conformal block decomposition that it should be straightforward to realize a bCFT with $\pi(v) \sim 1 - \chi^2 v^{2d-2}$ and $\alpha(v) \sim 1 + \chi^2 v^{2d}$, $\chi^2 < 1$, simply by taking advantage of the sums over blocks (5.125) and (5.129) with the opposite parity under $v \to 1/v$. An obvious question poses itself. Is it possible to realize physically interesting bCFTs with $\chi^2 \neq \mathbb{1}$? In this section we provide several examples below where we can move perturbatively away from the case where all the eigenvalues of χ are ± 1. Moreover, we will see that a model with perturbative corrections to $\chi^2 = \mathbb{1}$ provides a counter-example to the $b_2 = 8c$ relation in 4d.

The idea is to couple a free field in the bulk to a free field in the boundary with a classically marginal interaction that lives purely on the boundary. For simplicity, we will restrict the bulk fields to a scalar field and Maxwell field in four dimensions. For boundary fields, we will allow only scalars and fermions. The fermions require less fine tuning as their larger engineering dimension allows for fewer relevant interactions. We again consider a planar boundary located at $y = 0$ while the bulk fields live in $y > 0$. Here is our cast of characters:

1. A mixed dimensional Yukawa theory,

$$I = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x (\partial^\mu \phi)(\partial_\mu \phi) + \int_{\partial\mathcal{M}} \mathrm{d}^3x \left(i\bar{\psi}\partial\!\!\!/\psi - g\phi\bar{\psi}\psi \right) , \tag{5.201}$$

with the modified Neumann boundary condition $\partial_n \phi = -g\bar{\psi}\psi$. In our conventions, the unit normal n^μ points in the negative y-direction.

2. A mixed dimensional QED,

$$I = -\frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^4x F^{\mu\nu}F_{\mu\nu} + \int_{\partial\mathcal{M}} \mathrm{d}^3x \left(i\bar{\psi}D\!\!\!\!/\,\psi \right) , \tag{5.202}$$

where $D_\mu = \partial_\mu - igA_\mu$. The boundary conditions are a modification of the absolute boundary conditions discussed before, with $A_n = 0$, and $F_{nA} = \partial_n A_A = g\bar{\psi}\gamma_A\psi$.

3. A $d = 4$ mixed dimensional scalar theory,

$$I = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x (\partial^\mu \phi)(\partial_\mu \phi) - \int_{\partial\mathcal{M}} \mathrm{d}^3x \Big(\frac{1}{2}(\partial_A \eta)(\partial^A \eta) + (\partial_n \phi)(-\phi + g\eta^2) \Big) , \tag{5.203}$$

[14] The story was slightly more complicated for a vector of free scalars, ϕ^a, where additional pieces proportional to $\mathrm{tr}(\chi)$ appear. While we keep our discussion general, we remark that by having an equal number of Dirichlet and Neumann boundary conditions, we obtain $\mathrm{tr}(\chi) = 0$. In supersymmetric theories, an equal number of Neumann and Dirichlet boundary conditions appears to correlate with preserving a maximal amount of supersymmetry. In $\mathcal{N} = 4$ Super-Yang Mills theory in 3+1 dimensions, a $3 + 3$ splitting of the scalars preserves a $SO(3) \times SO(3) \subset SO(6)$ subgroup of the R-symmetry and a $OSp(4|4)$ subgroup of the $PSU(4|4)$ superalgebra [128, 129]. Similarly for ABJM theory, a $4 + 4$ splitting of the scalars preserves a $SO(4) \times SO(4) \subset SO(8)$ subgroup of the R-symmetry [130].

with modified Dirichlet boundary conditions $\phi = g\eta^2$. Another scalar field η is introduced on the boundary.

The boundary conditions are determined by having a well-posed variational principle for these classical actions. The coupling g is dimensionless. The limit $g \to 0$ results in two decoupled free theories, one living in the bulk space and another propagating on the boundary. We should perhaps emphasize that in each of these models, there is an alternate trivial choice of boundary conditions – Dirichlet, relative, and Neumann respectively – which leaves the boundary and bulk theories decoupled. In this case, only the free bulk theory contributes to central charges, since the free boundary theory can be defined independent of the embedding space, without "knowing" about extrinsic curvature or bulk curvature.

One can generalize these models to curved space with actions that are explicitly Weyl invariant. Here we have again focused on flat space. The improved stress tensors of these models are traceless on shell. This list is not meant to be exhaustive. In general, one can add additional classical marginal interactions on the boundary, but these toy models are sufficient to illustrate several interesting features of this class of interacting theories.

Among several other remarkable properties, the mixed QED theory is likely to be exactly conformal. For the other theories, using dimensional regularization and suitably tuning to eliminate relevant operators, we will find fixed points in the ϵ expansion using dimensional regularization

Apart from the mixed dimensional QED, to our knowledge none of these theories has been studied in the literature. The canonical example of an interacting bCFT appears to be scalar ϕ^4 theory in the bulk with no extra propagating degrees of freedom living on the boundary [87, 88, 95, 97].

The classically marginal interaction serves to alter slightly the boundary conditions on the bulk field away from Dirichlet or Neumann cases. One may think of these interactions as a coupling between an operator of dimension $\frac{d-2}{2}$ and an operator of dimension $\frac{d}{2}$. In the Neumann case, the operator of dimension $\frac{d-2}{2}$ is the boundary limit of the bulk field ϕ or A_A. In the Dirichlet case, the operator of dimension $\frac{d}{2}$ is the boundary limit of $\partial_n\phi$.

Recall in the discussion of crossing relations, we found the simple relation (5.116). The free fields we discussed in the previous section take advantage of this relation only in the limiting Dirichlet or Neumann cases $\chi \to \pm 1$ (or more generally when the eigenvalues of χ are ± 1). In these cases, the two-point function decomposes either into a single boundary block of dimension $\frac{d}{2}$ in the Dirichlet case or a single boundary block of dimension $\frac{d-2}{2}$ in the Neumann case. Indeed, the operator of the other dimension is missing because of the boundary conditions. Now we see, at least perturbatively, how the story will generalize. The boundary interaction adds back a little bit of the missing block, and the two-point function for the bulk free field will be characterized instead by a $\chi = \pm(1 - \mathcal{O}(g^2))$. (The story with the bulk Maxwell field is complicated by the lack of gauge invariance of $\langle A_\mu(x)A_\nu(x')\rangle$, but morally the story is the same.) Through Feynman diagram calculations below, we will confirm this over-arching picture.

With the modified two-point function of the bulk fields in hand, it will be straightforward to modify the corresponding two-point functions of the current and stress tensor, using Wick's theorem, to leading order in the interaction g. We just need to keep a general value of χ, instead of setting $\chi = \pm 1$. For the stress tensor, one finds the structure $\alpha(v) = 1 + \chi^2 v^{2d}$ instead of $\alpha(v) = 1 + v^{2d}$, and similarly for the current two-point function.

In the special case of mixed QED, where the theory is purported to be conformal in $d = 4$ dimensions, we have an example of a conformal field theory where $\alpha(1) < 2\alpha(0)$ and b_2 cannot be directly related to the the central charge c in the bulk trace anomaly. In fact, the situation is more subtle. In order to evaluate $\alpha(v)$ at $v = 1$, we take a near boundary limit. It is in fact not necessarily true that the $v \to 1$ limit commutes with the perturbative $g \to 0$ limit in these theories.

For the related function $\gamma(v)$, a similar perturbative computation indicates that $\gamma(1) = \mathcal{O}(g^2)$ where the nonzero contribution comes from T^{nA} exchange in the boundary conformal block decomposition. However, as mentioned before, we must have $T^{nA}|_{\rm bry} = 0$ as an operator statement since the dimension of T^{nA} is protected. Mathematically, one expects $\gamma(v) \sim g^2(1-v)^{\delta_T}$ where $\delta_T \sim \mathcal{O}(g^2)$, leading to noncommuting small g and $v \to 1$ limits and allowing $\gamma(1)$ to remain zero.[15]

From the conservation relations, one could worry there is a similar issue with $\alpha(1)$. But, looking more carefully, the behavior $\gamma(v) \sim g^2(1-v)^{\delta_T}$ leads to $\alpha(v) \sim g^2(1-v)^{1+\delta_T}$ which vanishes at $v = 1$ independent of the order of limits, and $\epsilon(v) \sim g^2 \delta_T \,(1-v)^{-1+\delta_T}$ whose associated divergence will only show up at the next order in perturbation theory. We therefore claim the $\mathcal{O}(g^2)$ contribution to $\alpha(1)$ we find is independent of the order of limits and comes from an alteration in the contribution of the displacement operator conformal block to the two-point function. Indeed, if we were to find a behavior of the form $\alpha(v) \sim g^2(1-v)^{\delta_T}$, which has the order of limits issue, that behavior through stress tensor conservation corresponds to an $\epsilon(v) \sim g^2\delta_T \,(1-v)^{-2+\delta_T}$ or equivalently exchange of a boundary spin two operator of dimension $d - 2 + \delta_T$ which is below the unitarity bound of $d-1$ for small δ_T. To check these arguments that $\alpha(1) \neq 2\alpha(0)$, ideally we should go to higher loop order in perturbation theory. We leave such calculations for the future.

It would be interesting furthermore to see if one can bound $\alpha(1)$ and correspondingly the boundary trace anomaly b_2. It is tempting to conjecture that free theories saturate an upper bound $\alpha(1) \leq 2\alpha(0)$ in four dimensions.[16] The phenomenon that $\alpha(1) = 2\alpha(0)$ at this point appears to be a special feature of free bCFTs.

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[15]We thank D. Gaiotto for discussions.

[16]Away from $d = 4$, there are already counterexamples. For ϕ^4 theory and Neumann (special) boundary conditions, $\alpha(1) > 2\alpha(0)$ both in the large N expansion in the range $5/2 < d < 4$ and also at leading order in the ϵ expansion for any N. See (7.31) and (7.23) of ref. [88]. In $d = 4$, the theory becomes free and one has $\alpha(1) = 2\alpha(0)$ or $b_2 = 8c$.

5.5.1 Mixed Yukawa Theory

Let us begin with a one loop analysis of the Yukawa-like theory,

$$I = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x(\partial^\mu\phi)(\partial_\mu\phi) + \int_{\partial\mathcal{M}} \mathrm{d}^3x \left(i\bar{\psi}\partial\!\!\!/\psi - g\phi\bar{\psi}\psi\right) \ , \tag{5.204}$$

with modified Neumann boundary conditions $\partial_n\phi = -g\bar{\psi}\psi$. Again, the normal coordinate will be denoted by y and the coordinates tangential to the boundary by $\mathbf{x}$: $x = (\mathbf{x}, y)$.

Our first task will be to calculate a β-function for the interaction $\phi\bar{\psi}\psi$ to see if we can find a conformal fixed point. We should comment briefly on the space of relevant operators and the amount of fine tuning we need to achieve our goal. The engineering dimension of the ψ field is one, and thus a $(\bar{\psi}\psi)^2$ term should be perturbatively irrelevant. One could in principle generate relevant ϕ and ϕ^2 and a classically marginal ϕ^3 interactions on the boundary through loop effects. We will assume that we can tune these terms away.

As we use dimensional regularization, we need the propagators for the scalar and spinor fields in arbitrary dimension. The Euclidean propagators are

$$G_\phi(x;x') = C_S\left(\frac{1}{\left((\mathbf{x}-\mathbf{x}')^2+(y-y')^2\right)^{\frac{d-2}{2}}} + \frac{1}{\left((\mathbf{x}-\mathbf{x}')^2+(y+y')^2\right)^{\frac{d-2}{2}}}\right) , \tag{5.205}$$

$$G_\psi(\mathbf{x}) = C_F\frac{\gamma_A x^A}{\mathbf{x}^{d-1}} = -\frac{C_F}{d-3}\gamma^A\partial_A\left(\frac{1}{\mathbf{x}^{d-3}}\right) . \tag{5.206}$$

A canonical normalization is $C_S = \kappa = 1/(d-2)\,\mathrm{Vol}(S^{d-1})$ for the scalar and $C_F = 1/\,\mathrm{Vol}(S^{d-2})$ for the boundary fermion, where $\mathrm{Vol}(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$. Note that, unlike what we did in section 5.1, here we have started with a propagator with $\chi = 1$, fixed by the required Neumann boundary condition (when $g = 0$) on a single scalar in this toy model.

For our Feynman diagram calculations, we need the Fourier transforms along the boundary directions:

$$\tilde{G}_\phi(p) \equiv \int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{x}\; e^{-ip\cdot\mathbf{x}} G_\phi(y,\mathbf{x};0,0) = \frac{e^{-py}}{p} \ , \tag{5.207}$$

$$\tilde{G}_\psi(p) \equiv \int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{x}\; e^{-ip\cdot\mathbf{x}} G_\psi(\mathbf{x}) = -i\frac{\gamma\cdot p}{p^2} \ . \tag{5.208}$$

While $\tilde{G}_\psi(p)$ takes its canonical, textbook form, the scaling of $\tilde{G}_\phi(p)$ is $1/p$ instead of the usual $1/p^2$. This shift leads to many of the physical effects we now consider. We will perform our Feynman diagram expansion in Lorentzian signature. Analytically continuing, we find the usual $-i/p\!\!\!/$ rule for an internal spinor line and a $-i/|p|$ for an internal scalar line. As the beginning and end point of the scalar line must lie on the $y = 0$ plane, we can remove the e^{-py} factor from the momentum space propagator.

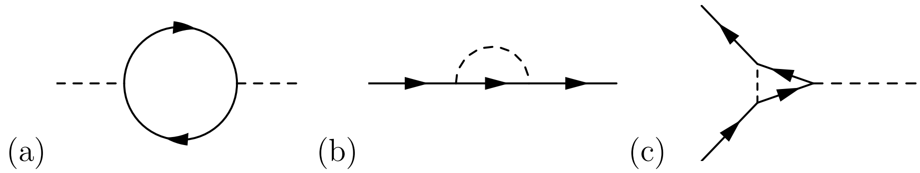

Figure 5.2: For the mixed dimensional Yukawa theory: (a) scalar one loop propagator correction; (b) fermion one loop propagator correction; (c) one loop vertex correction.

We now calculate the one loop corrections shown in figure 5.2. We begin with the scalar propagator. The diagram has a linear UV divergence which is invisible in dimensional regularization:

$$\begin{aligned} i\tilde{\Pi}_\phi(q) &= (-1)(-ig)^2 \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \frac{\mathrm{tr}[i\not{p}\, i(\not{p}+\not{q})]}{p^2(p+q)^2} && (5.209) \\ &= -ig^2 \frac{2^{5-2d}\pi^{2-\frac{d}{2}}}{\cos\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)} q^{d-3} \ , && (5.210) \end{aligned}$$

where we have used $\mathrm{tr}(\gamma_A\gamma_B) = -2\eta_{AB}$ and $\mathrm{tr}[\not{p}(\not{p}+\not{q})] = -2(p^2 + p\cdot q)$.[17] In $d = 4$, the self-energy reduces to

$$\tilde{\Pi}_\phi = -\frac{q}{8} g^2 \ . \qquad (5.211)$$

This result is in contrast to the usual self-energy correction for the 4d Yukawa theory, which has a logarithmic divergence. As the fermion momentum space propagators are the same in 3d and 4d, the difference comes from integrating over three rather than four momentum space dimensions.

The correction to the fermion propagator, in contrast, has a logarithmic divergence:

$$\begin{aligned} i\tilde{\Pi}_\psi(q) &= (-ig)^2 \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \frac{(i\not{p})(-i)}{p^2|p-q|} && (5.212) \\ &= -ig^2 \frac{4^{2-d}\pi^{\frac{1-d}{2}}\Gamma\left(2-\frac{d}{2}\right)\Gamma(d-2)}{\Gamma\left(d-\frac{3}{2}\right)} \frac{\gamma\cdot q}{q^{4-d}} \ . && (5.213) \end{aligned}$$

In $d = 4-\epsilon$, the result becomes

$$\tilde{\Pi}_\psi(q) = -\not{q} g^2 \left[\frac{1}{6\pi^2\epsilon} + \frac{1}{36\pi^2}(10 - 3\gamma - 3\log(q^2/\pi))\right] + \mathcal{O}(\epsilon) \ . \qquad (5.214)$$

The logarithmic divergence is evidenced by the $1/\epsilon$ in the dimensionally regulated expression, or we could have seen it explicitly by performing the original integral in $d = 4$ dimensions with a hard UV cut-off.

[17] In this section we take $\mathrm{tr}\,\mathbb{1} = 2$ for the three dimensional Clifford space.

Third, we look at the one loop correction to the vertex:

$$-ig\tilde{\Gamma}(q_1,q_2) \quad = \quad (-ig)^3 \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \frac{i(\not{p}+\not{q}_1)i(\not{p}+\not{q}_2)(-i)}{(p+q_1)^2(p+q_2)^2|p|} \ . \tag{5.215}$$

Using Feynman parameters, we can extract the most singular term. In $d = 4-\epsilon$ dimensions, we find that

$$g\tilde{\Gamma}(q_1,q_2) \quad = \quad -g^3 \frac{1}{2\pi^2\epsilon} + \text{finite} \ . \tag{5.216}$$

To compute the β-function for g, we introduce the wave function renormalization factors Z_ϕ and Z_ψ for the scalar and fermion kinetic terms as well as a vertex renormalization factor Z_g. The β-function follows from the relation

$$g_0 Z_\phi^{1/2} Z_\psi = g\mu^{\epsilon/2} Z_g \ , \tag{5.217}$$

where we can extract the Z-factors from our one loop computations:

$$Z_\psi \quad = \quad 1 + g^2\left(-\frac{1}{6\pi^2\epsilon} + \text{finite}\right) \ , \tag{5.218}$$

$$Z_\phi \quad = \quad 1 + g^2(\text{finite}) \ , \tag{5.219}$$

$$Z_g \quad = \quad 1 + g^2\left(\frac{1}{2\pi^2\epsilon} + \text{finite}\right) \ , \tag{5.220}$$

and g_0 denotes the bare coupling which is μ-independent. It follows that the β-function, $\beta(g(\mu)) = \mu\frac{\partial}{\partial\mu}g(\mu)$, is given by

$$\beta = -\frac{\epsilon}{2}g + \frac{2}{3\pi^2}g^3 + \mathcal{O}(g^4) \ . \tag{5.221}$$

For $d \geq 4$, the function remains positive which indicates that the coupling flows to zero at large distance. For $d < 4$, the coupling increases or decreases with the distance depending on the strengh of g. Given our fine tuning of relevant operators, we obtain an IR stable fixed point:

$$g_*^2 = \frac{3\pi^2}{4}\epsilon \ , \tag{5.222}$$

in $d < 4$ dimensions. Note that Z_ϕ has no divergent contribution. Indeed, a general feature of our collection of theories is that the bulk field will not be renormalized at one loop. In the case of the mixed dimensional QED theory, we can in fact make a stronger argument.

We claimed above that one effect of the classically marginal interaction was to shift slightly the form of the scalar-scalar two-point function. Let us see how that works by

Fourier transforming the result (5.211) back to position space[18]:

$$\Pi_\phi(x_1;x_2) = \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \tilde{\Pi}_\phi(p) \frac{e^{-p(y_1+y_2)}}{p^2} e^{ip\cdot\delta\mathbf{x}} \tag{5.223}$$

$$= -\frac{g^2}{16\pi^2} \frac{1}{(y_1+y_2)^2+\delta\mathbf{x}^2} \ . \tag{5.224}$$

As we started with a single component scalar with Neumann boundary conditions $\chi = 1$, this Fourier transform implies we have ended up with a two-point function with a slightly shifted χ:

$$\chi \to \chi = 1 - \mathcal{O}(g^2) \ . \tag{5.225}$$

The corrections to the current and stress tensor two-point functions will be controlled by the shift in the scalar two-point function, at this leading order $\mathcal{O}(g^2)$. Thus, we can read off the corresponding current and stress tensor two-point functions merely by inserting the modified value of χ in the formulae we found for the free scalar. Note this mixed Yukawa model becomes free in $d = 4$ where $\chi = 1$ is recovered. Our next example will be an interacting CFT in $d = 4$ where the parameter χ can be different from one.

5.5.2 Mixed Quantum Electrodynamics

The action for the mixed dimensional QED is

$$I = -\frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^4x F^{\mu\nu}F_{\mu\nu} + \int_{\partial\mathcal{M}} \mathrm{d}^3x \left(i\bar{\psi}\not{D}\psi\right) \ , \tag{5.226}$$

where $D_\mu = \partial_\mu - igA_\mu$. Note there is a potential generalization to include a Chern-Simons term on the boundary for this mixed QED model. We will work with a four component fermion to avoid generating a parity anomaly, and proceed with a standard evaluation of the one loop corrections (see figure 5.3) using the following Feynman rules: photon propagator, $-i\frac{e^{-py}}{p}\eta^{AB}$; fermion propagator, $\frac{i\not{p}}{p^2}$; interaction vertex, $ig\gamma^A$. The ghosts are decoupled in this abelian theory so below we do not need to consider them. A more general version of this calculation can be found in ref. [98].

The photon self-energy can be evaluated in a completely standard way:

$$i\tilde{\Pi}^{AB}_\gamma(q) = (-1)(ig)^2 \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \frac{\mathrm{tr}[\gamma^A\, i\not{p}\gamma^B\, i(\not{p}+\not{q})]}{p^2(p+q)^2} \tag{5.227}$$

$$= -2ig^2(q^2\eta^{AB} - q^Aq^B)\frac{(d-3)\pi^{2-\frac{d}{2}}}{4^{d-2}\cos\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}\right)}\frac{1}{q^{5-d}} \ . \tag{5.228}$$

[18]For the loop computation, we use a propagator from one point on the boundary to another where we set $y = 0$. When Fourier transforming back to real space, we are sewing on external propagators, taking us from points in the bulk (with non-zero y_1 and y_2) to points on the boundary.

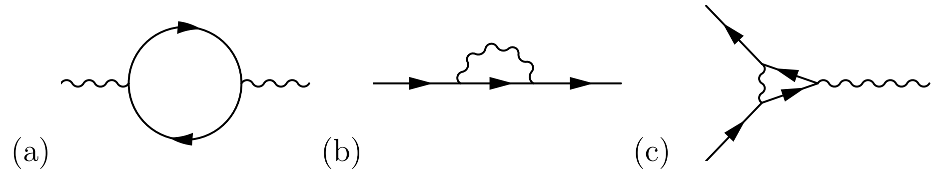

Figure 5.3: For the mixed dimensional QED: (a) photon one loop propagator correction; (b) fermion one loop propagator correction; (c) one loop vertex correction.

In $d = 4$, one gets the finite answer in dimensional regularization

$$\tilde{\Pi}_{\gamma}^{AB}(q) = -\frac{g^2}{8q}(q^2\eta^{AB} - q^A q^B) \ . \tag{5.229}$$

There is in fact never a logarithmic divergence at any order in the loop expansion for $\tilde{\Pi}_{\gamma}^{AB}(q)$, and the wave-function renormalization for the photon Z_γ will be finite in dimensional regularization. The usual topological argument shows that the photon self-energy diagrams have a linear superficial degree of divergence. Consider a general n-loop correction to the scalar propagator with ℓ internal propagators and v vertices. Momentum conservation tells us that $n - \ell + v = 1$. We can divide up ℓ into photon lines ℓ_γ and fermion lines ℓ_ψ. As each vertex involves two fermion lines and one photon, it must be that $\ell_\psi = v$ and (recalling that two photon lines are external) $\ell_\gamma = (v-2)/2$. Therefore $n = v/2$. The superficial degree of divergence of the photon self-energy diagrams is thus

$$n(d-1) - \ell_e - \ell_\gamma = n(d-1) - \frac{3v}{2} + 1 = n(d-4) + 1 \ , \tag{5.230}$$

which in $d = 4$ dimensions is equal to one. Gauge invariance implies that we can strip off a $q^A q^B - \eta^{AB} q^2$ factor from the self-energy. As a result, it is conventionally argued that the degree of divergence is reduced by 2. Thus the photon self-energy is finite in this mixed dimensional context. (In QED, the superficial degree of divergence is 2, and the gauge invariance argument changes the divergence to a log. There is then a corresponding renormalization of the photon wave-function.)

Let us again Fourier transform back to position space. There is a subtle issue associated with gauge invariance. Our Feynman gauge breaks conformal symmetry, and if we proceed naively, we will not be able to write the correlator $\langle A_\mu(x) A_\nu(x') \rangle$ as a function of the cross-ratio v, making it difficult to make use of the results from section 5.4. To fix things up, we have the freedom to perform a small gauge transformation that changes the bare propagator by a term of $\mathcal{O}(g^2)$. In fact, we claim we can tune this transformation such that there is a $\mathcal{O}(g^2)$ term in the bare propagator that cancels the $q^A q^B$ dependence of (5.229). The details are in appendix 5.7.3. In our slightly deformed gauge, the corrections to the position space correlation function become

$$\Pi_{\gamma}^{AB}(x;x') \ = \ -c \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \frac{e^{-p(y_1+y_2)+ip\cdot\delta\mathbf{x}}}{p^{5-d}} \eta^{AB} \tag{5.231}$$

where

$$c = (d-3)\frac{2g^2\pi^{2-\frac{d}{2}}}{4^{d-2}\cos\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}\right)} \ . \tag{5.232}$$

In four dimensions, we obtain

$$\Pi_\gamma^{AB}(x;x') \ = \ -\frac{g^2}{16\pi^2(\delta\mathbf{x}^2+(y_1+y_2)^2)}\eta^{AB} \ . \tag{5.233}$$

Analogous to the Yukawa theory, we can interpret this shift as a shift in the $\chi_\parallel$ parameter of the $\langle A_A(x)A_B(x')\rangle$ two-point function. The corresponding current and stress tensor two-point functions can then be deduced at leading order $\mathcal{O}(g^2)$ by making the appropriate substitutions for $\chi_\parallel$ in the Maxwell theory results obtained in section 5.4.

As in the Yukawa theory case, the corrections to the fermion propagator are modified slightly by the reduced dimensionality of the theory. The calculation is almost identical:

$$\begin{aligned} i\tilde{\Pi}_\psi(q) &= (ig)^2\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}}\frac{\gamma^A\, i\not{p}\gamma^B(-i)\eta_{AB}}{p^2|p-q|} && (5.234)\\ &= (ig)^2(d-3)\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}}\frac{i\not{p}(-i)}{p^2|p-q|} && (5.235)\\ &= -\not{q}g^2\frac{1}{6\pi^2\epsilon}+\text{finite} \ . && (5.236)\end{aligned}$$

The result is precisely the result for the fermion self-energy in the Yukawa theory.

Finally, we calculate the singular contributions to the one loop vertex correction:

$$ig\tilde{\Gamma}^A(q_1,q_2) \ = \ (ig)^3\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}}\frac{\gamma^C i(\not{p}+\not{q}_1)\gamma^A i(\not{p}+\not{q}_2)\gamma^B(-i)\eta_{CB}}{(p+q_1)^2(p+q_2)^2|p|} \ . \tag{5.237}$$

Evaluating this integral in $d=4-\epsilon$ dimensions yields

$$\tilde{\Gamma}^A(q_1,q_2) \ = \ g^2\gamma^A\frac{1}{6\pi^2\epsilon} \ . \tag{5.238}$$

There is a relative factor of -1/3 compared to the Yukawa theory. In fact, there is a well known and relevant Ward identity argument (see e.g. [131]) that can be employed here. Current conservation applied to the correlation function $\langle J^\mu(z)\bar{\psi}(x)\psi(y)\rangle$ implies that Z_g/Z_ψ is finite in perturbation theory. In the minimal subtraction scheme where all corrections to Z_g and Z_ψ are divergent, we conclude that $Z_g = Z_\psi$.

At one loop, we have all the information we need to compute the β-function:

$$g_0 Z_\gamma^{1/2}Z_\psi = g\mu^{\epsilon/2}Z_g \ , \tag{5.239}$$

where

$$\begin{aligned} Z_\psi &= 1-g^2\left(\frac{1}{6\pi^2\epsilon}+\text{finite}\right) \ , && (5.240)\\ Z_\gamma &= 1+g^2(\text{finite}) \ , && (5.241)\\ Z_g &= 1-g^2\left(\frac{1}{6\pi^2\epsilon}+\text{finite}\right) \ . && (5.242)\end{aligned}$$

Hence the beta function is

$$\beta = -\frac{\epsilon}{2} g + \mathcal{O}(g^4) \ . \tag{5.243}$$

In other words, the β-function vanishes in 4d at one loop. In fact, as we have sketched, the Ward identity argument $Z_\psi = Z_g$ and the non-renormalization $Z_\gamma = 1 + g^2$(finite) are expected to hold order by order in perturbation theory, and so we can tentatively conclude that this mixed dimensional QED is *exactly* conformal in four dimensions, making this theory rather special.

From the relation between b_2 and $\alpha(1)$ (5.155), the Fourier transformed propagator (5.233) and the two-point function of U(1) gauge fields in $d = 4$ (5.199), we obtain the boundary charge b_2 for the mixed conformal QED as

$$b_{2(\text{Mixed QED})} \ = \ \frac{2}{5}\Big(2 - \frac{g^2}{2} + \dots\Big) < \frac{4}{5} = 8c_{(\text{Mixed QED})} \ , \tag{5.244}$$

where $\frac{4}{5} = b_{2(\text{EM})}$ is the boundary charge for the standard bulk U(1) theory. This weakly interacting conformal model therefore provides an example of $b_2 \neq 8c$ in 4d bCFTs.

In addition to $\alpha(v)$, consider the behavior of $\gamma(v)$, defined in (5.52), and representing the correlation function of the boundary limit of T^{nA}. While for free theories, it vanishes universally, $\gamma(1) = 0$, in this mixed conformal QED we find instead that, from the one loop computation given here, $\gamma(1) = -\frac{3g^2}{2\pi^4}$. But, as mentioned earlier, we must have a vanishing T^{nA} in the boundary limit as an operator statement. We expect

$$\gamma(v) \sim -\frac{3g^2}{2\pi^4}(1-v)^{\delta_T} \ , \tag{5.245}$$

where $\delta_T \sim O(g^2)$ is the anomalous dimension. In this case, the small g and $v \to 1$ limits do not commute. While perturbatively, we might be fooled into thinking that $\gamma(1) \neq 0$, in point of fact $\gamma(1)$ should vanish.

While we do not do so here, there are two further calculations of great interest. The first is to look at the next loop order in the stress tensor two-point function. The stress tensor conservation equations suggest that the order of limits will not be an issue for evaluating $\alpha(1)$. It would be nevertheless nice to verify this claim by actually computing more Feynman diagrams. While we have no expectation that the value of $\alpha(1)$ is somehow protected in interacting theories, it would be fascinating if it were. The second project is to calculate the trace anomaly of this theory directly in curved space with a boundary to verify the relation between $\alpha(1)$ and b_2. We leave such projects for the future.

5.5.3 Mixed Scalar

In the two examples we considered so far, the boundary interaction modified a Neumann boundary condition. In this third example, the boundary interaction modifies a Dirichlet

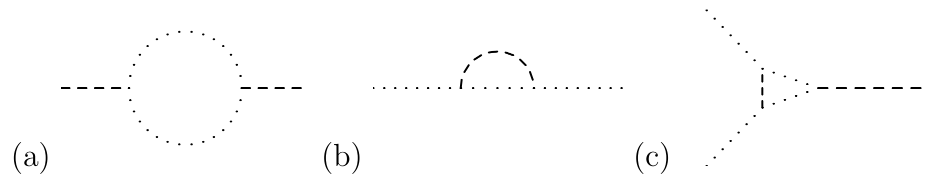

Figure 5.4: For the mixed dimensional scalar theory: (a) a 4d bulk scalar one loop self energy correction; (b) a 3d boundary scalar one loop self energy correction; (c) one loop vertex correction.

condition. There will be a corresponding all important change in sign in the correction to $\chi = -1$. The theory is

$$I = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4 x (\partial^\mu \phi)(\partial_\mu \phi) - \int_{\partial\mathcal{M}} \mathrm{d}^3 x \Big(\frac{1}{2}(\partial_A \eta)(\partial^A \eta) + (\partial_n \phi)(-\phi + g\eta^2)\Big) \ . \tag{5.246}$$

This theory has many possible relevant interactions on the boundary that can be generated by loop effects, e.g. ϕ^2, η^2, η^4, etc. We will assume we can fine tune all of these relevant terms away. We will also ignore additional classically marginal interactions such as $\phi^2\eta^2$ and η^6.

We proceed to a calculation of the three Feynman diagrams in figure 5.4. The propagator correction for the bulk scalar is

$$i\tilde{\Pi}_\phi = 2(ig)^2 \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \frac{(-i)^2}{p^2(p+q)^2} \tag{5.247}$$

$$= i\frac{g^2}{4q} \tag{5.248}$$

We can Fourier transform this result back to position space to see how the two-point function will be modified:

$$\Pi_\phi = \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \tilde{\Pi}_\phi(p) e^{-p(y_1+y_2)} e^{ip\cdot\delta\mathbf{x}} \tag{5.249}$$

$$= \frac{g^2}{8\pi^2(\delta\mathbf{x}^2 + (y_1+y_2)^2)} \ , \tag{5.250}$$

where in the last line, we set $d = 4$. Crucially, the sign here is different from (5.224) and (5.233), corresponding to a shift in the two-point function for the scalar away from Dirichlet conditions $\chi = -1 + \mathcal{O}(g^2)$ instead of away from Neumann conditions $\chi = 1 - \mathcal{O}(g^2)$. Note these results are consistent with the bounds on χ (5.117). At leading order $\mathcal{O}(g^2)$, we can compute the corrected current and stress tensor two-point functions as well, merely by making the appropriate replacement for χ in the free scalar result.

The correction to the boundary scalar propagator is

$$i\tilde{\Pi}_\eta = 4(-ig)^2 \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \frac{(-i)^2(-1)|p|}{(p+q)^2} \tag{5.251}$$

$$= -i\frac{2g^2q^2}{3\pi^2\epsilon} + \text{finite} . \tag{5.252}$$

Finally, we give the divergent contribution to the one loop vertex correction:

$$-ig\tilde{\Gamma}(q_1, q_2) = 8(-ig)^3 \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \frac{(-i)^3(-1)|p|}{(p+q_1)^2(p+q_2)^2} . \tag{5.253}$$

In $d = 4 - \epsilon$ dimensions, this reduces to

$$g\tilde{\Gamma}(q_1, q_2) = -g^3 \frac{4}{\pi^2\epsilon} + \text{finite} . \tag{5.254}$$

We compute the β-function for g using $g_0 Z_\phi^{1/2} Z_\eta = g\mu^{\epsilon/2} Z_g$ and[19]

$$Z_\eta = 1 - g^2\left(\frac{2}{3\pi^2\epsilon} + \text{finite}\right) , \tag{5.255}$$

$$Z_\phi = 1 + g^2(\text{finite}) , \tag{5.256}$$

$$Z_g = 1 + g^2\left(\frac{4}{\pi^2\epsilon} + \text{finite}\right) . \tag{5.257}$$

The result is that

$$\beta = -\frac{\epsilon}{2}g + \frac{14}{3\pi^2}g^3 + \mathcal{O}(g^4) . \tag{5.258}$$

There is an IR stable fixed point at

$$g_*^2 = \frac{3\pi^2}{28}\epsilon , \tag{5.259}$$

in $d < 4$ dimensions. In the $d = 4$ limit, the theory becomes free and one has $\alpha(1) = 2\alpha(0)$ and $b_2 = 8c$ relations.

5.6 Concluding Remarks

Motivated by recent classification of the boundary trace anomalies for bCFTs [3, 85, 90], we studied the structure of two-point functions in bCFTs. The main result of this chapter

[19] We note in passing that bulk fields are not renormalized in our one loop computations. Z_ϕ in (5.219) and (5.256) and Z_γ in (5.241) are finite. There should be an argument based on locality, that boundary interactions can never renormalize the bulk fields. We are not sure how to make precise the relationship between locality and the actual Feynman diagram computations, however. We thank D. Gaiotto for discussions on this point.

(5.155) states a relation between the b_2 boundary central charge in $d = 4$ bCFTs and the spin-zero displacement operator correlation function near the boundary. Since $\alpha(1) = 2\alpha(0)$ in free theories, we can explain the $b_2 = 8c$ relation observed in [85]. Indeed, from our study of free theories, we find that two-point functions of free bCFTs have a simple universal structure.

Going beyond free theory, we define a class of interacting models with the interactions restricted to the boundary. We computed their beta functions and pointed out the locations of the fixed points. In particular, the mixed dimensional QED is expected to be exactly conformal in $d = 4$. We have provided evidence that this model can be a counterexample of the $b_2 = 8c$ relation in 4d bCFTs. As we summarized before, this mixed QED theory is interesting for at least three other reasons as well: its connection with graphene, its connection with three dimensional QED, and its behavior under electric-magnetic duality. It doubtless deserves further exploration.

A feature of this graphene-like theory is that the near boundary limit of the stress tensor two-point function, characterized by $\alpha(1)$, depends on the exactly marginal coupling g. Given the claimed relationship between b_2 and $\alpha(1)$ (5.155), it follows that b_2 also depends on the exactly marginal coupling g. This dependence stands in contrast to the situation for the bulk charges a and c. Wess-Zumino consistency rules out the possibility of any such dependence for a [132]. The idea is to let $a(g(x))$ depend on the coupling g which we in turn promote to a coordinate dependent external field. Varying the Euler density must produce a total derivative. Any spatial dependence of a spoils this feature.

The situation is different for c (and hence also $\alpha(0)$). While the Euler density varies to produce a total derivative, the integrated W^2 term has zero Weyl variation. Thus in principle, one might be able to find examples of field theories where c depends on marginal couplings. In [133], an AdS/CFT model without supersymmetry is constructed suggesting the possibility that the c-charge can change under exactly marginal deformations. In practice, guaranteeing an exactly marginal direction in four dimensions is difficult and usually requires supersymmetry. Supersymmetry in turn fixes c to be a constant. For b_2, the situation is similar to the situation for c. The integrated KW boundary term also has a zero Weyl variation, and b_2 could in principle depend on marginal couplings. In constrast to the situation without a boundary, the presence of a boundary has allowed us to construct a non-supersymmetric theory with an exactly marginal direction in the moduli space – this mixed dimensional QED. Correspondingly, we are finding that $\alpha(1)$ and b_2 can depend on the position in this flat direction. A similar situation is that the boundary entropy g in two dimensional conformal field theories is known to depend on marginal directions in the moduli space [134].[20] There is a potential downside to this dependence. If we are looking for a quantity that orders quantum field theories under RG flow, it is inconvenient for that quantity to depend on marginal directions. We normally would like such a quantity to stay

[20] We would like to thank T. Dumitrescu for this remark.

constant on the space of exactly marginal couplings and only change when we change the energy scale. It is nevertheless interesting to understand better how these 4d boundary central charges behave under (boundary) RG flow.

5.7 Appendix

5.7.1 Null Cone Formalism

The null cone formalism is a useful tool for linearizing the action of the conformal group $O(1, d+1)$ [135]. The linearization in turn makes a derivation of the conformal blocks straightforward [119, 136, 137, 138] for higher spin operators, as we now review, drawing heavily on [95].

Points in physical space $x^\mu \in \mathbb{R}^d$ are in one-to-one correspondence with null rays in $\mathbb{R}^{1,d+1}$. Given a point written in light cone coordinates,

$$P^A = (P^+, P^-, P^1, \ldots, P^d) \in \mathbb{R}^{1,d+1} \ , \tag{5.260}$$

a null ray corresponds to the equivalence class $P^A \sim \lambda P^A$ such that $P^A P_A = 0$. A point in physical space can then be recovered via

$$x^\mu = \frac{P^\mu}{P^+} \ . \tag{5.261}$$

A linear $O(d+1, 1)$ transformation of $\mathbb{R}^{1,d+1}$ which maps null rays into null rays corresponds to a conformal transformation on the physical space.

We are further interested in correlation functions of symmetric traceless tensor fields $F_{\mu_1 \cdots \mu_n}$. For a tensor field lifted to embedding space $F_{A_1 \cdots A_n}(P)$ and inserted at P,

$$F_{A_1 \cdots A_n}(\lambda P) = \lambda^{-\Delta} F_{A_1 \cdots A_n}(P) \ , \tag{5.262}$$

we reduce this problem to that of correlation functions of scalar operators by contracting the open indices with a vector Z:

$$F(P, Z) = Z^{A_1} \cdots Z^{A_n} F_{A_1 \cdots A_n} \ . \tag{5.263}$$

Tracelessness means that we can take $Z^2 = 0$. In the embedding space, the tensor must be transverse $P^{A_1} F_{A_1 \cdots A_n} = 0$, which implies that $P \cdot \partial_Z F(P, Z) = 0$. Given the redundancy in the embedding space, we can also choose $Z \cdot P = 0$ without harm.

In the presence of a boundary, we have an extra unit normal vector $V = (0, \ldots, 0, 1)$ which breaks the symmetry $O(1, d+1)$ down to $O(1, d)$. For two-point functions with operators inserted at P and P', we can form the following scalar quantities invariant under $O(1, d)$:

$$P \cdot P' \ , \quad V \cdot P \ , \quad V \cdot P' \ , \quad Z \cdot P' \ , \quad Z' \cdot P \ , \quad V \cdot Z \ , \quad V \cdot Z' \ . \tag{5.264}$$

Note the cross ratio ξ can be written as

$$\xi = -\frac{P \cdot P'}{2(V \cdot P)(V \cdot P')} \ , \tag{5.265}$$

in this formalism. The game is then to write down functions of these invariants which correspond to a correlation function with the correct scaling weights and index structure. For the operator $F(P_i, Z_i)$ of weight Δ_i, we need one Z_i field for each index of the original $F_{\mu_1 \cdots \mu_n}$. Also, the expression should be homogeneous in P_i with degree $-\Delta_i$. Furthermore, we will need to make sure that the expressions satisfy transversality.

The one-point function of a scalar operator is

$$\langle O(P) \rangle = \frac{a_\Delta}{(2V \cdot P)^\Delta} \ . \tag{5.266}$$

Note the one-point function of an operator with spin l would introduce a factor $(V \cdot Z)^l$, which violates the transversality condition. Indeed, only the one-point function of a scalar is allowed in the presence of a boundary.

The scalar two-point function is

$$\langle O_1(P) O_2(P') \rangle = \frac{1}{(2V \cdot P)^{\Delta_1} (2V \cdot P')^{\Delta_2}} f(\xi) \ , \tag{5.267}$$

where

$$f(\xi) = \xi^{-\frac{(\Delta_1 + \Delta_2)}{2}} G(\xi) \ . \tag{5.268}$$

And, for current and stress tensor, we have

$$\langle Z \cdot J_1(P) Z' \cdot J_2(P') \rangle = \frac{P(\xi) S_1 + v^2 Q(\xi) S_2}{\xi^{d-1} (V \cdot P)^{\Delta_1} (V \cdot P')^{\Delta_2}} \ , \tag{5.269}$$

$$\langle Z \cdot T_1(P) \cdot Z \ Z' \cdot T_2(P') \cdot Z' \rangle = \frac{C(\xi) S_1^2 + 4v^2 B(\xi) S_1 S_2 + v^4 A(\xi) S_2^2}{(4\xi)^d (V \cdot P)^{\Delta_1} (V \cdot P_2)^{\Delta_2}} \ , \tag{5.270}$$

where

$$S_1 = \frac{(Z \cdot Z')(P \cdot P') - (Z \cdot P')(Z' \cdot P)}{P \cdot P'} \ , \tag{5.271}$$

$$S_2 = \left(\frac{(V \cdot P)(Z \cdot P')}{P \cdot P'} - V \cdot Z \right) \left(\frac{(V \cdot P')(Z' \cdot P)}{P \cdot P'} - V \cdot Z' \right) \ . \tag{5.272}$$

The conservation conditions can be expressed in terms of the Todorov differential operator

$$D_A^{(d)} = \left(\frac{d}{2} - 1 + Z \cdot \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial Z^A} - \frac{1}{2} Z_A \frac{\partial^2}{\partial Z \cdot \partial Z} \ . \tag{5.273}$$

Conservation for an operator $F(P, Z)$ means that $(\partial_P \cdot D^{(d)}) F = 0$. The conservation conditions will enforce that $\Delta_i = d - 1$ for the current and $\Delta_i = d$ for the stress tensor, but we leave them arbitrary for now.

The Todorov differential is also useful for writing the action of an element L_{AB} of the Lie algebra $\mathfrak{o}(1, d+1)$ on a symmetric traceless tensor:

$$L_{AB} F(P, Z) = \left(P_A \frac{\partial}{\partial P^B} - P_B \frac{\partial}{\partial P^A} + \frac{1}{\frac{d}{2} + s - 2} (Z_A D_B^{(d)} - Z_B D_A^{(d)}) \right) F(P, Z) \ . \quad (5.274)$$

The conformal Casimir equation is then

$$\frac{1}{2} L_{AB} L^{AB} F(P, Z) = -C_{\Delta,l} F(P, Z) \ , \quad (5.275)$$

where $C_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2)$. The conformal blocks in the bulk expansion are then determined by an equation of the form

$$\frac{1}{2}(L_{AB} + L'_{AB})(L^{AB} + L'^{AB}) G(P, Z, P', Z') = -C_{\Delta,0} G(P, Z, P', Z') \ , \quad (5.276)$$

acting on the two-point function $G(P, Z, P', Z')$ expressed in the null-cone formalism.

In the boundary conformal block expansion, we need to consider instead the generators of $O(1, d)$, $a, b = \pm, 1, \ldots, d-1$:

$$L_{ab} = P_a \frac{\partial}{\partial P^b} - P_b \frac{\partial}{\partial P^a} + \frac{1}{\frac{d-1}{2} + s - 2} (Z_a D_b^{(d-1)} - Z_b D_a^{(d-1)}) \ . \quad (5.277)$$

In this case, the conformal blocks in the boundary expansion are determined by

$$\frac{1}{2} L_{ab} L^{ab} G(P, Z, P', Z') = -\tilde{C}_{\Delta,l} G(P, Z, P', Z') \ , \quad (5.278)$$

where the Casimir operator acts on just the pair P and Z and $\tilde{C}_{\Delta,l} = \Delta(\Delta - d + 1) + l(l + d - 3)$.

We give some details of the derivation for the conserved current. (For conformal blocks of stress tensor two-point function, we refer the reader to [95] for details.) In this case, because of the linearity of the two-point function in Z and Z', the Todorov differentials can be replaced by ordinary partial differentials with respect to Z:

$$\frac{1}{\frac{d}{2} - 1} D_A^{(d)} \to \frac{\partial}{\partial Z^A} \ , \quad \frac{1}{\frac{d-1}{2} - 1} D_a^{(d-1)} \to \frac{\partial}{\partial Z^a} \ . \quad (5.279)$$

For what follows, we define the functions

$$\tilde{f} \equiv P \ , \quad \tilde{g} \equiv v^2 Q \ . \quad (5.280)$$

In the bulk conformal block decomposition, exchanging a scalar of dimension Δ with the boundary leads to the following pair of differential equations:

$$F: \quad 4\xi^2(1+\xi)\tilde{f}'' + 2\xi(2\xi + 2 - d)\tilde{f}' + [(d - \Delta)\Delta - (\Delta_1 - \Delta_2)^2]\tilde{f} - 2\tilde{g} = 0 \ , \quad (5.281)$$

$$G: \quad 4\xi^2(1+\xi)\tilde{g}'' + 2\xi(2\xi - 2 - d)\tilde{g}' + [(2 + d - \Delta)(2 + \Delta) - (\Delta_1 - \Delta_2)^2]\tilde{g} = 0 \ . \quad (5.282)$$

The tensor structure S_1 gives rise to the differential equation F while the structure S_2 gives the equation G. This system is compatible with the conservation relation. Restricting to $\Delta_i = d-1$, current conservation gives

$$J: \ (d+1)\tilde{g} - 2\xi\tilde{g}' - 2\xi^2(\tilde{f}' + \tilde{g}') = 0 \ . \tag{5.283}$$

One can construct a linear relation of the form $c_1 F' + c_2 G' + c_3 F + c_4 G + J'' + c_5 J' + c_6 J$, indicating that either of the second-order differential equations for $\tilde{f}$ and $\tilde{g}$ can be swapped for current conservation.

The differential equation G may be solved straightforwardly:

$$\tilde{g}_{\text{bulk}}(\Delta, \xi) = \xi^{1+\frac{\Delta}{2}} {}_2F_1\left(1 + \frac{\Delta + \Delta_1 - \Delta_2}{2}, 1 + \frac{\Delta - \Delta_1 + \Delta_2}{2}, 1 - \frac{d}{2} + \Delta, -\xi\right) \ , \tag{5.284}$$

where another solution with the behaviour $\sim \xi^{1-\frac{\Delta}{2}}$ is dropped. Note $\tilde{g}_{\text{bulk}}(\Delta, 0) = 0$. We introduce un-tilde'd functions that will simplify the equations for the boundary blocks:

$$\tilde{f}(\xi) = \xi^{(\Delta_1+\Delta_2)/2-d+1} f(\xi) \ , \tag{5.285}$$

$$\tilde{g}(\xi) = \xi^{(\Delta_1+\Delta_2)/2-d+1} g(\xi) \ . \tag{5.286}$$

Note the distinction disappears for conserved currents. Plugging the soluton (5.284) into the the conservation equation J one obtains

$$f_{\text{bulk}}(\Delta, \xi) + v^{-2} g_{\text{bulk}}(\Delta, \xi) = \frac{d-1}{\Delta} \xi^{\Delta/2} {}_2F_1\left(\frac{\Delta}{2}, 1 + \frac{\Delta}{2}, 1 - \frac{d}{2} + \Delta, -\xi\right) \ . \tag{5.287}$$

In the boundary block decomposition, we find the differential equations for f_{bry} and g_{bry} as

$$\begin{aligned} &\xi(1+\xi)g'' + \left(2\xi - \frac{d}{2}(3+2\xi)\right) g' \\ &\qquad + \left(\frac{2+d+d^2}{2\xi} - C_{\Delta,\ell}\right) g = (d-2) f \ , \end{aligned} \tag{5.288}$$

$$\begin{aligned} &\xi(1+\xi)f'' + \left(\xi(2-d) + 2 - \frac{3d}{2}\right) f' \\ &\qquad + \left(\frac{(d-2)(1+d+2\xi)}{2\xi} - C_{\Delta,\ell}\right) f = \frac{1+2\xi}{2\xi^2} g \ , \end{aligned} \tag{5.289}$$

where

$$C_{\Delta,\ell} = \ell(\ell + d - 3) + \Delta(\Delta - d + 1) \ . \tag{5.290}$$

As in the bulk case, these differential equations are compatible with the conservation condition, as can be verified by constructing a similar linear dependence between the equations.

We need to solve these equation for ($\ell = 0$ and $\Delta = d-1$) and also for ($\ell = 1$ and all Δ). In the first case

$$f^0_{\rm bry}(d-1,\xi) = \frac{1}{\xi}\left(\frac{\xi}{1+\xi}\right)^h = v^{d-2}(1-v^2)\ , \tag{5.291}$$

$$g^0_{\rm bry}(d-1,\xi) = \xi^h(1+\xi)^{-1-h}(d-2+2(d-1)\xi) = v^d(d-2+dv^2)\ . \tag{5.292}$$

There are similarly simple expressions for $\ell = 1$ and $\Delta = d-1$:

$$f^1_{\rm bry}(d-1,\xi) = \frac{1}{2}\xi^{h-1}(1+\xi)^{-h}(1+2\xi) = \frac{1}{2}v^{d-2}(1+v^2)\ , \tag{5.293}$$

$$g^1_{\rm bry}(d-1,\xi) = \frac{1}{2}\xi^h(1+\xi)^{-h-1}(d-2-2\xi) = \frac{1}{2}v^d(d-2-dv^2)\ . \tag{5.294}$$

In general, the spin one exchange is given by

$$g^1_{\rm bry}(\Delta,\xi) = -\xi^{d-1-\Delta}\,{}_3F_2\left(\begin{matrix} 1+\Delta, 3-d+\Delta, 1-\frac{d}{2}+\Delta \\ 2-d+\Delta, 2-d+2\Delta \end{matrix} ; -\frac{1}{\xi}\right)\ , \tag{5.295}$$

$$\begin{aligned} f^1_{\rm bry}(\Delta,\xi) = \frac{\xi^{d-\Delta-2}}{2(\Delta+2-d)}\Big[&2\xi(\Delta+1-d)\,{}_2F_1\left(\Delta, -\frac{d}{2}+\Delta+1; -d+2\Delta+2; -\frac{1}{\xi}\right) \\ &+(2\xi+1)\,{}_2F_1\left(\Delta+1, -\frac{d}{2}+\Delta+1; -d+2\Delta+2; -\frac{1}{\xi}\right)\Big]\ . \end{aligned} \tag{5.296}$$

5.7.2 Variation Rules

Here we give a brief review on the definitions of the Weyl tensor and extrinsic curvature. We list relevant metric perturbation formulae.

Under the metric perturbation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, the transformed Christoffel connection is given by

$$\delta^{(n)}\Gamma^\lambda_{\mu\nu} = \frac{n}{2}\delta^{(n-1)}(g^{\lambda\rho})\Big(\nabla_\mu\delta g_{\rho\nu} + \nabla_\nu\delta g_{\rho\mu} - \nabla_\rho\delta g_{\mu\nu}\Big)\ . \tag{5.297}$$

The Riemann and Ricci curvature tensors transform as

$$\delta R^\lambda{}_{\mu\sigma\nu} = \nabla_\sigma\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\mu\sigma}\ , \tag{5.298}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\Big(\nabla^\lambda\nabla_\mu\delta g_{\lambda\nu} + \nabla^\lambda\nabla_\nu\delta g_{\mu\lambda} - g^{\lambda\rho}\nabla_\mu\nabla_\nu\delta g_{\lambda\rho} - \Box\delta g_{\mu\nu}\Big)\ , \tag{5.299}$$

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\Big(\nabla^\nu\delta g_{\mu\nu} - g^{\lambda\rho}\nabla_\mu\delta g_{\lambda\rho}\Big)\ . \tag{5.300}$$

The Weyl tensor in d-dimensions (for $d > 3$) is defined as

$$W^{(d)}_{\mu\sigma\rho\nu} = R_{\mu\sigma\rho\nu} - \frac{2}{d-2}\Big(g_{\mu[\rho}R_{\nu]\sigma} - g_{\sigma[\rho}R_{\nu]\mu} - \frac{g_{\mu[\rho}g_{\nu]\sigma}}{(d-1)}R\Big)\ . \tag{5.301}$$

Note $W_{\mu\sigma\rho\nu} = W_{[\mu\sigma][\rho\nu]}$, $W_{\mu[\sigma\rho\nu]} = 0$ and $W^{\mu}{}_{\sigma\rho\mu} = 0$. One can write the transformation of the Weyl tensor as

$$\delta W_{\mu\sigma\rho\nu} = -2P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\partial^{\gamma}\partial^{\delta}\delta g^{\alpha\beta} \ , \tag{5.302}$$

where $P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}$ is a projector given by

$$\begin{aligned} P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} \ = \ & \tfrac{1}{12}\big(\delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\sigma\gamma}\delta_{\rho\delta} + \delta_{\mu\delta}\delta_{\sigma\beta}\delta_{\rho\alpha}\delta_{\nu\gamma} - \mu\leftrightarrow\sigma, \nu\leftrightarrow\rho\big) \\ & + \tfrac{1}{24}\big(\delta_{\mu\alpha}\delta_{\nu\gamma}\delta_{\rho\delta}\delta_{\sigma\beta} - \mu\leftrightarrow\sigma, \nu\leftrightarrow\rho, \alpha\leftrightarrow\gamma, \delta\leftrightarrow\beta\big) \\ & - \tfrac{1}{8(d-2)}\big(\delta_{\mu\rho}\delta_{\alpha\delta}\delta_{\sigma\gamma}\delta_{\nu\beta} + \delta_{\mu\rho}\delta_{\alpha\delta}\delta_{\sigma\beta}\delta_{\nu\gamma} - \mu\leftrightarrow\sigma, \nu\leftrightarrow\rho, \alpha\leftrightarrow\gamma, \delta\leftrightarrow\beta\big) \\ & + \tfrac{1}{2(d-1)(d-2)}\big(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\rho\sigma}\big)\big(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\delta\gamma}\big) \ . \end{aligned} \tag{5.303}$$

For a symmetric tensor or operator $t^{\gamma\delta}$ one has the following symmetric property:

$$P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}t^{\gamma\delta} = P_{\mu\sigma\rho\nu,\beta\gamma\delta\alpha}t^{\gamma\delta} \ , \tag{5.304}$$

while in general $P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} \neq P_{\mu\sigma\rho\nu,\beta\gamma\delta\alpha}$.

Defining the induced metric by $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, where n_μ is the outward-pointing normal vactor, the extrinsic curvature is

$$K_{\mu\nu} = h^{\lambda}_{\mu}h^{\sigma}_{\nu}\nabla_\lambda n_\sigma = \nabla_\mu n_\nu - n_\mu a_\nu \ , \tag{5.305}$$

where $a^\mu = n^\lambda\nabla_\lambda n^\mu$. On the boundary we have the following variations in general coordinates:

$$\begin{aligned} \delta n_\mu &= \frac{1}{2}n_\mu\delta g_{nn} \ , && (5.306) \\ \delta n^\mu &= -\frac{1}{2}n^\mu\delta g_{nn} - h^{\mu\nu}\delta g_{n\nu} \ , && (5.307) \\ \delta K_{\mu\nu} &= \frac{K_{\mu\nu}}{2}\delta g_{nn} + \Big(n_\mu K^{\lambda}_{\nu} + n_\nu K^{\lambda}_{\mu}\Big)\delta g_{\lambda n} - \frac{h^{\lambda}_{\mu}h^{\rho}_{\nu}n^{\alpha}}{2}\Big(\nabla_\lambda\delta g_{\alpha\rho} + \nabla_\rho\delta g_{\lambda\alpha} - \nabla_\alpha\delta g_{\lambda\rho}\Big) \ , \\ \delta K &= -\frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} - \frac{1}{2}n^\mu\Big(\nabla^\nu\delta g_{\mu\nu} - g^{\nu\lambda}\nabla_\mu\delta g_{\nu\lambda}\Big) - \frac{1}{2}\mathring{\nabla}_A(h^{AB}\delta g_{Bn}) \ , && (5.308) \end{aligned}$$

where $\mathring{\nabla}^\mu$ denotes the covariant derivative compatible with the boundary metric.

We can foliate the spacetime with hypersurfaces labelled by $y \equiv n^\mu x_\mu$ and adopt the Gaussian normal coordinates. The metric reads

$$ds^2 = dy^2 + h_{AB}(y, x_A)dx^A dx^B \ . \tag{5.309}$$

In the Gaussian normal coordinate $a^\mu = 0$, and one has

$$K_{AB} = \frac{1}{2}\partial_n h_{AB} \ , \tag{5.310}$$

and $\Gamma^y_{AB} = -K_{AB}$, $\Gamma^A_{yB} = K^A_B$, $\Gamma^A_{yy} = \Gamma^y_{yA} = \Gamma^y_{yy} = 0$. The transformation rules of the extrinsic curvature become

$$\begin{aligned}
\delta K_{AB} &= \frac{1}{2}\nabla_n \delta g_{AB} + \frac{1}{2}K^C_A \delta g_{BC} + \frac{1}{2}K^C_B \delta g_{AC} - \frac{1}{2}K_{AB}\delta g_{nn} - \mathring{\nabla}_{(A}\delta g_{B)n} \ , && (5.311)\\
\delta K &= \frac{1}{2}h^{AB}\nabla_n \delta g_{AB} - \frac{1}{2}K\delta g_{nn} - \mathring{\nabla}^A \delta g_{An} \ , && (5.312)\\
\nabla_n \delta g_{AB} &= \partial_n \delta g_{AB} - K^C_A \delta g_{BC} - K^C_B \delta g_{AC} \ . && (5.313)
\end{aligned}$$

5.7.3 Gauge Fixing Mixed Dimensional QED

In the presence of a planar boundary, which already breaks the full Lorentz invariance of the theory, it can be more convenient to consider a more general type of gauge fixing, characterized by two constants η and ζ instead of just the usual ξ:

$$I = \int_{\mathcal{M}} \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\eta\partial_n A^n - \zeta\partial_A A^A)^2 \right) + \int_{\partial\mathcal{M}} \mathrm{d}^3x \left(i\bar{\psi}\not{D}\psi \right) \ , \qquad (5.314)$$

where the boundary fermions do not affect the discussion of the gauge field Green's function in what follows. Standard Feynman gauge is achieved by setting $\zeta = \eta = 1$. We will kill the off-diagonal terms in the equations of motion by setting $\eta = 1/\zeta$.

Our strategy will be to first proceed by ignoring the presence of a boundary and then to take it into account at a later stage using the method of images. The (Euclidean) Green's function is defined by the equation:

$$\begin{pmatrix} \partial^2\delta_{AB} + (\zeta^2 - 1)\partial_A\partial_B & 0 \\ 0 & \partial^2 + (\zeta^{-2} - 1)\partial_n^2 \end{pmatrix} G^{\mu\nu}(x, x') = \delta^{(4)}(x - x') \ . \qquad (5.315)$$

Fourier transforming, we obtain

$$\begin{pmatrix} k^2\delta_{AB} + (\zeta^2 - 1)k_A k_B & 0 \\ 0 & k^2 + (\zeta^{-2} - 1)k_n^2 \end{pmatrix} \tilde{G}^{\mu\nu}(k) = -1 \ . \qquad (5.316)$$

Inverting this matrix, we don't quite get the usual result because $k^2 \neq k_A k^A$. The full result is a bit messy. Instead, let us take $\eta^2 = 1 + \delta\eta$ and expand to linear order in $\delta\eta$. We find

$$\tilde{G}_{\mu\nu}(k) = \delta_{\mu\nu}\frac{1}{k^2} + \frac{\delta\eta}{k^4}\begin{pmatrix} -k_A k_B & 0 \\ 0 & k_n^2 \end{pmatrix} + \mathcal{O}(\delta\eta^2) \ . \qquad (5.317)$$

The next step is to undo the Fourier transform in the normal direction. We have a handful of contour integrals to perform:

$$\begin{aligned}
I_0 &= \int \frac{\mathrm{d}q}{2\pi}\frac{e^{iq\,\delta y}}{\mathbf{k}^2 + q^2} = \frac{e^{-|\mathbf{k}||\delta y|}}{2|\mathbf{k}|} \ , && (5.318)\\
I_1 &= \int \frac{\mathrm{d}q}{2\pi}\frac{e^{iq\,\delta y}}{(\mathbf{k}^2 + q^2)^2} = \frac{e^{-|\mathbf{k}||\delta y|}(1 + |\mathbf{k}||\delta y|)}{4|\mathbf{k}|^3} \ , && (5.319)\\
I_2 &= \int \frac{\mathrm{d}q}{2\pi}\frac{q^2 e^{iq\,\delta y}}{(\mathbf{k}^2 + q^2)^2} = \frac{e^{-|\mathbf{k}||\delta y|}(1 - |\mathbf{k}||\delta y|)}{4|\mathbf{k}|} \ , && (5.320)
\end{aligned}$$

where we denote $q = k_n$. In the absence of a boundary, we can then write the partially Fourier transformed Green's function in the form

$$\tilde{G}_{\mu\nu}(\mathbf{k}, \delta y) \;\; = \;\; \delta_{\mu\nu} \frac{e^{-|\mathbf{k}||\delta y|}}{2|\mathbf{k}|} + \frac{\delta\eta \, e^{-|\mathbf{k}||\delta y|}}{4|\mathbf{k}|^3} \begin{pmatrix} -k_A k_B (1 + |\mathbf{k}||\delta y|) & 0 \\ 0 & |\mathbf{k}|^2 (1 - |\mathbf{k}||\delta y|) \end{pmatrix} \tag{5.321}$$

Recall that $\delta y = y - y'$. In the presence of a boundary, depending on our choice of absolute or relative boundary conditions, we can add or subtract the reflected Green's function $\tilde{G}_{\mu\nu}(\mathbf{k}, y + y')$. Let the resulting Green's function be $\tilde{G}^{(B)}_{\mu\nu}(\mathbf{k}, y, y')$. To make contact with the mixed QED theory considered in the text, we would like absolute boundary conditions, i.e. Dirichlet for A_n and Neumann for A_B. In this case, the partially transformed Green's function restricted to the boundary is

$$\tilde{G}^{(B)}_{\mu\nu}(\mathbf{k}, 0, 0) \;\; = \;\; \delta_{\mu\nu} \frac{1}{|\mathbf{k}|} + \frac{\delta\eta}{2|\mathbf{k}|^3} \begin{pmatrix} -k_A k_B & 0 \\ 0 & 0 \end{pmatrix} . \tag{5.322}$$

We can thus adopt a small gauge transformation to compensate for the additional $\mathcal{O}(g^2)\; k_A k_B$ dependence in the photon self-energy (5.229) when performing the Fourier transform (5.231).

Chapter 6

Displacement Operators and Constraints on Boundary Central Charges

This chapter is an edited version of my publication [5], written in collaboration with Christopher Herzog and Kristan Jensen.

The motivation of this chapter is to generalize the discussions in the previous chapter to consider other boundary charges in $d = 3$ and $d = 4$ CFTs. Let us begin with a quick review of the boundary trace anomalies including definitions of the anomaly coefficients $a_{(3d)}$, b, b_1, and b_2.

In $d = 3$ spacetime dimensions with a two-dimensional boundary, the anomaly only appears on the boundary, and it is given by [66]

$$\langle T^\mu{}_\mu \rangle^{d=3} = \frac{\delta(x_\perp)}{4\pi} \left(a_{(3d)} \mathring{R} + b \, \mathrm{tr}\, \hat{K}^2 \right) , \tag{6.1}$$

where $\delta(x_\perp)$ is a Dirac delta function with support on the boundary, and $\mathrm{tr}\, \hat{K}^2 = \mathrm{tr}\, K^2 - \frac{1}{2} K^2$; $\mathring{R}$ is the boundary Ricci scalar. For free fields, the values of these boundary charges were computed in the literature [116, 56, 117]: $a^{s=0,(D)}_{(3d)} = -\frac{1}{96}$, $a^{s=0,(R)}_{(3d)} = \frac{1}{96}$ and $a^{s=\frac{1}{2}}_{(3d)} = 0$, where $(D)/(R)$ denotes Dirichlet/Robin boundary condition. (In our notation, s is the spin of the free field.)

The structure becomes much richer in $d = 4$ CFTs. Dropping a regularization dependent term, the trace anomaly reads

$$\begin{aligned} \langle T^\mu{}_\mu \rangle^{d=4} &= \frac{1}{16\pi^2} \left(c W^2_{\mu\nu\lambda\rho} - a_{(4d)} E_4 \right) \\ &+ \frac{\delta(x_\perp)}{16\pi^2} \left(a_{(4d)} E_4^{(\mathrm{bry})} - b_1 \, \mathrm{tr}\, \hat{K}^3 - b_2 h^{\alpha\gamma} \hat{K}^{\beta\delta} W_{\alpha\beta\gamma\delta} \right) , \end{aligned} \tag{6.2}$$

where E_4 is the bulk Euler density in $d = 4$, and $W_{\mu\nu\rho\sigma}$ is the Weyl tensor. In the presence of a boundary, the boundary term of the Euler characteristic, $E^{(\text{bry})}$, is added in order to preserve the topological invariance. Let us here repeat and list the values of the b_1 charge for free fields: $b_1^{s=0,(D)} = \frac{2}{35}$ [71], $b_1^{s=0,(R)} = \frac{2}{45}$ [75], $b_1^{s=\frac{1}{2}} = \frac{2}{7}$ [117], $b_1^{s=1} = \frac{16}{35}$ [117].

The general strategy is similar to that adopted in the previous chapter: one simply looks at the correlation functions of the displacement operator in flat space. But there are several differences when compared with the computation of the b_2 charge. The first difference is that these b and b_1 boundary charges do not talk to bulk charges, while the b_2 structure is intimately related to the surface term generated from varying the bulk c-type anomaly effective action, as we considered in the previous chapter. The second difference is that in order to compute b_1 in $d = 4$, one has to look not at two-point functions but at a boundary three-point function.

We will in this chapter prove that the coefficients b and b_1 are related to two- and three-point functions of the displacement operator. The main results of this chapter are (6.11) and (6.20). We will conjecture that the $a_{(3d)}$ coefficient satisfies a related constraint (6.23), from which follows a lower bound (6.24) on $a_{(3d)}/b$. We will demonstrate that our relations hold for free theories.

6.1 Displacement Operator and General Relations

To set notation, let W be the generating functional for connected Green's functions. The stress tensor in Euclidean signature is

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g}}\frac{\delta W}{g^{\mu\nu}(x)} \ . \tag{6.3}$$

Let us first consider $d = 3$ CFTs with a boundary. Denote $\widetilde{W}$ as the anomalous part of W. The anomaly effective action in dimensional regularization is

$$\widetilde{W} = \frac{\mu^\epsilon}{\epsilon}\frac{1}{4\pi}\left(a_{(3d)}\int_{\partial\mathcal{M}} \mathring{R} + b\int_{\partial\mathcal{M}} \operatorname{tr}\hat{K}^2\right) \ . \tag{6.4}$$

Consider the special case where $\partial\mathcal{M}$ is almost the planar surface at $y = 0$, and can be described by a small displacement $\delta y(x^A)$, which is a function of the directions tangent to the boundary, denoted by x^A. In this situation, the normal vector is well-approximated by

$$n_\mu = (\partial_A \delta y, 1) \ . \tag{6.5}$$

The extrinsic curvature then becomes $K_{AB} = \partial_A\partial_B\delta y$, and we have

$$\int_{\partial\mathcal{M}} \operatorname{tr}\hat{K}^2 = \frac{1}{2}\int_{\partial\mathcal{M}} \delta y \mathring{\Box}^2 \delta y \ , \tag{6.6}$$

where $\mathring{\Box}^2 = \partial^A \partial_A$ acts only on the boundary. Correlation functions of the displacement operator $D^n(\mathbf{x})$ can be generated by varying W with respect to $\delta y(x^A)$. Note that diffeomorphisms act on both the metric and the embedding function $\delta y(x^A)$. As the effective action W is diffeomorphism invariant, there is a Ward identity that relates the stress tensor to the displacement operator, an integrated version of which in the flat limit becomes

$$T^{nn}|_{\partial\mathcal{M}} = D^n \ . \tag{6.7}$$

Because the displacement operator lives inside the boundary surface and we have conformal symmetry in this surface, the two point function is fixed up to a constant, which we call c_{nn}:

$$\langle D^n(\mathbf{x}) D^n(0) \rangle = \frac{c_{nn}}{\mathbf{x}^{2d}} \ . \tag{6.8}$$

(In the notation of the previous chapter, c_{nn} was called $\alpha(1)$ through its relation to the two point function of the stress tensor.) Replacing the expression (6.8) with a regularized version [122, 89] in the case of interest $d = 3$,

$$\langle D^n(\mathbf{x}) D^n(0) \rangle = \frac{c_{nn}^{(3d)}}{512} \mathring{\Box}^3 (\log \mu^2 \mathbf{x}^2)^2 \ , \tag{6.9}$$

the scale-dependent part is then

$$\mu \frac{\partial}{\partial \mu} \langle D^n(\mathbf{x}) D^n(0) \rangle = \pi \frac{c_{nn}^{(3d)}}{32} \mathring{\Box}^2 \delta(\mathbf{x}) \ . \tag{6.10}$$

Equating the scale dependent pieces yields

$$b = \frac{\pi^2}{8} c_{nn}^{(3d)} \ . \tag{6.11}$$

A similar calculation for the case of a codimension-two defect in four-dimensions was presented in ref. [103] in the context of entanglement entropy. Note that the b-charge can change under marginal deformations, although here we do not discuss a 3d example.

Next we consider $d = 4$. The constraint on the b_2 boundary charge was found in the previous chapter, and it reads

$$b_2 = \frac{2\pi^4}{15} c_{nn}^{(4d)} \ . \tag{6.12}$$

In flat space, the two-point function is not enough to constrain the b_1 boundary charge, since the related Weyl anomaly has a $\mathcal{O}(K^3)$ structure. Thus, we will need to consider the three-point function. The relevant anomaly effective action is

$$\widetilde{W}^{(b_1)} = \frac{b_1}{16\pi^2} \frac{\mu^\epsilon}{\epsilon} \int_{\partial\mathcal{M}} \operatorname{tr} \hat{K}^3 \ . \tag{6.13}$$

We again consider $\partial\mathcal{M}$ to be nearly flat and described by a small displacement, $\delta y(x^A)$. Approximating the normal vector by $n_\mu = (\partial_A \delta y, 1)$, we obtain

$$\begin{aligned}\int_{\partial\mathcal{M}} \mathrm{tr}\, \hat{K}^3 = \int_{\partial\mathcal{M}} \Big(&\mathrm{tr}\left[(\partial_A\partial_B\delta y)^3\right] \\ &- (\mathring{\Box}\delta y)\,\mathrm{tr}\left[(\partial_A\partial_B\delta y)^2\right] + \frac{2}{9}(\mathring{\Box}\delta y)^3\Big) \ .\end{aligned} \tag{6.14}$$

We will relate this b-charge with the displacement operator three-point function defined by

$$\langle D^n(\mathbf{x}) D^n(\mathbf{x}') D^n(\mathbf{0})\rangle = \frac{c_{nnn}}{|\mathbf{x}|^4|\mathbf{x}'|^4|\mathbf{x}-\mathbf{x}'|^4} \ , \tag{6.15}$$

where c_{nnn} is a constant. The full structure of the stress tensor three-point function with a boundary has not been studied yet. But, as mentioned earlier, to constrain these boundary charges one can simply look at the purely normal-normal component of the stress-tensor correlation functions that represent the displacement operator contributions.

While it is not obvious how to proceed in position space, we note that the Fourier transform of the three-point function of operators O_1, O_2 and O_3 is generally [139, 140]

$$C_{123}\int_0^\infty dx\, x^\alpha \prod_{j=1}^{3} p_j^{\beta_j} K_{\beta_j}(p_j x) \ , \tag{6.16}$$

where $K_{\beta_j}(x)$ denotes the modified Bessel function of the second kind, and $\alpha = \frac{\delta}{2}-1, \beta_j = \Delta_j - \frac{\delta}{2}$; Δ_j is the conformal dimension of operator O_j and δ is the dimension of the CFT. In this case, we are interested in the CFT living on the boundary, so $\delta = 3$ while the scaling dimension of the displacement operator is $\Delta_j = 4$. Taking c_{123} as the corresponding coefficient of the position space three-point function, one has [139]

$$c_{nnn} = \frac{105}{\sqrt{2}\pi^{5/2}} C_{nnn} \ . \tag{6.17}$$

The $1/x$ term in a small x expansion of the integrand gives rise to a logarithm in the position space three-point function and a corresponding anomalous scale dependence. Observe the $1/x$ term is

$$\frac{3\pi^{3/2}}{32\sqrt{2}x}\Big(p_1^6+p_2^6+p_3^6-p_1^2p_2^4-p_1^2p_3^4-p_2^2p_1^4-p_2^2p_3^4-p_3^2p_1^4-p_3^2p_2^4-\frac{2}{3}p_1^2p_2^2p_3^2\Big) \ . \tag{6.18}$$

Through integration by parts along the boundary, the above expression can be rewritten as

$$\frac{9\pi^{3/2}}{4\sqrt{2}x}\Big((p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_1) - p_1^2(p_2\cdot p_3)^2 + \frac{2}{9}p_1^2p_2^2p_3^2\Big) \ . \tag{6.19}$$

The result matches exactly the derivative form (6.14) computed from the b_1 boundary trace anomaly. Including a factor $\frac{1}{3!}$ coming from varying with respect to δy three times, we obtain $b_1 = \frac{1}{3!}\cdot 16\pi^2\left(\frac{9\pi^{3/2}}{4\sqrt{2}}\right)\left(\frac{\sqrt{2}\pi^{5/2}}{105}\right)c_{nnn}$, which gives

$$b_1 = \frac{2\pi^6}{35} c_{nnn} \ . \tag{6.20}$$

This boundary charge in $d = 4$ can depend on marginal interactions. In particular, if the charge b_2 of the mixed-dimensional quantum electrodynamics (QED) depends on the marginal interactions (see the previous chapter), so does b_1.

6.2 Conjecture for $a_{(3d)}$

As discussed in the previous chapter, we can write down expressions for the near-boundary limit of the stress-tensor two-point function:

$$\langle T_{\mu\nu}(\mathbf{x}, y) T_{\rho\sigma}(\mathbf{0}, y')\rangle = A_{\mu\nu,\rho\sigma}(\mathbf{x}, y, y') \frac{1}{|\mathbf{x}|^{2d}} , \tag{6.21}$$

where

$$\begin{aligned} A_{nn,nn}(\mathbf{x}, y, y') &= \alpha(v) , \\ A_{nA,nB}(\mathbf{x}, y, y') &= -\gamma(v) I_{AB}(\mathbf{x}, y, y'), \\ A_{AB,CD}(\mathbf{x}, y, y') &= \alpha(v) \frac{d}{d-1} I^{(d)}_{AB,CD} \\ &\quad + \left(2\epsilon(v) - \frac{d}{d-1}\alpha(v)\right) I^{(d-1)}_{AB,CD} , \end{aligned} \tag{6.22}$$

where $I_{AB}(x) = \delta_{AB} - 2\frac{x_A x_B}{x^2}$ and $I^{(d)}_{AB,CD} = \frac{1}{2}(I_{AC} I_{BD} + I_{AD} I_{BC}) - \frac{1}{d}\delta_{AB}\delta_{CD}$. The quantity v is a cross-ratio $v = \frac{(x-x')^2}{(x-x')^2 + 4yy'}$, which behaves as $\sim 1 - \frac{4yy'}{|\mathbf{x}|^2}$ near the boundary at $v = 1$. The functions α, γ and ϵ are related to each other by two differential constraints. Conservation of the stress tensor at the boundary, conformal invariance, and unitarity together impose that γ smoothly vanishes as $v \to 1$, while α is smooth, and ϵ can blow up as $(1-v)^{\delta-1}$ for a small anomalous dimension $\delta > 0$. Both α and ϵ may have $O(1-v)^0$ terms, which we refer to as $\alpha(1)$ and $\epsilon(1)$. (Note the relation between $\alpha(v)$ and the D^n two-point function, $\alpha(1) = c_{nn}$.)

The symmetries also allow for a boundary stress tensor which would only arise from decoupled boundary degrees of freedom. If present it appears as a distributional term in the two-point function $C I^{(d-1)}_{AB,CD}\delta(y)\delta(y')$.

We conjecture that the boundary anomaly coefficient $a_{(3d)}$ is a linear combination of $\alpha(1)$, $\epsilon(1)$, and C. The dependence on C is already fixed by the argument relating the trace anomaly of a two-dimensional CFT to the two-point function of its stress tensor. More precisely, $c_{(2d)} = 2\pi C$, where $c_{(2d)}$ is the 2d central charge in the Euler anomaly $\langle T^A_A\rangle = \delta(y)\frac{c_{(2d)}}{24\pi}\mathring{R}$. The coefficient C vanishes for a theory of free $3d$ scalars and for free $3d$ fermions since these theories do not have extra decoupled boundary degrees of freedom. We fix the dependence on $\alpha(1)$ and $\epsilon(1)$ by the known values for the conformal scalar with Dirichlet and Robin boundary conditions, giving

$$a_{(3d)} = \frac{\pi^2}{9}\left(\epsilon(1) - \frac{3}{4}\alpha(1) + 3C\right) . \tag{6.23}$$

Note this conjecture gives the correct result for free fermions, reproducing $a_{(3d)}^{s=\frac{1}{2}}=0$.

In a general interacting bCFT we suspect only $\alpha(1)$ to be nonzero for the following reason. Interactions coupling boundary degrees of freedom to the bulk ought to lead to a unique stress tensor, leading to $C=0$. Meanwhile, $\epsilon(1)$ corresponds to a dimension-3 boundary operator appearing in the boundary operator product expansion of T_{AB}, but the boundary conformal symmetry does not guarantee the existence of such an operator.

Reflection positivity means that the functions $\alpha(v)$ and $\epsilon(v)$ are non-negative, as discussed in the previous chapter. The coefficient C is also non-negative. If $\epsilon(v)$ is regular near the boundary, then $\epsilon(1)$ is non-negative, and comparing with the new result (6.11) for b, we obtain the bounds

$$\text{d}=3\ \ \text{bCFTs}:\quad \frac{a_{(3d)}}{b}\geq-\frac{2}{3}\,,\quad (b\geq 0)\,. \tag{6.24}$$

These bounds recall the Hofman-Maldacena [141] bounds on $d=4$ bulk central charges. However, if $\epsilon(v)$ is singular near the boundary, then there is no constraint on the sign of $\epsilon(1)$, and thus, no definite bound on $a_{(3d)}$ charge. We note that $a_{(3d)}$ and b have been computed in a bottom-up holographic model [142] and their ratio falls below our proposed bound.

6.3 Two- and Three-Point Functions in Free Theories

We would like to verify the general relations (6.11) and (6.20) in free theories, including a conformal scalar, a Dirac fermion and, in $d=4$, Maxwell theory. The stress tensor two-point functions with a planar boundary for the scalar and fermion were already considered in ref. [87]. In the previous chapter, we have computed the two-point functions for a Maxwell field. We will list the relevant two-point function results for completeness, and consider three-point functions with a boundary in free theories. These latter results are, to our knowledge, new.

Considering first a vector of scalar fields, i.e $\phi\to\phi^a$ (the index a will be suppressed), we introduce complementary projectors $\Pi_\pm$ satisfying $\Pi_++\Pi_-=\mathbb{1}$ and $\Pi_\pm^2=\Pi_\pm$. The boundary conditions are $\partial_n(\Pi_+\phi)|_{y=0}=0$ and $\Pi_-\phi|_{y=0}=0$. The scalar displacement operator is

$$T_{nn}=(\partial_n\phi)^2-\frac{1}{4}\frac{1}{d-1}\left((d-2)\partial_n^2+\Box\right)\phi^2\ , \tag{6.25}$$

which is the boundary limit of the normal-normal component of the improved stress tensor. The two-point function of the scalar field can be found using the image method:

$$\langle\phi(x)\phi(x')\rangle=\kappa\Big(\frac{\mathbb{1}}{|x-x'|^{d-2}}+\frac{\chi}{((\mathbf{x}-\mathbf{x}')^2+(y+y')^2)^{(d-2)/2}}\Big)\ , \tag{6.26}$$

where the parameter $\chi=\Pi_+-\Pi_-$ is determined by boundary conditions. We have adopted the normalization $\kappa=\frac{1}{(d-2)\mathrm{Vol}(S^{d-1})}$ where $\mathrm{Vol}(S^{d-1})=\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. Note $\chi^2=\mathbb{1}$, and that an eigenvalue of χ is 1 for Neumann and -1 for Dirichlet boundary conditions.

To keep the expressions simple, we will focus on the displacement operator two-point function in $d = 3$ and the three-point function in $d = 4$. These two quantities are required in computing the boundary central charges from the relations (6.11) and (6.20).

A straightforward application of Wick's theorem gives

$$\langle D^n(\mathbf{x})D^n(\mathbf{0})\rangle_{3d}^{s=0} = \frac{\operatorname{tr}(\mathbb{1})}{8\pi^2\mathbf{x}^6} \ , \tag{6.27}$$

$$\langle D^n(\mathbf{x})D^n(\mathbf{x}')D^n(\mathbf{0})\rangle_{4d}^{s=0} = \frac{1}{9\pi^6}\frac{8\operatorname{tr}(\mathbb{1}) - \operatorname{tr}(\chi)}{|\mathbf{x}|^4|\mathbf{x}'|^4|\mathbf{x}-\mathbf{x}'|^4} \ . \tag{6.28}$$

The result (6.27) implies that the b boundary charge (in $d = 3$) does not depend on boundary conditions for a free scalar. Indeed, using the relation (6.20), we recover the known value of the b charge for a $d = 3$ free scalar, $b = \frac{1}{64}$. On the other hand, clearly b_1 is sensitive to boundary conditions through the $\operatorname{tr}(\chi)$. Using the relation (6.20), we can verify that b_1 is $\frac{2}{35}$ for a Dirichlet scalar and $\frac{2}{45}$ for a Neumann scalar.

Next we consider a Dirac fermion. In Minkowski (mostly plus) signature, $\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$. The fermion's displacement operator and two-point function are

$$T_{nn} \ = \ \frac{i}{2}\left(\dot{\bar{\psi}}\gamma_n\psi - \bar{\psi}\gamma_n\dot{\psi}\right) \ , \quad (\dot{\psi} \equiv \partial_n\psi) \tag{6.29}$$

$$\langle\psi(x)\bar{\psi}(x')\rangle \ = \ -\kappa_f\Big(\frac{i\gamma\cdot(x-x')}{|x-x'|^d} + \chi\frac{i\gamma\cdot(\bar{x}-x')}{|\bar{x}-x'|^d}\Big) \ , \tag{6.30}$$

where $\bar{x} = (-y, \mathbf{x})$ and $\kappa_f = 1/\operatorname{Vol}(S^{d-1})$ and $\bar{\psi} = \psi^\dagger\gamma^0$. The χ parameter satisfies

$$\chi\gamma_n = -\gamma_n\bar{\chi} \ , \quad \chi\gamma_A = \gamma_A\bar{\chi} \ , \quad \chi^2 = \bar{\chi}^2 = \mathbb{1} \ , \tag{6.31}$$

where $\bar{\chi} = \gamma^0\chi^\dagger\gamma^0$. Focusing on the fermion displacement operator two-point function in $d = 3$ and the three-point function in $d = 4$, we find

$$\langle D(\mathbf{x})D(\mathbf{0})\rangle_{3d}^{s=\frac{1}{2}} \ = \ \frac{3}{16\pi^2}\frac{\operatorname{tr}_\gamma(\mathbb{1})}{\mathbf{x}^6} \ , \tag{6.32}$$

$$\langle D(\mathbf{x})D(\mathbf{x}')D(\mathbf{0})\rangle_{4d}^{s=\frac{1}{2}} \ = \ \frac{5}{4\pi^6}\frac{\operatorname{tr}_\gamma(\mathbb{1})}{\mathbf{x}^4\mathbf{x}'^4(\mathbf{x}-\mathbf{x}')^4} \ , \tag{6.33}$$

where $\operatorname{tr}_\gamma(\mathbb{1})$ depends on the Clifford algebra one uses; we will take $\operatorname{tr}_\gamma(\mathbb{1}) = 2^{\lfloor d/2\rfloor}$. As $\chi^2 = \mathbb{1}$, the boundary dependence drops out of these two- and three-point functions. We can again verify the relations (6.11) and (6.20) for the fermion.

Finally, we consider a Maxwell field in Feynman gauge. As the field in $d = 3$ is not conformal, we focus on the $d = 4$ case. The displacement operator is

$$T_{nn} = \frac{1}{2}F_{nA}F_n{}^A - \frac{1}{4}F_{AB}F^{AB} \ , \tag{6.34}$$

and the gauge field two-point function is

$$\langle A_\mu(x)A^\nu(x')\rangle = \kappa\Big(\frac{\delta_\mu^\nu}{(x-x')^2} + \frac{\chi_\mu^\nu}{((\mathbf{x}-\mathbf{x}')^2 + (y+y')^2)^2}\Big) \ . \tag{6.35}$$

The χ_μ^ν parameter determines the boundary condition; it is equal to δ_μ^ν up to a sign. For gauge fields one can consider the absolute boundary condition where the normal component of the field strength is zero, which gives $\partial_n A_A = 0$ and $A_n = 0$, or the relative boundary condition where $A_A = 0$ which gives $\partial_n A^n = 0$ when recalling the gauge fixing. See the previous chapter for more details. We find

$$\langle D^n(\mathbf{x}) D^n(\mathbf{x}') D^n(\mathbf{0}) \rangle_{4d}^{s=1} = \frac{512\kappa^3}{|\mathbf{x}|^4 |\mathbf{x}'|^4 |\mathbf{x}-\mathbf{x}'|^4} , \tag{6.36}$$

independent of the choice of boundary conditions. From the relation (6.20) we recover the value of b_1 charge for the $d = 4$ Maxwell field with a boundary.

6.4 Discussion

We presented new results for the boundary terms in the trace anomaly for CFTs in 3d and 4d. By relating b (6.11), b_1 (6.20), b_2 (6.12), and $a_{(3d)}$ (6.23) to two- and three-point functions of the displacement operator in flat space, these results make the boundary coefficients more straightforward to compute. Ultimately, perhaps building on the bound (6.24), we hope that a classification scheme for bCFT can be organized around these coefficients.

Let us conclude by listing some open problems:

- What can one say about these boundary charges for the maximally supersymmetric Yang-Mills theory in 4d in the presence of a boundary?
- Search for stronger bounds on boundary charges, building perhaps on the reflection positivity.
- Understand how these 4d boundary charges behave under boundary RG flow.
- Compute directly the b_1 and b_2 charges for mixed QED in curved space.
- Search for new interacting bCFTs in 4 and other dimensions.
- Consider the stress tensor two-point function with a codimension-2 surface. Such geometry has an important relationship to quantum entanglement.
- Classify the structure of three-point functions in bCFTs.

Clearly, there is much to be done.

Bibliography

[1] C. P. Herzog and K.-W. Huang, "Stress Tensors from Trace Anomalies in Conformal Field Theories," Phys. Rev. D 87, 081901 (2013) [arXiv:1301.5002 [hep-th]].

[2] K.-W. Huang, "Weyl Anomaly Induced Stress Tensors in General Manifolds," Nucl. Phys. B 879, 370 (2014) [arXiv:1308.2355 [hep-th]].

[3] C. P. Herzog, K.-W. Huang and K. Jensen, "Universal Entanglement and Boundary Geometry in Conformal Field Theory," JHEP 1601, 162 (2016) [arXiv:1510.00021 [hep-th]].

[4] C. P. Herzog and K.-W. Huang, "Boundary Conformal Field Theory and a Boundary Central Charge," JHEP 1710, 189 (2017) [arXiv:1707.06224 [hep-th]].

[5] C. Herzog, K.-W. Huang and K. Jensen, "Displacement Operators and Constraints on Boundary Central Charges," Phys. Rev. Lett. 120, no. 2, 021601 (2018) [arXiv:1709.07431 [hep-th]].

[6] C. P. Herzog, K.-W. Huang and R. Vaz, "Linear Resistivity from Non-Abelian Black Holes," JHEP 1411, 066 (2014) [arXiv:1405.3714 [hep-th]].

[7] K.-W. Huang, "Central Charge and Entangled Gauge Fields," Phys. Rev. D 92, no. 2, 025010 (2015) [arXiv:1412.2730 [hep-th]].

[8] K.-W. Huang, "Boundary Anomalies and Correlation Functions," JHEP 1608, 013 (2016) [arXiv:1604.02138 [hep-th]].

[9] C. P. Herzog and K.-W. Huang, "Boundary Fluctuations and A Reduction Entropy," Phys. Rev. D 95, no. 2, 021901 (2017) [arXiv:1610.08970 [hep-th]].

[10] K.-W. Huang, R. Roiban and A. A. Tseytlin, "Self-dual 6d 2-form fields coupled to non-abelian gauge field: quantum corrections," JHEP 1806, 134 (2018) [arXiv:1804.05059 [hep-th]].

[11] C. P. Herzog, K.-W. Huang, I. Shamir and J. Virrueta, "Superconformal Models for Graphene and Boundary Central Charges," arXiv:1807.01700 [hep-th].

[12] A. B. Zamolodchikov, "Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory," JETP Lett. 43, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. 43, 565 (1986)].

[13] J. L. Cardy, "Is There a c Theorem in Four-Dimensions?," Phys. Lett. B 215, 749 (1988).

[14] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions," JHEP 1112, 099 (2011) [arXiv:1107.3987 [hep-th]].

[15] H. Elvang, D. Z. Freedman, L.-Y. Hung, M. Kiermaier, R. C. Myers and S. Theisen, "On renormalization group flows and the a-theorem in 6d," JHEP 1210, 011 (2012) [arXiv:1205.3994 [hep-th]].

[16] I. R. Klebanov and A. A. Tseytlin, "Entropy of near extremal black p-branes," Nucl. Phys. B 475, 164 (1996) [hep-th/9604089].

[17] M. Henningson and K. Skenderis, "The Holographic Weyl anomaly," JHEP 9807, 023 (1998) [hep-th/9806087].

[18] H. W. J. Blöte, J. L. Cardy and M. P. Nightingale, "Conformal Invariance, the Central Charge, and Universal Finite Size Amplitudes at Criticality," Phys. Rev. Lett. 56, 742 (1986).

[19] L. S. Brown and J. P. Cassidy, "Stress Tensors and their Trace Anomalies in Conformally Flat Space-Times," Phys. Rev. D 16, 1712 (1977).

[20] S. Deser, M. J. Duff and C. J. Isham, "Nonlocal Conformal Anomalies," Nucl. Phys. B 111, 45 (1976).

[21] D. N. Page, "Thermal Stress Tensors in Static Einstein Spaces," Phys. Rev. D 25, 1499 (1982).

[22] A. Cappelli and A. Coste, "On The Stress Tensor Of Conformal Field Theories In Higher Dimensions," Nucl. Phys. B 314, 707 (1989).

[23] A. Schwimmer and S. Theisen, "Diffeomorphisms, anomalies and the Fefferman-Graham ambiguity," JHEP 0008, 032 (2000) [hep-th/0008082].

[24] S. Deser and A. Schwimmer, "Geometric classification of conformal anomalies in arbitrary dimensions," Phys. Lett. B 309, 279 (1993) [hep-th/9302047];

[25] L. Bonora, P. Pasti and M. Bregola, "Weyl Cocycles," Class. Quant. Grav. 3, 635 (1986).

[26] J. Erdmenger, "Conformally covariant differential operators: Properties and applications," Class. Quant. Grav. 14, 2061 (1997) [hep-th/9704108].

[27] F. Bastianelli, S. Frolov and A. A. Tseytlin, "Conformal anomaly of (2,0) tensor multiplet in six-dimensions and AdS / CFT correspondence," JHEP 0002, 013 (2000) [hep-th/0001041].

[28] N. Boulanger and J. Erdmenger, "A Classification of local Weyl invariants in D=8," Class. Quant. Grav. 21, 4305 (2004) [hep-th/0405228].

[29] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge University Press (1982).

[30] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," Commun. Math. Phys. 208, 413 (1999) [hep-th/9902121].

[31] R. Emparan, C. V. Johnson and R. C. Myers, "Surface terms as counterterms in the AdS / CFT correspondence," Phys. Rev. D 60, 104001 (1999) [hep-th/9903238].

[32] S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of spacetime and renormalization in the AdS / CFT correspondence," Commun. Math. Phys. 217, 595 (2001) [hep-th/0002230].

[33] A. M. Awad and C. V. Johnson, "Higher dimensional Kerr - AdS black holes and the AdS / CFT correspondence," Phys. Rev. D 63, 124023 (2001) [hep-th/0008211].

[34] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. 323, 183 (2000) [hep-th/9905111].

[35] T. Maxfield and S. Sethi, "The Conformal Anomaly of M5-Branes," JHEP 1206, 075 (2012) [arXiv:1204.2002 [hep-th]].

[36] L. H. Ford, "Quantum Vacuum Energy in a Closed Universe," Phys. Rev. D 14, 3304 (1976).

[37] M. Marino, "Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories," J. Phys. A 44, 463001 (2011) [arXiv:1104.0783 [hep-th]].

[38] G. W. Gibbons, M. J. Perry and C. N. Pope, "AdS/CFT Casimir energy for rotating black holes," Phys. Rev. Lett. 95, 231601 (2005) [hep-th/0507034].

[39] Y. Nakayama, "Does anomalous violation of null energy condition invalidate holographic c-theorem?," Phys. Lett. B 720, 265 (2013) [arXiv:1211.4628 [hep-th]].

[40] H. Casini, M. Huerta and R. C. Myers, "Towards a derivation of holographic entanglement entropy," JHEP 1105, 036 (2011) [arXiv:1102.0440 [hep-th]].

[41] F. Bastianelli, G. Cuoghi and L. Nocetti, "Consistency conditions and trace anomalies in six-dimensions," Class. Quant. Grav. 18, 793 (2001) [hep-th/0007222].

[42] M. R. Brown, A. C. Ottewill, D. N. Page "Conformally invariant quantum field theory in static Einstein space-times" Physical Review D, 1986

[43] T. J. Osborne and M. A. Nielsen, "Entanglement in a simple quantum phase transition," Phys. Rev. A 66, 032110 (2002) [quant-ph/0202162].

[44] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, "Entanglement in quantum critical phenomena," Phys. Rev. Lett. 90, 227902 (2003) [quant-ph/0211074].

[45] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, "Quantum source of entropy for black holes," Phys. Rev. D 34, 373 (1986).

[46] M. Srednicki, "Entropy and area," Phys. Rev. Lett. 71, 666 (1993) [hep-th/9303048].

[47] H. Casini and M. Huerta, "A c-theorem for the entanglement entropy," J. Phys. A 40, 7031 (2007) [cond-mat/0610375].

[48] H. Casini and M. Huerta, "On the RG running of the entanglement entropy of a circle," Phys. Rev. D 85, 125016 (2012) [arXiv:1202.5650 [hep-th]].

[49] S. N. Solodukhin, "Entanglement entropy, conformal invariance and extrinsic geometry," Phys. Lett. B 665, 305 (2008) [arXiv:0802.3117 [hep-th]].

[50] P. Buividovich and M. Polikarpov, "Entanglement entropy in gauge theories and the holographic principle for electric strings," Phys. Lett. B 670, 141 (2008) [arXiv:0806.3376 [hep-th]].

[51] H. Casini, M. Huerta and J. Rosabal, "Remarks on entanglement entropy for gauge fields," Phys. Rev. D 89, 085012 (2014) [arXiv:1312.1183 [hep-th]].

[52] W. Donnelly and A. C. Wall, "Entanglement entropy of electromagnetic edge modes," Phys. Rev. Lett. 114, no. 11, 111603 (2015) [arXiv:1412.1895 [hep-th]].

[53] D. Fursaev, "Quantum Entanglement on Boundaries," JHEP 1307 119 (2013) [arXiv:1305.2335 [hep-th]].

[54] J. Polchinski, *String Theory: Volume 1*, Cambridge University Press, 1998.

[55] M. R. Brown and A. C. Ottewill, "Effective actions and conformal transformations," Phys. Rev. D 31, 2514 (1985).

[56] K. Jensen and A. O'Bannon, "Constraint on Defect and Boundary Renormalization Group Flows," Phys. Rev. Lett. **116**, no. 9, 091601 (2016) [arXiv:1509.02160 [hep-th]].

[57] V. Korepin and B.-Q. Jin, "Quantum Spin Chain, Toeplitz Determinants and Fisher-Hartwig Formula," J. Stat. Phys. 116, 79 (2004) [quant-ph/0304108].

[58] P. Calabrese and J. L. Cardy, "Entanglement entropy and quantum field theory," J. Stat. Mech. 0406, P06002 (2004) [hep-th/0405152].

[59] Z. Komargodski, "The Constraints of Conformal Symmetry on RG Flows," JHEP 1207, 069 (2012) [arXiv:1112.4538 [hep-th]].

[60] J. Wess and B. Zumino, "Consequences of anomalous Ward identities," Phys. Lett. B 37, 95 (1971).

[61] A. Schwimmer and S. Theisen, "Spontaneous Breaking of Conformal Invariance and Trace Anomaly Matching," Nucl. Phys. B 847, 590 (2011) [arXiv:1011.0696 [hep-th]].

[62] S. M. Christensen and M. J. Duff, "Quantum Gravity in Two + ϵ Dimensions," Phys. Lett. B 79, 213 (1978).

[63] C. Holzhey, F. Larsen and F. Wilczek, "Geometric and renormalized entropy in conformal field theory," Nucl. Phys. B 424, 443 (1994) [hep-th/9403108].

[64] S. N. Solodukhin, "The Conical singularity and quantum corrections to entropy of black hole," Phys. Rev. D 51, 609 (1995) [hep-th/9407001].

[65] V. P. Frolov, W. Israel and S. N. Solodukhin, "On one loop quantum corrections to the thermodynamics of charged black holes," Phys. Rev. D 54, 2732 (1996) [hep-th/9602105].

[66] C. R. Graham and E. Witten, "Conformal anomaly of submanifold observables in AdS/CFT correspondence," Nucl. Phys. B 546, 52 (1999) [hep-th/9901021].

[67] A. Schwimmer and S. Theisen, "Entanglement Entropy, Trace Anomalies and Holography," Nucl. Phys. B 801, 1 (2008) [arXiv:0802.1017 [hep-th]].

[68] T. Eguchi, P. B. Gilkey and A. J. Hanson, "Gravitation, Gauge Theories and Differential Geometry," Phys. Rept. 66, 213 (1980).

[69] K. Jensen, R. Loganayagam and A. Yarom, "Anomaly inflow and thermal equilibrium," JHEP 1405, 134 (2014) [arXiv:1310.7024].

[70] O. Miskovic and R. Olea, "Counterterms in Dimensionally Continued AdS Gravity," JHEP 0710, 028 (2007) [arXiv:0706.4460 [hep-th]].

[71] J. Melmed, "Conformal invariance and the regularised one-loop effective action," J. Phys. A: Math. Gen. 21, L1131 (1988).

[72] J. S. Dowker and J. P. Schofield, "Conformal Transformations and the Effective Action in the Presence of Boundaries," J. Math. Phys. 31, 808 (1990).

[73] D. M. McAvity and H. Osborn, "Quantum field theories on manifolds with curved boundaries: Scalar fields," Nucl. Phys. B 394, 728 (1993) [cond-mat/9206009].

[74] T. P. Branson, P. B. Gilkey and D. V. Vassilevich, "The Asymptotics of the Laplacian on a manifold with boundary. 2," Boll. Union. Mat. Ital. 11B, 39 (1997) [hep-th/9504029].

[75] I. G. Moss, "Boundary terms in the heat kernel expansion," Class. Quant. Grav. 6, 759 (1989).

[76] K. Jensen and A. O'Bannon, "Holography, Entanglement Entropy, and Conformal Field Theories with Boundaries or Defects," Phys. Rev. D 88, 106006 (2013) [arXiv:1309.4523].

[77] J. Estes, K. Jensen, A. O'Bannon, E. Tsatis and T. Wrase "On Holographic Defect Entropy," JHEP 1405, 084 (2014) [arXiv:1403.6475].

[78] D. Gaiotto, "Boundary F-maximization", [arXiv:1403.8052].

[79] R. C. Myers, "Higher Derivative Gravity, Surface Terms and String Theory," Phys. Rev. D 36, 392 (1987).

[80] P. Candelas and J. S. Dowker, "Field Theories On Conformally Related Space-times: Some Global Considerations," Phys. Rev. D 19, 2902 (1979).

[81] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen and D. Martelli, "The Casimir Energy in Curved Space and its Supersymmetric Counterpart," JHEP 07, 043 (2015) [arXiv:1503.05537 [hep-th]].

[82] R. C. Myers and A. Sinha, "Holographic c-theorems in arbitrary dimensions," JHEP 1101, 125 (2011) [arXiv:1011.5819 [hep-th]].

[83] R. C. Myers and A. Sinha, "Seeing a c-theorem with holography," Phys. Rev. D 82, 046006 (2010) [arXiv:1006.1263 [hep-th]].

[84] The On-line Encyclopedia of Integer Sequences, http://oeis.org.

[85] D. Fursaev, "Conformal anomalies of CFT's with boundaries," JHEP 1512, 112 (2015) [arXiv:1510.01427 [hep-th]].

[86] J. Cardy, "Conformal invariance and surface critical behavior," Nucl. Phys. B 240, 4, (1984)

[87] D. M. McAvity and H. Osborn, "Energy momentum tensor in conformal field theories near a boundary," Nucl. Phys. B 406, 655 (1993) [hep-th/9302068].

[88] D. M. McAvity and H. Osborn, "Conformal field theories near a boundary in general dimensions," Nucl. Phys. B 455, 522 (1995) [cond-mat/9505127].

[89] H. Osborn and A. C. Petkou, "Implications of conformal invariance in field theories for general dimensions," Annals Phys. 231, 311 (1994) [hep-th/9307010].

[90] S. N. Solodukhin, "Boundary terms of conformal anomaly," Phys. Lett. B 752, 131 (2016) [arXiv:1510.04566 [hep-th]].

[91] C.-S. Chu, R.-X. Miao and W.-Z. Guo, "On New Proposal for Holographic BCFT," JHEP 1704, 089 (2017) [arXiv:1701.07202 [hep-th]].

[92] R.-X. Miao, C.-S. Chu and W.-Z. Guo, "New proposal for a holographic boundary conformal field theory," Phys. Rev. D 96, no. 4, 046005 (2017) [arXiv:1701.04275 [hep-th]].

[93] A. Faraji Astaneh and S. N. Solodukhin, "Holographic calculation of boundary terms in conformal anomaly," Phys. Lett. B 769, 25 (2017) [arXiv:1702.00566 [hep-th]].

[94] J. Erdmenger and H. Osborn, "Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions," Nucl. Phys. B 483, 431 (1997) [hep-th/9605009].

[95] P. Liendo, L. Rastelli and B. C. van Rees, "The Bootstrap Program for Boundary CFT_d," JHEP 1307, 113 (2013) [arXiv:1210.4258 [hep-th]].

[96] E. Eisenriegler, M. Krech and S. Dietrich, "Absence of hyperuniversality in critical films," Phys. Rev. Lett. 70, 619 (1993)

[97] F. Gliozzi, P. Liendo, M. Meineri and A. Rago, "Boundary and Interface CFTs from the Conformal Bootstrap," JHEP 1505, 036 (2015) [arXiv:1502.07217 [hep-th]].

[98] S. Teber, "Electromagnetic current correlations in reduced quantum electrodynamics," Phys. Rev. D 86, 025005 (2012) [arXiv:1204.5664 [hep-ph]].

[99] S. Teber, "Two-loop fermion self-energy and propagator in reduced $\mathrm{QED}_{3,2}$," Phys. Rev. D 89, no. 6, 067702 (2014) [arXiv:1402.5032 [hep-ph]].

[100] D. Gaiotto, S. Gukov and N. Seiberg, "Surface Defects and Resolvents," JHEP **1309**, 070 (2013) [arXiv:1307.2578 [hep-th]].

[101] O. J. C. Dias, G. T. Horowitz, N. Iqbal and J. E. Santos, "Vortices in holographic superfluids and superconductors as conformal defects," JHEP 1404, 096 (2014) [arXiv:1311.3673 [hep-th]].

[102] M. Billó, V. Gonçalves, E. Lauria and M. Meineri, "Defects in conformal field theory," JHEP 1604, 091 (2016) [arXiv:1601.02883 [hep-th]].

[103] L. Bianchi, M. Meineri, R. C. Myers and M. Smolkin, "Rényi entropy and conformal defects," JHEP 1607, 076 (2016) [arXiv:1511.06713 [hep-th]].

[104] S. Balakrishnan, S. Dutta and T. Faulkner, "Gravitational dual of the Rényi twist displacement operator," Phys. Rev. D 96, no. 4, 046019 (2017) [arXiv:1607.06155 [hep-th]].

[105] N. Drukker, D. Martelli and I. Shamir, "The energy-momentum multiplet of supersymmetric defect field theories," JHEP 1708, 010 (2017) [arXiv:1701.04323 [hep-th]].

[106] E. V. Gorbar, V. P. Gusynin and V. A. Miransky, "Dynamical chiral symmetry breaking on a brane in reduced QED," Phys. Rev. D 64, 105028 (2001) [hep-ph/0105059].

[107] A. V. Kotikov and S. Teber, "Two-loop fermion self-energy in reduced quantum electrodynamics and application to the ultrarelativistic limit of graphene," Phys. Rev. D 89, no. 6, 065038 (2014) [arXiv:1312.2430 [hep-ph]].

[108] A. V. Kotikov and S. Teber, "Critical behaviour of reduced $\mathrm{QED}_{4,3}$ and dynamical fermion gap generation in graphene," Phys. Rev. D 94, no. 11, 114010 (2016) [arXiv:1610.00934 [hep-th]].

[109] D. T. Son, "Quantum critical point in graphene approached in the limit of infinitely strong Coulomb interaction," Phys. Rev. B 75, no. 23, 235423 (2007) [cond-mat/0701501 [cond-mat.str-el]].

[110] M. A. H. Vozmediano, "Renormalization group aspects of graphene," Phil. Trans. Roy. Soc. Lond. A 369, 2625 (2011) [arXiv:1010.5057 [cond-mat.str-el]].

[111] D. T. Son, "Is the Composite Fermion a Dirac Particle?," Phys. Rev. X 5, no. 3, 031027 (2015) [arXiv:1502.03446 [cond-mat.mes-hall]].

[112] W.-H. Hsiao and D. T. Son, "Duality and universal transport in mixed-dimension electrodynamics," Phys. Rev. B 96, no. 7, 075127 (2017) [arXiv:1705.01102 [cond-mat.mes-hall]].

[113] C. P. Herzog, P. Kovtun, S. Sachdev and D. T. Son, "Quantum critical transport, duality, and M-theory," Phys. Rev. D 75, 085020 (2007) [hep-th/0701036].

[114] M. J. Duff, "Twenty years of the Weyl anomaly," Class. Quant. Grav. 11, 1387 (1994) [hep-th/9308075].

[115] J. Cardy, "Boundary conformal field theory," Encyclopedia of Mathematical Physics, (Elsevier, 2006) [hep-th/0411189].

[116] M. Nozaki, T. Takayanagi and T. Ugajin, "Central Charges for BCFTs and Holography," JHEP 1206, 066 (2012) [arXiv:1205.1573 [hep-th]].

[117] D. V. Fursaev and S. N. Solodukhin, "Anomalies, entropy and boundaries," Phys. Rev. D 93, no. 8, 084021 (2016) [arXiv:1601.06418 [hep-th]].

[118] H. Osborn and A. Stergiou, "C_T for non-unitary CFTs in higher dimensions," JHEP 1606, 079 (2016) [arXiv:1603.07307 [hep-th]].

[119] F. A. Dolan and H. Osborn, "Conformal partial waves and the operator product expansion," Nucl. Phys. B 678, 491 (2004) [hep-th/0309180].

[120] M. Hogervorst and B. C. van Rees, "Crossing symmetry in alpha space," JHEP 1711, 193 (2017) [arXiv:1702.08471 [hep-th]].

[121] M. Hogervorst, "Crossing Kernels for Boundary and Crosscap CFTs," arXiv:1703.08159 [hep-th].

[122] D. Z. Freedman, K. Johnson and J. I. Latorre, "Differential regularization and renormalization: A New method of calculation in quantum field theory," Nucl. Phys. B 371 (1992) 353.

[123] D. V. Vassilevich, "Heat kernel expansion: User's manual," Phys. Rept. 388, 279 (2003) [hep-th/0306138].

[124] H. Luckock,"Mixed boundary conditions in quantum field theory," J. Math. Phys. 32 (1991) 1755.

[125] F. Bastianelli, S. Frolov and A. A. Tseytlin, "Three point correlators of stress tensors in maximally supersymmetric conformal theories in $D = 3$ and $D = 6$," Nucl. Phys. B 578, 139 (2000) [hep-th/9911135].

[126] A. Cappelli and G. D'Appollonio, "On the trace anomaly as a measure of degrees of freedom," Phys. Lett. B 487, 87 (2000) [hep-th/0005115].

[127] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton U. Press (1992)

[128] E. D'Hoker, J. Estes and M. Gutperle, "Interface Yang-Mills, supersymmetry, and Janus," Nucl. Phys. B 753, 16 (2006) [hep-th/0603013].

[129] D. Gaiotto and E. Witten, "Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory," J. Statist. Phys. 135, 789 (2009) [arXiv:0804.2902 [hep-th]].

[130] E. D'Hoker, J. Estes, M. Gutperle and D. Krym, "Janus solutions in M-theory," JHEP 0906, 018 (2009) [arXiv:0904.3313 [hep-th]].

[131] M. Srednicki, *Quantum Field Theory*, Cambridge U. Press (2006).

[132] H. Osborn, "Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories," Nucl. Phys. B 363, 486 (1991)

[133] Y. Nakayama, "Can we change c in four-dimensional CFTs by exactly marginal deformations?," JHEP 1707, 004 (2017) [arXiv:1702.02324 [hep-th]].

[134] S. Elitzur, E. Rabinovici and G. Sarkissian, "On least action D-branes," Nucl. Phys. B 541, 246 (1999) [hep-th/9807161].

[135] P. A. M. Dirac, "Wave equations in conformal space," Annals Math. 37, 429 (1936).

[136] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, "Spinning Conformal Correlators," JHEP 1111, 071 (2011) [arXiv:1107.3554 [hep-th]].

[137] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, "Spinning Conformal Blocks," JHEP 1111, 154 (2011) [arXiv:1109.6321 [hep-th]].

[138] S. Weinberg, "Six-dimensional Methods for Four-dimensional Conformal Field Theories," Phys. Rev. D 82, 045031 (2010) [arXiv:1006.3480 [hep-th]].

[139] A. Bzowski, P. McFadden and K. Skenderis, "Implications of conformal invariance in momentum space," JHEP 1403, 111 (2014) [arXiv:1304.7760 [hep-th]].

[140] A. Bzowski, P. McFadden and K. Skenderis, "Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies," JHEP 1603, 066 (2016) [arXiv:1510.08442 [hep-th]].

[141] D. M. Hofman and J. Maldacena, "Conformal collider physics: Energy and charge correlations," JHEP 0805, 012 (2008) [arXiv:0803.1467 [hep-th]].

[142] R.-X. Miao and C.-S. Chu, "Universality for Shape Dependence of Casimir Effects from Weyl Anomaly," JHEP 1803, 046 (2018) [arXiv:1706.09652 [hep-th]].