

Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

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General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

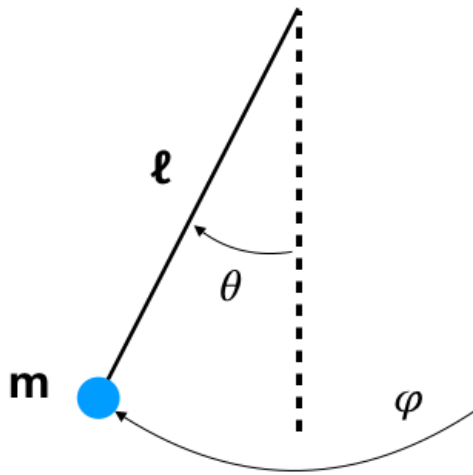
Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used.**

Classical Mechanics 1

Spherical Pendulum

Consider a spherical pendulum: a mass m on a rope of length ℓ attached a frictionless pivot that allows the mass freedom to move in two angular directions. You may assume that the length of the rope is fixed so that the motion of the mass is confined to a sphere with radius ℓ . Do not make small angle approximations in this problem.



- (4 points) Determine the Lagrangian of this system in terms of angular variables θ , the angle of the rope w.r.t to the vertical axis, and ϕ the azimuthal motion of the mass.
- (4 points) Using the symmetries of the Lagrangian, identify two constants of motion.
- (4 points) Using your results from (b), find an implicit solution for the equations of motion $\theta(t)$ and $\phi(t)$ (your results should be in terms of integrals that you do not need to do).
- (4 points) Using the variational principle, determine the equation(s) of motion for the pendulum and identify the location of stable orbits.
- (4 points) Under what conditions are the orbits from (d) stable? Determine the frequency of small oscillations around these orbits (you may leave your answer in terms of θ_c the value of the θ coordinate for a stable circular orbit). Does your answer reduce to the usual frequency of a simple 1-D pendulum in the small θ_c limit? Why or why not?

Solution

(a) First, let's determine the Lagrangian of the pendulum in cartesian coordinates x, y, z with the origin at the pivot point:

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - mg(\ell + z) \quad (1)$$

where $\dot{} = d/dt$ and g is the gravitational acceleration. Now, let's impose the constraint that the pendulum motion is restricted to the surface of a sphere of radius ℓ ,

$$x = \ell \sin \theta \cos \phi \quad y = \ell \sin \theta \sin \phi \quad z = -\ell \cos \theta \quad (2)$$

One can do this via Lagrange multipliers or direct substitution and the result is

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2 \sin^2 \theta \dot{\phi}^2 - mg\ell(1 - \cos \theta) \quad (3)$$

(b) The Lagrangian has no explicit t dependence and no explicit ϕ dependence. The ϕ -independence allows the z -component of the angular momentum to be conserved

$$L_z = \frac{\partial L}{\partial \dot{\phi}} = m\ell^2 \sin^2 \theta \dot{\phi} \quad (4)$$

and the t -independence means that

$$E = \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} + \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L \quad (5)$$

$$= \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2 \sin^2 \theta \dot{\phi}^2 + mg\ell(1 - \cos \theta) \quad (6)$$

is also conserved. Using the conserved angular momentum, this can be rewritten as

$$E = \frac{1}{2}m\ell^2\dot{\theta}^2 + \underbrace{\frac{L_z^2}{2m\ell^2 \sin^2 \theta}}_{\equiv V_{eff}(\theta)} + mg\ell(1 - \cos \theta) \quad (7)$$

(c) From the conserved energy in (b), we can write

$$\frac{d\theta}{dt} = \sqrt{\frac{2}{m\ell} (E - V_{eff}(\theta))} \quad (8)$$

so that

$$\Delta t = \int \frac{d\theta}{\sqrt{\frac{2}{m\ell} (E - V_{eff}(\theta))}} \quad (9)$$

and, using the conserved angular momentum

$$\frac{d\phi}{dt} = \frac{L_z}{m\ell^2 \sin^2 \theta} \quad (10)$$

so that

$$\phi = \int \frac{L_z}{m\ell^2 \sin^2 \theta} dt \quad (11)$$

$$= \int \frac{L_z}{m\ell^2 \sin^2 \theta} \frac{d\theta}{\dot{\theta}} \quad (12)$$

$$= \int \frac{L_z}{m\ell^2 \sin^2 \theta \sqrt{\frac{2}{m\ell} (E - V_{eff}(\theta))}} d\theta \quad (13)$$

(d) Varying the action with respect to $\phi(t)$ just gives

$$\frac{d}{dt} L_z = 0 \quad (14)$$

as expected. Varying the action with respect to $\theta(t)$ gives

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \quad (15)$$

which gives

$$m\ell^2 \ddot{\theta} = m\ell^2 \sin \theta \cos \theta \dot{\phi}^2 - mgl \sin \theta \quad (16)$$

$$= m\ell^2 \sin \theta \cos \theta \frac{L_z^2}{m^2 \ell^4 \sin^4 \theta} - mgl \sin \theta \quad (17)$$

$$= \frac{L_z^2 \cos \theta}{m\ell^2 \sin^3 \theta} - mgl \sin \theta \quad (18)$$

Note that this can just be rewritten as

$$m\ell^2 \ddot{\theta} = - \frac{\partial V_{eff}}{\partial \theta} \quad (19)$$

Solutions with θ_c defined via $\partial V_{eff} / \partial \theta(\theta_c) = 0$,

$$\frac{L_z^2}{gm^2 \ell^3} = \frac{\sin^4 \theta_c}{\cos \theta_c} \quad (20)$$

will have $\ddot{\theta} = 0$. That is, for each pair L_z and θ_c satisfying Eq. (20), there is a circular "orbit."

(e) To check the stability of the circular orbits we consider $\theta(t) = \theta_c + \delta\theta$, which has equation of motion

$$\ddot{\delta\theta} = - \frac{1}{m\ell^2} \left. \frac{\partial^2 V_{eff}}{\partial \theta^2} \right|_{\theta=\theta_c} \delta\theta \quad (21)$$

so the stability of the orbits can be assessed by considering the sign of $\frac{\partial^2 V_{eff}}{\partial \theta^2}$ at $\theta = \theta_c$,

$$\frac{\partial^2 V_{eff}}{\partial \theta^2}(\theta_c) = gml \cos \theta_c + \frac{L_z^2}{m\ell^2 \sin^2 \theta_c} + 3 \frac{L_z^2 \cos^2 \theta_c}{m\ell^2 \sin^4 \theta_c} \quad (22)$$

$$= 4gml \cos \theta_c + gml \frac{\sin^2 \theta_c}{\cos \theta_c} \quad (23)$$

$$= gml \cos \theta_c (4 + \tan^2 \theta_c) \quad (24)$$

So, the orbits are stable as long as $0 \geq \theta_c < \pi/2$ and the frequency of small oscillations is given by

$$\omega_c = \sqrt{\frac{g \cos \theta_c}{\ell} (4 + \tan^2 \theta_c)} \quad (25)$$

For $\theta_c \ll 1$, this reduces to

$$\omega_c = 2\sqrt{\frac{g}{\ell}} + \mathcal{O}(\theta_c^2) \quad (26)$$

A 1D pendulum would have $\omega_{1D} = \sqrt{g/\ell}$, in going from Eq. (22) to Eq. (23) we assumed $L_z \neq 0$, but motion in a single vertical plane would have $\phi = \text{const.}$ so that $L_z = 0$ and we have divided by 0. Yet, starting from Eq. (22) and setting $L_z = 0$ recovers the familiar 1D equation

$$\delta\ddot{\theta} \approx -g/\ell\delta\theta \quad (27)$$

Classical Mechanics 2

Isotropic oscillator in a magnetic field

Consider a particle of mass m and positive charge q moving in the x, y plane in an isotropic harmonic potential $V(x, y) = \frac{1}{2}m\omega_0^2(x^2 + y^2)$. In addition, the particle is placed in a uniform magnetic field of magnitude B_0 in the z direction.

- (a) (2 points) Write down the Lagrangian of the system in cartesian and cylindrical coordinates $x = r \cos \phi$ and $y = r \sin \phi$.

Hint: Use the gauge $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, and define the magnetic frequency¹ $\omega_B \equiv eB/(2m)$ to simplify the algebra in what follows.

- (b) (2 points) Determine the equations of motion in cartesian and cylindrical coordinates.
- (c) (4 points) Identify the constant integrals of motion, and write down explicit expressions for these quantities in cartesian and cylindrical coordinates.

- (d) (5 points) What are the radii of the stable circular orbits of the particle and the associated angular velocities $\dot{\phi}$. Explicitly interpret the allowed values of $\dot{\phi}$ by drawing a well labeled free body diagram indicating the forces on the particle.

- (e) (5 points) Determine the general solution to the equations of motion for $x(t)$ and $y(t)$.

Hint: Write down equations of motion for $z \equiv x + iy$ and solve this linear differential equation. Express the final result for $x(t)$ and $y(t)$ in terms of real functions and real constants of integration.

- (f) (2 points) Evaluate the angular momentum of the system for the general solution of part (e) and interpret the result.

¹In Gaussian or Heaviside Lorentz units $\omega_B \equiv eB/(2mc)$.

Solution

(a) The Lagrangian is

$$L = \frac{1}{2}mv^2 - \frac{1}{2}m\omega_0^2(x^2 + y^2) + \frac{q\mathbf{v} \cdot \mathbf{A}}{c}. \quad (1)$$

Then writing

$$\mathbf{A} = \frac{\mathbf{B}}{2} \times \mathbf{r} = \frac{B_0}{2}(-y, x), \quad (2)$$

and $\mathbf{v} = (\dot{x}, \dot{y})$ we have

$$L = \frac{1}{2}m\dot{v}^2 - \frac{1}{2}m\omega_0^2(x^2 + y^2) + m\omega_B(-\dot{x}y + \dot{y}x). \quad (3)$$

Here and below we have defined

$$\omega_B \equiv \frac{qB_0}{2mc} = \text{half the cyclotron frequency}. \quad (4)$$

The cyclotron frequency $\omega_c \equiv qB_0/mc$ determines the period of circular orbits of charged particles in a magnetic field, period = $2\pi/\omega_c$. Note this period is independent of the radius of the orbit.

In cylindrical coordinates

$$x = r \cos \phi, \quad y = r \sin \phi, \quad (5)$$

we have

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}m\omega_0^2r^2 + mr^2\omega_B\dot{\phi}. \quad (6)$$

(b) First we will compute the equation of motion in cylindrical coordinates. We note that ϕ is cyclic a coordinate and thus

$$\frac{dp_\phi}{dt} = 0, \quad \text{where} \quad p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + mr^2\omega_B \equiv \ell. \quad (7)$$

Then the equation of motion for r follows from the Euler Lagrange equations, and takes the form

$$\frac{d(m\dot{r})}{dt} = \frac{\partial L}{\partial r} = -\frac{\partial V_{\text{eff}}(r)}{\partial r}, \quad (8)$$

where

$$V_{\text{eff}}(r) = \frac{\ell^2}{2mr^2} + \frac{1}{2}m(\omega^2 + \omega_B^2)r^2 + \ell\omega_B. \quad (9)$$

The form of the effective potential can be intuited from the first integral. Energy conservation (the first integral) reads:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m\omega_0^2r^2, \quad (10)$$

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r). \quad (11)$$

In cartesian coordinates we have simply:

$$\frac{d}{dt} [m(\dot{x} - \omega_B y)] = -\omega_0^2 x + m\omega_B \dot{y}, \quad (12)$$

$$\frac{d}{dt} [m(\dot{y} + \omega_B x)] = -\omega_0^2 y - m\omega_B \dot{x}. \quad (13)$$

Collecting terms for later use we find

$$\ddot{x} + \omega_0^2 x - 2\omega_B \dot{y} = 0, \quad (14a)$$

$$\ddot{y} + \omega_0^2 y + 2\omega_B \dot{x} = 0. \quad (14b)$$

(c) The integrals of motion are the energy and the angular momentum around the z axis. The conserved energy is given by the Hamiltonian function, which for a general Lagrangian reads:

$$h(q, \dot{q}, t) = \sum_a \frac{\partial L}{\partial \dot{q}^a} \dot{q}^a - L. \quad (15)$$

For the problem at hand this evaluates to

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\omega_0^2(x^2 + y^2), \quad (16)$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m\omega_0^2 r^2. \quad (17)$$

Notice that the energy is independent of the magnetic field. This follows physically from the fact that the magnetic field does no work. Mathematically it is a statement that for a Lagrangian of the form

$$L = \sum_i \frac{1}{2}m_i(q)(\dot{q}^i)^2 + b_i(q)\dot{q}^i - V(q), \quad (18)$$

the first integral is independent of $b_i(q)$

$$h = \sum_i \frac{1}{2}m_i(q)(\dot{q}^i)^2 + V(q). \quad (19)$$

The Hamiltonian $H(p, q, t)$ is a function of p_i and q_i

$$H = \sum_i \frac{(p_i - b_i(q))^2}{2m_i(q)} + V(q), \quad (20)$$

and, in contrast to $h(q, \dot{q}, t)$, is a function of the magnetic field $b_i(q)$.

Now we will evaluate the angular momentum around the z -axis. The angular momentum around the z axis is associated with the cyclic coordinate ϕ

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + mr^2\omega_B. \quad (21)$$

One can either translate this result into cartesian coordinates or work directly with the cartesian Lagrangian

$$p_\phi = xp_y - yp_x, \quad (22a)$$

$$= xm(\dot{y} + \omega_B x) - ym(\dot{x} - \omega_B y), \quad (22b)$$

$$= m(xy\dot{y} + y\dot{x}) + m\omega_B(x^2 + y^2). \quad (22c)$$

where we used that $p_x = \partial L / \partial \dot{x} = m\dot{x} - \omega_B y$.

(d) The circular orbits are determined by the minimum of the effective potential

$$\left. \frac{\partial V_{\text{eff}}}{\partial r} \right|_{r_0} = -\frac{\ell^2}{mr_0^3} + m(\omega^2 + \omega_B^2)r_0 = 0, \quad (23)$$

which yields

$$r_0^2 = \frac{|\ell|}{m\sqrt{\omega_0^2 + \omega_B^2}}. \quad (24)$$

Of course the value of ℓ is arbitrary, so there are stable circular orbits at any radius. The radius is determined by $|\ell|$. The associated angular velocities are

$$\frac{d\phi}{dt} = \frac{p_\phi}{mr_0^2} - \omega_B, \quad (25)$$

$$= \frac{\ell}{|\ell|} \sqrt{\omega_0^2 + \omega_B^2} - \omega_B. \quad (26)$$

If the angular momentum ℓ is positive, the circular orbit is counter clockwise, and the particle moves relatively slowly with angular velocity $\omega_+ = \sqrt{\omega_0^2 + \omega_B^2} - \omega_B$. If ℓ is negative, the particle moves relatively quickly with angular velocity $\omega_- = -(\sqrt{\omega_0^2 + \omega_B^2} + \omega_B)$, where the leading negative sign indicates a clockwise orbit. The freebody diagram is shown in Fig. 1. Setting $F_{\text{net}} = ma$ for Fig. 1(a) gives

$$-m\omega_0^2 r_0 + \frac{qB_0}{c} \omega r_0 = -m\omega^2 r_0, \quad (27)$$

which determines the allowed rotational frequencies

$$\omega = \pm \sqrt{\omega_B^2 + \omega_0^2} - \omega_B. \quad (28)$$

Note that the Newtonian equation of motion Eq. (27) does not constrain the radius. Any radius is allowed.

(e) Adding the equation of motion for x and the equation of motion for iy , leads to a single equation for $z = x + iy$

$$\ddot{z} + \omega_0^2 z + 2i\omega_B \dot{z} = 0. \quad (29)$$

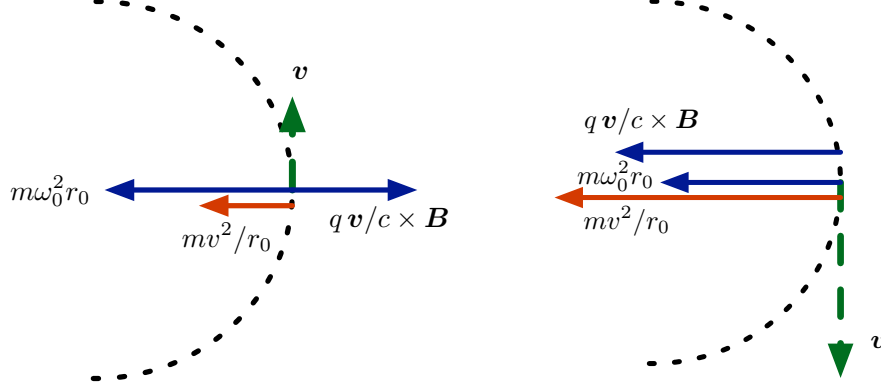


Figure 1: The free-body diagram for the circular orbits. The blue arrows show the spring and magnetic forces. The velocity is shown by the green arrows. The net force is $ma = mv^2/r_0 = m\omega^2 r_0$.

We will look for a solution to this equation of the form $Ae^{i\omega t}$. Substituting $Ae^{i\omega t}$ into Eq. (29) leads to the characteristic equation

$$-\omega^2 + \omega_0^2 - 2\omega_B\omega = 0. \quad (30)$$

This equation has two roots

$$\omega_{\pm} = \pm\sqrt{\omega_0^2 + \omega_B^2} - \omega_B, \quad (31)$$

which results in the general solution

$$z = A_1 e^{i\omega_+ t} + A_2 e^{i\omega_- t}. \quad (32)$$

Taking the real and imaginary parts the general solution for $x(t)$ and $y(t)$ reads

$$x(t) = \text{Re } z(t) \quad (33a)$$

$$= |A_1| \cos(\omega_+ t + \varphi_1) + |A_2| \cos(\omega_- t + \varphi_2) \quad (33b)$$

$$y(t) = \text{Im } z(t) \quad (33c)$$

$$= |A_1| \sin(\omega_+ t + \varphi_1) + |A_2| \sin(\omega_- t + \varphi_2) \quad (33d)$$

The first term has $\omega_+ > 0$ and corresponds to a slower counter-clockwise orbit with $p_\phi > 0$. The second term has $\omega_- < 0$ and corresponds to the faster clockwise orbit with $p_\phi < 0$. As we will see in the next exercise the angular momentum in the system is $p_\phi = m\omega'(|A_1|^2 - |A_2|^2)$

Here we have guessed that the solution for $x + iy$ is simple. If one did not have this intuition, one could recognize that the equations in Eq. (14) are linear second order differential equations. In such cases one simply substitutes $(x, y) = (x_0, y_0)e^{-i\omega t}$ and then find the eigenvectors (x_0, y_0) . For instance this leads matrix equation

$$\begin{pmatrix} -\omega^2 + \omega_0^2 & i2\omega_B\omega \\ -2i\omega_B\omega & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0. \quad (34)$$

Then the non-trivial solutions are when

$$\det \begin{pmatrix} -\omega^2 + \omega_0^2 & i2\omega_B\omega \\ -2i\omega_B\omega & -\omega^2 + \omega_0^2 \end{pmatrix} = 0. \quad (35)$$

This equation determines the allowed frequencies and gives the same results as the procedure outlined above.

(f) Now lets evaluate the angular momentum

$$p_\phi = m(xy - yx) + m(x^2 + y^2)\omega_B. \quad (36)$$

The second term is proportional to

$$x^2 + y^2 = z\bar{z} = |A_1|^2 + |A_2|^2 + (A_1A_2^*e^{i((\omega_+)-\omega_-)t} + \text{c.c.}), \quad (37)$$

where c.c. denotes the complex conjugate of the first term in brackets. while the first term is proportional to

$$(xy - yx) = \frac{-1}{2i} (z\dot{\bar{z}} - \bar{z}\dot{z}) = \omega_+|A_1|^2 + \omega_-|A_2|^2 + \frac{1}{2}(\omega_+ + \omega_-) (A_1A_2^*e^{i(\omega_+-\omega_-)t} + \text{c.c.}). \quad (38)$$

Noting that $(\omega_+ + \omega_-)/2 = -\omega_B$, we have finally

$$p_\phi = m(\omega_B + \omega_+)|A_1|^2 + m(\omega_B + \omega_-)|A_2|^2, \quad (39a)$$

$$= m\omega' (|A_1|^2 - |A_2|^2). \quad (39b)$$

where $\omega' = \sqrt{\omega_0^2 + \omega_B^2}$.

The interpretation is the following: the general solution is a superposition of two circular orbits with different velocities and directions. The counter-clockwise orbit is slower, and has radius $|A_1|$. The angular momentum associated with this orbit is $m\omega'|A_1|^2$. The clockwise orbit is faster, has radius $|A_2|$, and has angular momentum $-m\omega'|A_2|^2$.

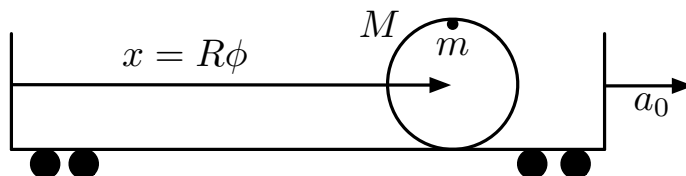
Classical Mechanics 3

An accelerating frame

A cylinder with mass M , moment of inertial I , and radius R , starts rolling without slipping from rest in a train that accelerates with constant acceleration a_0 . The axis of the cylinder is perpendicular to the motion of the train.

- (2 points) Calculate the acceleration of the cylinder using Newton's laws in the lab frame (the ground). Draw a well labelled free body diagram indicating the forces and acceleration in this frame.
- (2 points) Calculate the acceleration of the cylinder in the frame of the accelerating train by using Newton's laws in this frame. Again, draw a well labeled free-body diagram indicating the forces and acceleration in this frame.
- (5 points) Write down a Lagrangian for the cylinder and calculate its acceleration by solving the Euler Lagrange equations. Is the acceleration consistent with parts (a) and (b)?

Now consider a cylinder-like contraption consisting of a cylindrical ring of mass M and radius R , and a small weight of mass m fixed to the rim of the ring (see below). At time $t = 0$ the cylinder starts to roll without slipping from rest in the accelerating train, and the weight is at the top of its arc as shown in the figure below.



- (8 points) Determine the Lagrangian for the angle $\phi(t)$, where $x \equiv R\phi$ is the position of the center of the cylinder relative to the back of the train (see figure). Show that the Lagrangian may (up to total derivatives) be written in a time independent form

$$L = \frac{1}{2} m_{\text{eff}}(\phi) R^2 \dot{\phi}^2 - U(\phi), \quad (1)$$

where $m_{\text{eff}}(\phi)$ and $U(\phi)$ are specific functions of ϕ .

Hint: In the train's frame the acceleration functions like an additional gravitational field of magnitude a_0 pulling the negative x direction.

- (3 points) What is the speed of the cylinder after it rolls for two complete turns.

Solution

(a) The floor exerts a force F . Therefore, according to Newton, $F = Ma$ and $-FR = I\alpha$. The rolling without slipping constraint says that $a = a_0 + R\alpha$, So that the bottom of the wheel has acceleration $a - R\alpha = a_0$, i.e. moves with the train. This is equivalent to saying

$$x(t) = x_0(t) + R\phi \quad (2)$$

Then solving we find

$$a = \frac{a_0}{1 + \frac{MR^2}{I}} \quad (3)$$

We note for later that the acceleration in the frame of the train \mathcal{A} is

$$\mathcal{A} = a - a_0 = -a_0 \frac{MR^2/I}{1 + MR^2/I} \quad (4)$$

(b) In the train frame the forces are the non-inertial force $F_{\text{eff}} = -Ma_0$ and friction F with a total force $F_{\text{net}} = -Ma_0 + F$. Newton's law reads

$$-Ma_0 + F = M\mathcal{A} \quad (5)$$

The net torque is only from friction

$$-FR = I\alpha \quad (6)$$

We must have that the bottom of the rim has no acceleration in this frame so $\mathcal{A} + R\alpha = 0$. Solving we find

$$\mathcal{A} = -a_0 \frac{MR^2/I}{1 + MR^2/I} \quad (7)$$

(c) The Lagrangian with the constraint $x = x_0 + R\phi$ reads

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + F(x - x_0 - R\phi) \quad (8)$$

Here F is the Lagrange multiplier, and will later be interpreted as the force of static friction. The Euler-Lagrange equations are given by

$$M\ddot{x} = F \quad (9)$$

$$I\ddot{\phi} = -RF \quad (10)$$

$$x = x_0 + R\phi \quad (11)$$

Using the constraint $\ddot{x} = \ddot{x}_0 + R\ddot{\phi}$, we may solve for a yielding

$$a = \frac{a_0}{1 + MR^2/I} \quad (12)$$

(d) In the train's frame the acceleration acts like an additional gravitational potential $\phi = a_0x$. The Lagrangian of the contraption is thus

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) - Ma_0x - ma_0x_1 - mgy_1 \quad (13)$$

where $x = R\phi$. The coordinates of the weight are

$$x_1 = R\phi + R\sin\phi \quad (14)$$

$$y_1 = R\cos\phi \quad (15)$$

So with $I = MR^2$ and $x = R\phi$ we have after minor algebra

$$L = \frac{1}{2}m_{\text{eff}}(\phi)R^2\dot{\phi}^2 - U(\phi) \quad (16)$$

where

$$m_{\text{eff}}(x) = 2M + 2m(1 + \cos\phi) \quad (17)$$

$$U = Ma_0x + ma_0(x + R\sin\phi) + mgR\cos\phi \quad (18)$$

Here it is understood that $x = R\phi$.

Additional discussion: In the previous solution we have intuited that the constant acceleration acts like an additional potential. We could derive this by working in the lab frame. Let us first show how this works for a free particle with coordinate $X(t) = x_0(t) + x(t)$. Here $x_0(t) = \frac{1}{2}a_0t^2$ is the position of a fixed point on the accelerating train, and $x(t)$ are the coordinates relative to this point on the train, and $X(t)$ is the position in the "lab" frame. The Lagrangian for this particle is

$$S = \int dt \frac{1}{2}m\dot{X}^2, \quad (19)$$

$$= \int dt \frac{1}{2}m(\dot{x}_0^2 + 2\dot{x}_0\dot{x} + \dot{x}^2). \quad (20)$$

The first term is a total derivative and independent of x and may be ignored. The second term may be integrated by parts yielding

$$S = \int dt \frac{1}{2}m(-2\ddot{x}_0x + \dot{x}^2), \quad (21)$$

$$= \int dt \frac{1}{2}m\dot{x}^2 - ma_0x. \quad (22)$$

These steps show that the particle in the accelerating frame has the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - ma_0x. \quad (23)$$

Here we have treated one particle. But it clearly extends to an ensemble of particles, and then to a rigid body yielding the Lagrangian

$$L = \frac{1}{2}M_{\text{tot}}\dot{x}_{\text{cm}}^2 + \frac{1}{2}I\omega^2 - M_{\text{tot}}a_0x - U_0(x), \quad (24)$$

where $U_0(x)$ is potential in the absence of acceleration.

(e) The system has a first integral and therefore

$$E = \frac{1}{2}m_{\text{eff}}(x)\dot{x}^2 + U(x) \quad (25)$$

is constant. Thus

$$\left. \frac{1}{2}m_{\text{eff}}(x)\dot{x}^2 + U(x) \right|_{\text{final}} = \left. \frac{1}{2}m_{\text{eff}}(x)\dot{x}^2 + U(x) \right|_{\text{initial}} \quad (26)$$

Since the initial kinetic energy is zero and the initial potential energy is mgR

$$\frac{1}{2}m_{\text{eff}}(x)\dot{x}^2 = mgR - U(x) \quad (27)$$

After two turns $x = -2(2\pi R)$ and $\phi = 4\pi$. Thus $U(x) = -(M + m)a_0(4\pi R) + mgR$, while the mass is

$$m_{\text{eff}}(x_{\text{final}}) = 2M + 4m \quad (28)$$

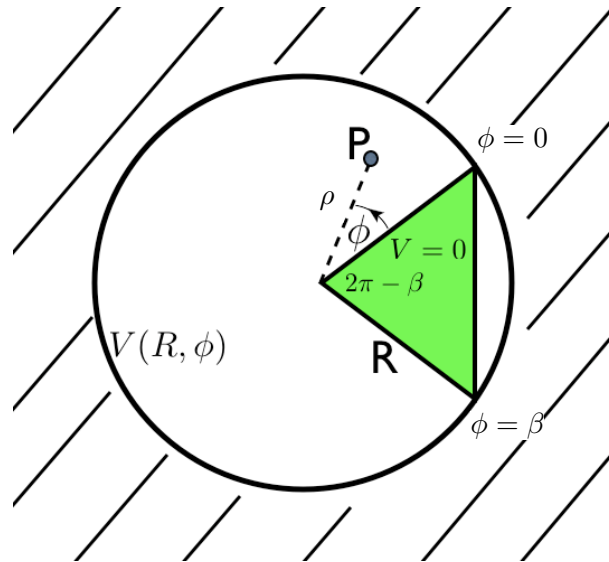
So we find

$$\dot{x} = \sqrt{\frac{4\pi a(M + m)R}{2M + 4m}} \quad (29)$$

Electromagnetism 1

A conducting wedge inside a cylinder

A hollow cylinder of radius R has a potential $V(R, \phi)$ maintained on its surface, as shown in the figure below. A conducting wedge with surfaces at $\phi = 0$ and $\phi = \beta$ and its apex at the symmetry axis of the cylinder (as shown by the shaded region), is placed inside the cylinder and held at a potential of $V = 0$ ($\rho = \sqrt{x^2 + y^2}$ denotes the radial coordinate).



- (a) (6 points) Derive the most general form for the potential inside the cylinder for $0 < \phi < \beta$ for a general boundary-condition $V(R, \phi)$.²
- (b) (2 points) Determine the potential for $0 < \phi < \beta$ resulting from the boundary condition

$$V(R, \phi) = \bar{V}_1 \sin\left(\frac{\pi\phi}{\beta}\right) + \bar{V}_3 \sin\left(\frac{3\pi\phi}{\beta}\right) \quad (1)$$

where $\bar{V}_{1,3}$ are constants.

- (c) (6 points) Determine the electric field at each point inside the cylinder for $0 < \phi < \beta$, and determine the surface charge per area on the $\phi = 0$ and $\phi = \beta$ surfaces of the wedge for the boundary conditions of (b).
- (d) (3 points) Describe qualitatively the behavior of the electric field (sketch the field lines) and the surface charge per area of part (c) near the tip of the wedge as a function of the wedge apex-angle, $2\pi - \beta$.

²Possibly useful:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

- (e) (3 points) For the situation where $\beta = \pi$ and $\bar{V}_3 = 0$, draw the electric field lines and equi-potential surfaces inside the cylinder.

Solution

(a) Separating the Laplace equation we try solutions of the form $\varphi = R(\rho)\Phi(\phi)$

$$-\frac{\rho^2}{\varphi}\nabla^2\varphi = \frac{-\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{-1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0, \quad (2)$$

which yields two equations

$$\frac{\partial^2\Phi}{\partial\phi^2} = -\alpha^2\Phi, \quad (3)$$

and

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) = \alpha^2R. \quad (4)$$

The solutions to the first equations which satisfy $\Phi(0) = \Phi(\beta) = 0$ are given by

$$\Phi_m = \sin(m\pi\phi/\beta) \quad m = 1, 2, 3, \dots \quad (5)$$

Then, the radial direction takes the form $R = \rho^\gamma$ with $\gamma = \pm m\pi/\beta$. This yields a general solution. Only the positive value of γ yields a regular solution near the origin, leading to the following general form for the regular solution

$$\varphi = \sum_{m=1}^{\infty} C_m \rho^{m\pi/\beta} \sin(m\pi\phi/\beta). \quad (6)$$

(b) For the specific case at hand, we have only the $m = 1$ and $m = 3$ terms in the expansion and thus the solution takes the form

$$\varphi = \bar{V}_1 \left(\frac{\rho}{R}\right)^{\pi/\beta} \sin(\pi\phi/\beta) + \bar{V}_3 \left(\frac{\rho}{R}\right)^{3\pi/\beta} \sin(3\pi\phi/\beta). \quad (7)$$

(c) The electric field is given by the gradient of the φ

$$\mathbf{E} = -\frac{\partial\varphi}{\partial\rho}\hat{\boldsymbol{\rho}} + -\frac{1}{\rho}\frac{\partial\varphi}{\partial\phi}\hat{\boldsymbol{\phi}}. \quad (8)$$

Differentiating we find the electric field takes the form

$$\mathbf{E}(\rho, \phi) = \mathbf{E}_1 + \mathbf{E}_3, \quad (9)$$

where

$$\mathbf{E}_m = \frac{-\bar{V}_1}{R} \left(\frac{m\pi}{\beta}\right) \left(\frac{\rho}{R}\right)^{m\pi/\beta-1} \left[\sin(m\pi\phi/\beta)\hat{\boldsymbol{\rho}} + \cos(m\pi\phi)\hat{\boldsymbol{\phi}} \right]. \quad (10)$$

Similarly, the charge density at $\phi = 0$ is of the form

$$\sigma = \sigma_1 + \sigma_3, \quad (11)$$

where

$$\sigma_m = \hat{\phi} \cdot \mathbf{E}_m \Big|_{\phi=0}, \quad (12)$$

$$= \frac{-\bar{V}_1}{R} \left(\frac{m\pi}{\beta} \right) \left(\frac{\rho}{R} \right)^{m\pi/\beta-1}. \quad (13)$$

The charge density at $\phi = \beta$ is the same as at $\phi = 0$ by symmetry. Technically this works out because the normal to the surface \mathbf{n} is $-\hat{\phi}$.

$$\sigma_m = -\hat{\phi} \cdot \mathbf{E}_m \Big|_{\phi=\beta}, \quad (14)$$

$$= \frac{-\bar{V}_1}{R} \left(\frac{m\pi}{\beta} \right) \left(\frac{\rho}{R} \right)^{m\pi/\beta-1}. \quad (15)$$

(d) The geometry and boundary conditions are reflection symmetric around $\phi = \beta/2$. Also near the tip, the field and surface densities are dominated by the first term in the expansion

$$\mathbf{E} \simeq \mathbf{E}_1, \quad \sigma \simeq \sigma_1. \quad (16)$$

We will take \bar{V}_1 to be negative so that the field lines are outward directed, and normal to the surface of the wedge. This fact, together with the symmetry of the geometry (which guarantees that the field lines do not cross the $\phi = \beta/2$ line), leads to Fig. 1.

The surface charge density is singular if $\beta > \pi$, and therefore increases without bound as $\rho \rightarrow 0$ as $\rho^{\pi/\beta-1}$. The surface charge density is bounded for $\beta < \pi$.

(e) For $\beta = \pi$ and $V_3 = 0$ the potential is just

$$\varphi = \rho \sin(\phi) = y. \quad (17)$$

This potential is just that of a constant electric field in the ‘‘y’’ direction, i.e. perpendicular to the wedge surfaces. The surface charge density is just a constant. This is shown in Fig. 2.

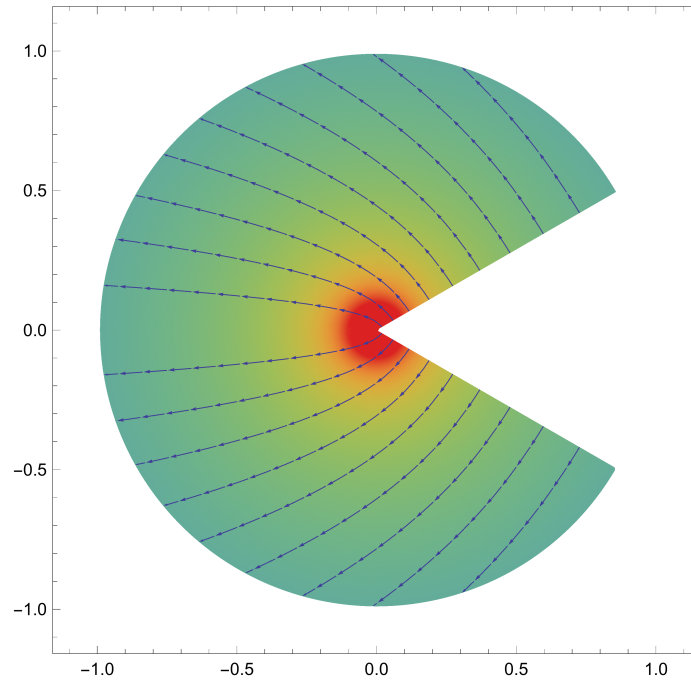


Figure 1: The field lines near the tip of the geometry. The colors indicate the magnitude of the electric field. The density of field lines also indicates the magnitude of the field.

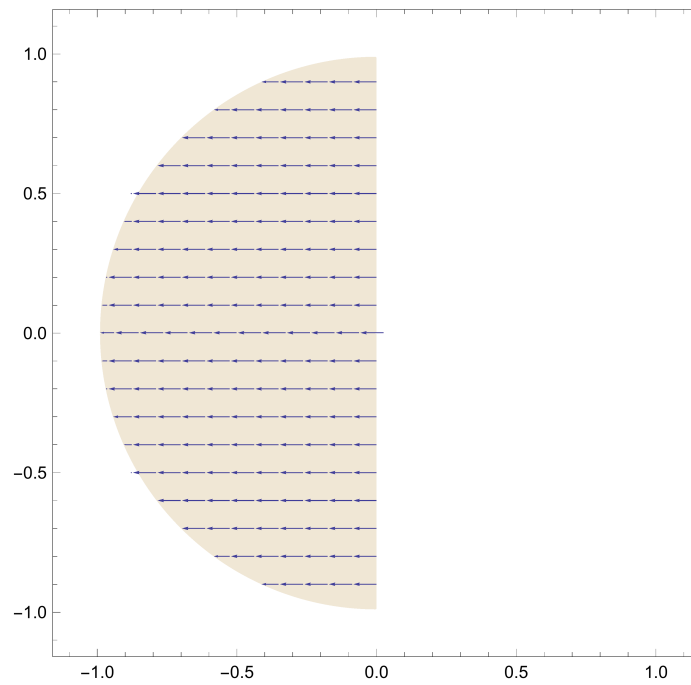


Figure 2: The field lines near the tip of the geometry for $\beta = \pi$. The electric field is constant.

Electromagnetism 2

Line with rising current

A neutral wire along the z -axis carries a current $I(t)$ that varies with time t as

$$I(t) = \begin{cases} \alpha t & t \geq 0 \\ 0 & t < 0 \end{cases}, \quad (1)$$

where α is a positive constant.

- a. 9 points** Determine the time-dependence of the electric and magnetic fields around the wire at a point $(r, \phi = 0, z = 0)$, in a cylindrical coordinate system where $r = \sqrt{x^2 + y^2}$.
- b. 4 points** Use your result to determine the fields for long times. Give a physical interpretation of your answer.
- c. 4 points** Use your result to determine the fields for short times. Give a physical interpretation of your answer.
- d. 3 points** Describe briefly the overall physical onset of the fields in time.

Solution

a. Since the wire remains neutral as the current flows, the retarded scalar potential φ is null at all times. Also, since the flow is linear in the z -direction, the retarded vector potential reads

$$A_z(t, x) = \int dx' \frac{J_z(t - R/c, x')}{cR} = \int_{-z_0}^{+z_0} dz \frac{I dz}{cR} \quad (2)$$

with $R = |x - x'|$. The last equation follows for an observer at $(r, 0, 0)$, with a current line along $(0, 0, z)$ and $R^2 = r^2 + z^2$ in cylindrical coordinates. The integration boundary in z follows from the causal support of the light cone $(ct)^2 > R^2 = r^2 + z^2$ or $|z| < z_0 = ((ct)^2 - r^2)^{\frac{1}{2}}$. More specifically

$$A_z(t, r, 0, 0) = \int_{-z_0}^{+z_0} \frac{\alpha t}{c((ct)^2 - r^2)^{\frac{1}{2}}} = \frac{2\alpha t}{c} \ln\left(\frac{z_0 + ct}{r}\right) \quad (3)$$

The electric and magnetic fields follow from

$$\begin{aligned} B_\phi &= (\nabla \times A)_\phi = -\frac{\partial A_z}{\partial r} = \frac{2\alpha z_0}{c^2 r} \\ E_z &= \left(-\nabla\varphi - \frac{1}{c} \frac{\partial A_z}{\partial t}\right)_z = -\frac{1}{c} \frac{\partial A_z}{\partial t} = -\frac{2\alpha}{c^2} \ln\left(\frac{z_0 + ct}{r}\right) \end{aligned} \quad (4)$$

b. For long times and fixed r , i.e. $ct \gg r$ and $z_0 \approx ct$ we have

$$\begin{aligned} B_\phi &= \frac{2\alpha t}{cr} = \frac{2I(t)}{cr} = B_0(t) \\ E_z &= -\frac{2\alpha}{c^2} \ln\left(\frac{2ct}{r}\right) = -B_0(t) \frac{r}{ct} \ln\left(\frac{2ct}{r}\right) \ll B_0(t) \end{aligned} \quad (5)$$

with $B_0(t) = 2I(t)/cr$ the instantaneous magnetic field produced by the current $I(t) = \alpha t$. Magnetostatics is recovered at large times.

c. For short times we can set $ct = r + \epsilon$ with $\epsilon \ll r$, so that $z_0 \approx \sqrt{2\epsilon r}$. In this regime,

$$\begin{aligned} B_\phi &= +\frac{2\alpha}{c^2} \sqrt{\frac{2\epsilon}{r}} \\ E_z &= -\frac{2\alpha}{c^2} \ln\left(\frac{r + \epsilon + \sqrt{2\epsilon r}}{r}\right) \approx -\frac{2\alpha}{c^2} \sqrt{\frac{2\epsilon}{r}} = -B_\phi \end{aligned} \quad (6)$$

the fields are radiative: E, B are equal in magnitude, orthogonal and transverse to the line of sight. They travel outwardly at the speed of light. Note that the fall-off is $1/\sqrt{r}$ due to

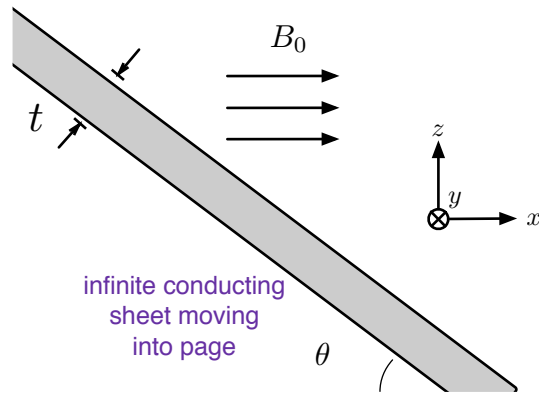
the cylindrical character of the radiation.

d. The fields develop after only a time $ct \geq r$ by causality. They are initially radiative and cylindrical and move outward at the speed of light. However, for fixed r as time passes, magnetostatics settles.

Electromagnetism 3

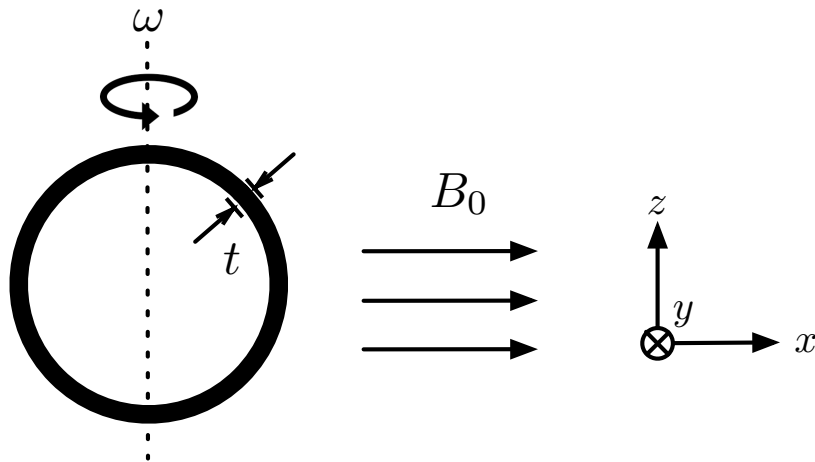
A rotating sphere

Consider an infinitely large metal sheet of thickness t and conductivity σ , in a homogeneous magnetic field pointing in the x direction, $\mathbf{B} = B_0\hat{x}$. The sheet has an inclination angle θ relative the x axis (see below), and is moving non-relativistically with velocity v in the y direction (into the page)



- (4 points) Determine the charge per area on the surfaces of the sheet. Draw a sketch.
- (4 points) Determine the current density in the sheet and the energy dissipated per surface area.

Now consider a thin metal spherical shell of thickness t , conductivity σ , and radius R . The shell is placed in the same magnetic field directed along the x axis, $\mathbf{B} = B_0\hat{x}$. The sphere is rotated with angular velocity ω around the z axis.



- (4 points) Use part (b) to *estimate* the total energy dissipated in the sphere per time in terms of σ, ω, R, t and B_0 up to an order one numerical factor.

- (d) (3 points) Determine charge per area on the inner and outer surfaces of the sphere and the radial component of the electric field in the metal.
- (e) (3 points) Determine the electrostatic potential in the metal consistent with (d) by solving the Laplace equation. Find all components of the electric field in the metal.
- Some formulas on the Laplace equation in spherical coordinates are compiled below.
- (f) (2 points) Determine the current density \mathbf{J} in the sphere.

The Laplace equation in spherical coordinates:

- Gradient in spherical coordinates:

$$\nabla\psi = \frac{\partial\psi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial\psi}{\partial\theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial\phi} \hat{\boldsymbol{\phi}}$$

- Laplacian in spherical coordinates:

$$\nabla^2\psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2}$$

- Separated solutions to the Laplace equation take the form:

$$\psi(r, \theta, \phi) = \sum_{\ell m} \left(\frac{A_{\ell m}}{r^{\ell+1}} + B_{\ell m} r^\ell \right) Y_{\ell m}(\theta, \phi).$$

Here $Y_{\ell m}(\theta, \phi)$ are spherical harmonics and $A_{\ell m}$ and $B_{\ell m}$ are constants.

- The lowest spherical harmonics are:

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

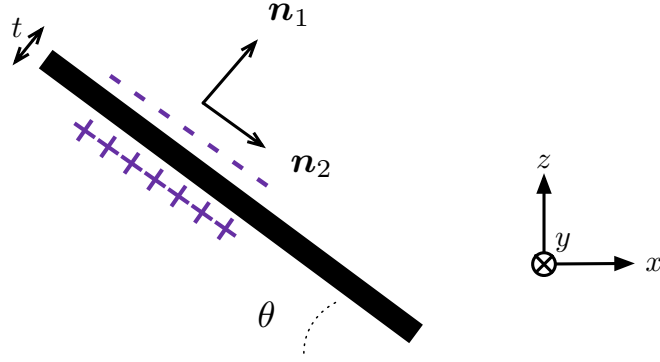
$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}} (3 \cos^2\theta - 1)$$

$$Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{\pm i\phi}$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm i2\phi}$$



Solution

(a) Let \mathbf{n}_1 be the normal to the sheet (“outward” directed) and \mathbf{n}_2 be parallel to the sheet as shown above. The current density is

$$\mathbf{J} = \sigma(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}). \quad (7)$$

Let us first momentarily assume that E is zero. Then writing

$$\mathbf{B} = B_0 \hat{\mathbf{x}} \quad \mathbf{v} = v \hat{\mathbf{y}} \quad (8)$$

we find

$$\sigma \frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{\sigma v B}{c} \hat{\mathbf{z}} = \frac{\sigma v B}{c} (\sin \theta \mathbf{n}_2 - \cos \theta \mathbf{n}_1). \quad (9)$$

Of course the current can not flow off the sheet. So the charge must build up on both sides of the sheet, so that the component of the current normal to the surface vanishes. The induced electric field thus has a component perpendicular to the sheet

$$\mathbf{E} = \frac{\sigma v B}{c} \cos \theta \mathbf{n}_1, \quad (10)$$

so that the \mathbf{n}_1 components of $\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}$ is zero.

Recognizing a formal similarity with a parallel plate capacitor, on the top half of the sheet we have³

$$\kappa = -\frac{\sigma v B}{c} \cos \theta, \quad (11)$$

while on the bottom half of the sheet we have

$$\kappa = +\frac{\sigma v B}{c} \cos \theta. \quad (12)$$

³We use κ instead of the σ to denote the surface charge density.

(b) The current is

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad (13)$$

and so

$$\mathbf{J} = \frac{\sigma v B}{c} \sin \theta \mathbf{n}_2. \quad (14)$$

The energy dissipated per volume is \mathbf{J}^2/σ , and thus the energy lost per area \mathcal{A} is

$$\frac{1}{\mathcal{A}} \frac{dW}{dt} = \sigma \left(\frac{v B \sin \theta}{c} \right)^2 t. \quad (15)$$

(c) Estimating the typical velocity of the sphere $v \sim \omega R$, and the area $\mathcal{A} \sim R^2$, we find

$$\frac{dW}{dt} \sim \sigma \left(\frac{\omega R B_0}{c} \right)^2 R^2 t. \quad (16)$$

(d) The current density is given by

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \quad (17)$$

Let us momentarily assume that \mathbf{E} is zero. Then writing

$$\mathbf{B} = B_0 \hat{\mathbf{x}} \quad \mathbf{v} = \omega r \sin \theta \hat{\boldsymbol{\phi}}, \quad (18)$$

and using

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}, \quad (19)$$

we find that the B-field tries to create the current

$$\frac{\sigma}{c} \mathbf{v} \times \mathbf{B} = -\frac{\sigma \omega B_0}{c} r \sin \theta \cos \phi \hat{\mathbf{z}} = -\frac{\sigma \omega B_0}{c} r \sin \theta \cos \phi \left(-\sin \theta \hat{\boldsymbol{\theta}} + \cos \theta \hat{\mathbf{r}} \right). \quad (20)$$

The current will cause the charge to build up on the surface of the sphere, until the the component of the current normal to the surface will vanish. Thus we require that at the surfaces of the sphere the r component is

$$\sigma E_r = \frac{\sigma \omega B_0}{2c} r (2 \sin \theta \cos \theta \cos \phi) \equiv \frac{\sigma \omega B_0}{2c} Y(\theta, \phi) \quad (21)$$

where we have defined $Y \equiv 2 \sin \theta \cos \theta \cos \phi$. Then using the electrostatic boundary conditions⁴

$$\kappa = \hat{\mathbf{r}} \cdot (\mathbf{E}_2 - \mathbf{E}_1) \quad (22)$$

we find that on the outer surface the charge is

$$\kappa = -\frac{\omega B_0}{2c} r Y(\theta, \phi) \quad (23)$$

⁴To avoid confusion with the conductivity, we will use κ rather than σ to denote the surface charge density.

while on the inner surface the charge is of opposite sign

$$\kappa = \frac{\omega B_0}{2c} r Y(\theta, \phi) \quad (24)$$

We have tacitly assumed that the electric field vanishes outside of the metal. This is a valid assumption provided the thickness is small. The inner and outer surface act like a parallel plate capacitor.

(e) We also require that $\mathbf{E} = -\nabla\varphi$ where φ satisfies the Laplace equation. Recognizing that the span of the $\ell = 2$ spherical harmonics contains the function $Y = 2 \sin\theta \cos\theta \cos\phi \propto Y_{21} + Y_{2-1}$, and knowing that the general solution to the Laplace equation takes the form

$$\varphi = \sum_{\ell m} \left(\frac{A_{\ell m}}{r^{\ell+1}} + r^\ell B_{\ell m} \right) Y_{\ell m}, \quad (25)$$

we conclude that solution takes the form

$$\varphi = \left(\frac{A}{r^3} + r^2 B \right) Y. \quad (26)$$

Now want $-\partial_r\varphi = E_r$ with E_r given by Eq. (21), which fixes the constant $A = 0$ and B , leading to

$$\varphi = -\frac{1}{2} r^2 \frac{\omega B_0}{2c} Y. \quad (27)$$

The electric field is $\mathbf{E} = -\nabla\varphi$

$$\mathbf{E} = -\partial_r\varphi \hat{\mathbf{r}} - \frac{1}{r} \partial_\theta\varphi \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \partial_\phi\varphi \hat{\boldsymbol{\phi}}, \quad (28)$$

and we thus find

$$\sigma \mathbf{E} = \frac{\sigma \omega B_0}{2c} r \left(Y \hat{\mathbf{r}} + \frac{1}{2} \partial_\theta Y \hat{\boldsymbol{\theta}} + \frac{1}{2 \sin\theta} \partial_\phi Y \hat{\boldsymbol{\phi}} \right). \quad (29)$$

(f) We now only need to compute $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$. Writing

$$\sigma \frac{\mathbf{v}}{c} \times \mathbf{B} = \frac{\sigma \omega B_0}{2c} r \left(2 \sin^2\theta \cos\phi \hat{\boldsymbol{\theta}} - Y \hat{\mathbf{r}} \right), \quad (30)$$

$$= \frac{\sigma \omega B_0}{2c} r \left((\cos\phi - \frac{1}{2} \partial_\theta Y) \hat{\boldsymbol{\theta}} - Y \hat{\mathbf{r}} \right), \quad (31)$$

we find a current flow which is parallel to the surface

$$\mathbf{J} = \frac{\sigma \omega B_0}{2c} R \left(\cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}} \right). \quad (32)$$

One can check that $\nabla \cdot \mathbf{J} = 0$. The current flow is displayed graphically in Fig. 1.

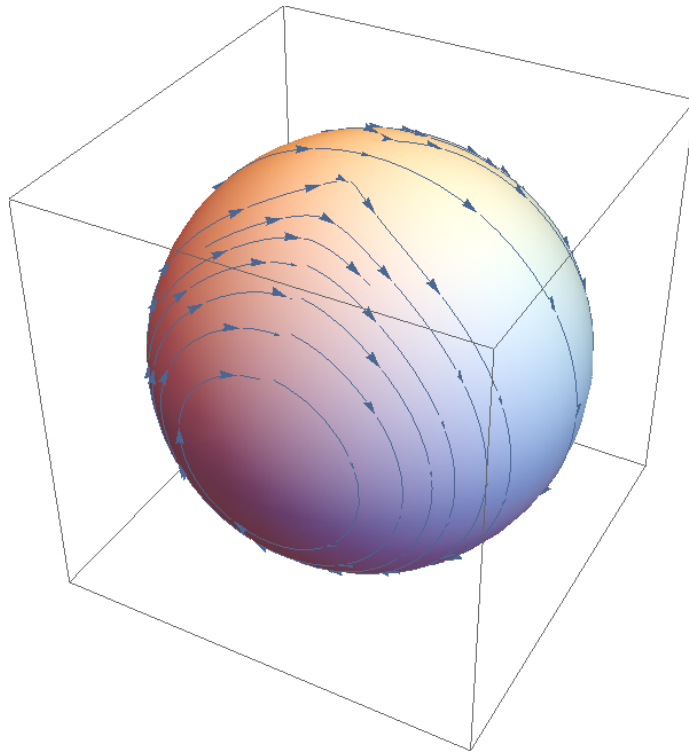


Figure 1: Current flow in the rotating sphere

Not on exam: So then, the total energy dissipated per time is

$$\frac{dW}{dt} = \int dV \frac{\mathbf{J}^2}{\sigma}. \quad (33)$$

Writing the volume as $dV = 4\pi R^2 t (d\Omega/4\pi)$ we have

$$\frac{dW}{dt} = \sigma \left(\frac{\omega R B_0}{2c} \right)^2 4\pi R^2 t \int \frac{d\Omega}{4\pi} (\cos^2 \phi + \cos^2 \theta \sin^2 \phi), \quad (34)$$

yielding a final result

$$\frac{dW}{dt} = \sigma \left(\frac{\omega R B_0}{c} \right)^2 R^2 t \left(\frac{2\pi}{3} \right). \quad (35)$$

This determines the order one numerical factor of part (c).

Quantum Mechanics 1

Angular momentum and Wigner functions

(a) (5 pts) Use the properties of the angular momentum operators $\vec{J} = \{J_x, J_y, J_z\}$, and the properties of the standard angular momentum eigenstates $|j, m\rangle$ to find the average and the magnitude of the fluctuations of the x component of the angular momentum in the eigenstates $|j, m\rangle$:

$$\langle j, m | J_x | j, m \rangle, \quad \langle j, m | J_x^2 | j, m \rangle.$$

For a given j , find the value of m which minimizes the standard deviation σ of J_x . Provide a brief (no more than two sentences) qualitative interpretation of the result.

(b) (5 pts) Using again the properties of \vec{J} , calculate the rotated operator

$$e^{i\beta J_y/\hbar} J_z e^{-i\beta J_y/\hbar}.$$

Compare the obtained expression to the classical vector rotation.

(c) (7 pts) From the results of parts (a) and (b), calculate the following characteristics of the Wigner functions $d_{m'm}(\beta)$:

$$\sum_{m'} m' |d_{m'm}(\beta)|^2, \quad \sum_{m'} m'^2 |d_{m'm}(\beta)|^2.$$

As a reminder, $d_{m'm}(\beta) = \langle j, m' | e^{-i\beta J_y/\hbar} | j, m \rangle$.

(d) (3 pts) Characteristics of the Wigner functions calculated in part (c) can be viewed as the average and the fluctuations of an operator. Identify the operator and calculate its standard deviation. Make a brief comparison to part (a).

Solution

(a) The x component of the momentum can be expressed in terms of the standard operators “raising” and “lowering” the projection m of the angular momentum on the z axis:

$$J_x = \frac{1}{2}(J_+ + J_-),$$

with the known matrix elements in the $|j, m\rangle$ basis:

$$\langle j, m' | J_{\pm} | j, m \rangle = [(j \mp m)(j \pm m + 1)]^{1/2} \delta_{m', m \pm 1}.$$

These relations show explicitly that the average of J_x vanishes in any of the states $|j, m\rangle$:

$$\langle j, m | J_x | j, m \rangle = 0.$$

and

$$\langle j, m | J_x^2 | j, m \rangle = \frac{1}{4} \langle j, m | (J_+ J_- + J_- J_+) | j, m \rangle = \frac{\hbar^2}{4} [(j+m)(j-m+1) + (j-m)(j+m+1)] = \frac{\hbar^2}{2} [j(j+1) - m^2].$$

One can see immediately that the standard deviation σ of J_x is reached for $m = \pm j$:

$$\sigma = \langle j, m | J_x^2 | j, m \rangle^{1/2} |_{m=\pm j} = \hbar \sqrt{j/2}.$$

In this case, the angular momentum vector is aligned to the largest extent possible along the z axis, either in the positive or negative direction. Because of this, its fluctuations in the x direction orthogonal to z are minimized.

(b) The most direct way to find the rotated operator J_z is to use the Baker-Hausdorff formula:

$$e^{i\beta J_y/\hbar} J_z e^{-i\beta J_y/\hbar} = J_z + i\frac{\beta}{\hbar} [J_y, J_z] + \frac{(i\beta)^2}{2\hbar^2} [J_y, [J_y, J_z]] + \dots + \frac{(i\beta)^k}{k!\hbar^k} [J_y, [J_y, \dots [J_y, J_z] \dots]] + \dots,$$

where the last (general) terms contains k commutators. Using the fact that

$$[J_y, J_z] = i\hbar J_x, \quad [J_y, J_x] = -i\hbar J_z,$$

we see that the operator part of all odd terms in the series is equal to J_x , while of all even terms $-J_z$, while the numerical factors sum up to sine and cosine, respectively:

$$e^{i\beta J_y/\hbar} J_z e^{-i\beta J_y/\hbar} = J_z \cos \beta - J_x \sin \beta.$$

This formula agrees precisely with the classical expression for the components of the vector oriented originally in the positive z direction and rotated around the y axis, if we interpret the direction of rotation in this expression to be clockwise.

(c) To determine the properties of the Wigner functions $d_{m'm}(\beta)$, we start with the definition:

$$d_{m'm}(\beta) = \langle j, m' | e^{-i\beta J_y/\hbar} | j, m \rangle,$$

which shows that the sum we need to find can be expressed as follows:

$$\begin{aligned} \sum_{m'} m' |d_{m'm}(\beta)|^2 &= \sum_{m'} m' d_{m'm}(\beta) d_{m'm}^*(\beta) = \sum_{m'} m' d_{m'm}(\beta) (d^\dagger)_{mm'}(\beta) \\ &= \sum_{m'} \langle j, m | e^{i\beta J_y/\hbar} |j, m'\rangle m' \langle j, m' | e^{-i\beta J_y/\hbar} |j, m\rangle = \frac{1}{\hbar} \langle j, m | e^{i\beta J_y/\hbar} J_z e^{-i\beta J_y/\hbar} |j, m\rangle. \end{aligned}$$

As was shown in part (b), the last expression can be written as

$$\frac{1}{\hbar} \langle j, m | J_z \cos \beta - J_x \sin \beta |j, m\rangle = m \cos \beta.$$

This result can be viewed as a direct consequence of the vector character of the angular momentum \vec{J} : if the initial z component of the momentum is m , it becomes equal to $m \cos \beta$ after the rotation by angle β .

In the same way, we can express the sum as the matrix element of the square of the rotated J_z operator:

$$\sum_{m'} m'^2 |d_{m'm}(\beta)|^2 = \frac{1}{\hbar^2} \langle j, m | \left[e^{i\beta J_y/\hbar} J_z e^{-i\beta J_y/\hbar} \right]^2 |j, m\rangle = \frac{1}{\hbar^2} \langle j, m | (J_z \cos \beta - J_x \sin \beta)^2 |j, m\rangle.$$

Using the results of part (a) we transform this further as

$$m^2 \cos^2 \beta + \sin^2 \beta \frac{1}{\hbar^2} \langle j, m | J_x^2 |j, m\rangle = \frac{1}{2} \left[j(j+1) \sin^2 \beta + m^2 (3 \cos^2 \beta - 1) \right].$$

(d) From the derivations in part (c), we see that the operator in question is the rotated J_z operator from part (b). The standard deviation σ of this operator is:

$$\begin{aligned} \sigma &= [\langle j, m | (J_z \cos \beta - J_x \sin \beta)^2 |j, m\rangle - \langle j, m | J_z \cos \beta - J_x \sin \beta |j, m\rangle^2]^{1/2} \\ &= [(j(j+1) - m^2)/2]^{1/2} \sin \beta \end{aligned}$$

We see that apart from the overall factor $\sin \beta$, this is the same expression as in part (a). This is natural, since the J_z part of the rotated operator does not contribute to the fluctuations, since the average is taken over the eigenstates of J_z .

Quantum Mechanics 2

Two particles in a box – perturbative, adiabatic & diabatic changes

Two spin-1/2 fermions of mass μ interact only through a “ferromagnetic” spin-spin interaction:

$$V = -u\vec{\sigma}_1 \cdot \vec{\sigma}_2, \quad u > 0,$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) (3 pts) What is the spin contribution $E_{s,t}$ to the particle energy in the singlet and the triplet states, respectively.

(b) (5 pts) The particles are confined to the one-dimensional interval $0 < x < a$ (box with width a). What are the (total) eigenenergies $E_{n,m}$ and the wavefunctions $\psi_{n,m}(x_1, x_2)$ of the two particles? Indicate when the ground state is singlet or triplet and explain why.

(c) (4 pts) A perturbation potential

$$U(x) = \begin{cases} \delta & x > a/2 \\ -\delta & x < a/2 \end{cases}.$$

is applied to the particles. Find the first-order correction to the energy of a singlet ground state.

(d) (4 pts) Consider making changes to the box potential by adjusting the width to a different value b . If this width is changed smoothly and slowly, how does the wave function and energy of the particles change with time if they start in the ground state in the original box?

(e) (4 pts) Consider starting with the system in the lowest energy singlet state and adjusting the size of the box suddenly to $b = 2a$. How does the wave function and energy of the particles change with time? Calculate the probability p of still finding the particles in the (new) ground state right after the sudden expansion.

Solution

(a) For Fermions, the total wavefunction must be antisymmetric with respect to exchange of the two particles. If one aims to construct the total wavefunction as a product of spin and spatial portions, then the spin portion can be symmetric or antisymmetric, corresponding to triplet or singlet states. The triplet state, with total spin $M = 0, \pm 1$ along the z -axis, is symmetric in the spins

$$\chi_3(M) = \begin{cases} |\uparrow\uparrow\rangle & M = 1, \\ |\downarrow\downarrow\rangle & M = -1, \\ \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) & M = 0. \end{cases} \quad (1)$$

while the singlet state is antisymmetric in the spins

$$\chi_1 = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \quad (2)$$

The expectation value of V for the triplet states is $-u$, whereas for the singlet state it is u .

(b) For a single particle in a box of width a , the wave function is given by:

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin(n\pi x/a) \quad (3)$$

For two identical particles in a box, the spatial portion of the total wavefunction is given by:

$$\psi_{n,m}^{\pm}(x_1, x_2) = \frac{1}{a} (\sin(n\pi x_1/a) \sin(m\pi x_2/a) \pm \sin(m\pi x_1/a) \sin(n\pi x_2/a)). \quad (4)$$

The complete wavefunction of the triplet state is

$$\Psi_{n,m;M}(x_1, x_2) = \psi_{n,m}^-(x_1, x_2) \chi_3(M), \quad (5)$$

while the complete wavefunction of the singlet state is

$$\Psi_{n,m}(x_1, x_2) = \psi_{n,m}^+(x_1, x_2) \chi_1. \quad (6)$$

The total energies are given by:

$$E_{n,m} = \frac{n^2 h^2}{8\mu a^2} + \frac{m^2 h^2}{8\mu a^2} \pm u \quad (7)$$

For a triplet state, $n \neq m$, so the energy of the state with the lowest n, m values is given by:

$$E_{t,1,2} = \frac{5h^2}{8\mu a^2} - u \quad (8)$$

For a singlet state, $n = m = 1$, and the energy is given by:

$$E_{s,1,2} = \frac{2h^2}{8\mu a^2} + u \quad (9)$$

The ground state is triplet if:

$$E_{1,2} = \frac{5h^2}{8\mu a^2} - u < \frac{2h^2}{8\mu a^2} + u \quad (10)$$

i.e., if

$$E_{1,2} = \frac{3h^2}{8\mu a^2} < 2u \quad (11)$$

(c) The first order shift to the singlet ground state is trivially zero

$$E^{(1)} = \langle \Psi | H' | \Psi \rangle = \frac{4\delta}{a^2} \int_0^a \int_0^a dx_1 dx_2 \sin^2(\pi x_1/a) \sin^2(\pi x_2/a) (U(x_1) + U(x_2)) = 0 \quad (12)$$

This follows because the wave functions are even and the perturbing potential is odd.

(d) The spatial portion of the wave function smoothly evolves to:

$$\psi_{n,m}(x_1, x_2) = \frac{1}{b} (\sin(n\pi x_1/b) \sin(m\pi x_2/b) \pm \sin(m\pi x_1/b) \sin(n\pi x_2/b)) \quad (13)$$

and the energies are given by

$$E_{n,m} = \frac{n^2 h^2}{8\mu b^2} + \frac{m^2 h^2}{8\mu b^2} \pm u \quad (14)$$

(e) In the sudden approximation, the wave function does not have time to adjust to the change in potential and remains unchanged, projecting onto the eigenstates of the new potential. The probability of finding the particles in the new ground state is given by the projection of the initial ground state state onto the final one:

$$\begin{aligned} p &= \left| \langle \psi_{1,1}^a(x_1, x_2) | \psi_{1,1}^b(x_1, x_2) \rangle \right|^2 \\ &= \left| \frac{2}{a} \frac{2}{b} \int_0^a \int_0^a dx_1 dx_2 (\sin(\pi x_1/a) \sin(\pi x_2/a) \sin(\pi x_1/b) \sin(\pi x_2/b)) \right|^2 \\ &= \left| \frac{4}{ab} \int_0^a \int_0^a dx_1 dx_2 (\sin(\pi x_1/a) \sin(\pi x_2/a) \sin(\pi x_1/2a) \sin(\pi x_2/2a)) \right|^2 \\ &= \frac{1}{4} \left| \frac{8}{3\pi} \right|^4 = 0.13 \end{aligned} \quad (15)$$

Quantum Mechanics 3

Isotropic 3D harmonic oscillator

The Hamiltonian of an isotropic 3D harmonic oscillator of mass μ and frequency ω is:

$$H = \frac{p^2}{2\mu} + \frac{\mu\omega^2 r^2}{2}.$$

(a) (5 pts) Show that the time-independent Schrödinger equation separates in Cartesian coordinate system. Determine the eigenenergies E_n and their degeneracies g_n of the stationary states of the oscillator.

(b) (5 pts) Express the Cartesian components of the angular momentum operator $\vec{L} = \{L_x, L_y, L_z\}$ in terms of the raising and lowering operator a_j^\dagger, a_j , where $j = x, y, z$. Find the time dependence of \vec{L} in Heisenberg representation.

(c) (5 pts) Calculate the operator L^2 of the angular momentum and express it through the operator n of the total number of the excitations, and operators Q and Q^\dagger :

$$n = \sum_j a_j^\dagger a_j, \quad Q = \sum_j a_j^2.$$

(d) (5 pts) Consider the subspace of the degenerate energy eigenstates $|n_x, n_y, n_z\rangle$ of the oscillator with energy $E = (7/2)\hbar\omega$. Using the previous results in this problem, construct the state with vanishing angular momentum, $|L^2 = 0\rangle$, in this subspace.

Solution

(a) In Cartesian coordinates, one sees immediately that the Hamiltonian of a 3D harmonic oscillator is a sum of the three equivalent 1D oscillators, each for one coordinate:

$$H = H_x + H_y + H_z.$$

By the standard logic, this means that the time-independent Schrödinger equation can be reduced to three independent Schrödinger equations, one for each coordinate, and the eigenenergies E_n of the stationary states of the 3D oscillator are the sums of the three energies of each of the 1D oscillators. This means that

$$E_n = \hbar\omega\left(n + \frac{3}{2}\right) \quad n = n_x + n_y + n_z, \quad n_j = 0, 1, 2, \dots, \quad j = x, y, z.$$

Degeneracy g_n of the energy level E_n is then given by the number of way in which a positive integer n can be represented as a sum of the three non-negative integers, and can be found directly. If the number of excitations in the x oscillator is n_x , the remaining $n - n_x$ quanta can be distributed between the y and z oscillators in the $n - n_x + 1$ ways. Therefore, the total number of ways to distribute n quanta over the three coordinates is

$$g_n = \sum_{n_x=0}^n (n - n_x + 1) = (n + 1)^2 - \frac{n(n + 1)}{2} = \frac{(n + 1)(n + 2)}{2}.$$

(b) One starts by expressing the operators of coordinates and the corresponding momenta in terms of the raising and lowering operators, e.g.

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_x + a_x^\dagger), \quad p_x = -i\sqrt{\frac{m\omega\hbar}{2}}(a_x - a_x^\dagger).$$

Using these relations in the definition of the component of the angular momentum, e.g., $L_z = xp_y - yp_x$, we find:

$$L_z = \frac{i\hbar}{2} [(a_y + a_y^\dagger)(a_x - a_x^\dagger) - (a_x + a_x^\dagger)(a_y - a_y^\dagger)] = i\hbar(a_x a_y^\dagger - a_y a_x^\dagger).$$

note that the order of operators in each of the products is arbitrary, since raising and lowering operators for different coordinates commute. The other two component are obtained similarly

$$L_x = i\hbar(a_y a_z^\dagger - a_z a_y^\dagger), \quad L_y = i\hbar(a_z a_x^\dagger - a_x a_z^\dagger).$$

One can check that these expressions satisfy the necessary momentum commutation relations

$$[L_x, L_y] = i\hbar L_z,$$

plus cyclic permutations of x, y, z . Using these expressions, one can check explicitly that every component of the angular momentum operator \vec{L} commutes with the Hamiltonian:

$$[H, L_x] = [H, L_y] = [H, L_z] = 0.$$

This means that the equation of motion of the \vec{L} in the Heisenberg representation is

$$d\vec{L}/dt = 0,$$

i.e. angular momentum is a constant of motion, as one should expect for a spherically symmetric system.

(c) The operator L^2 is calculated directly by squaring each of the components:

$$L_z^2 = \hbar^2 [a_y^\dagger a_y a_x a_x^\dagger + a_x^\dagger a_x a_y a_y^\dagger - a_y^\dagger a_y^\dagger a_x a_x - a_x^\dagger a_x^\dagger a_y a_y],$$

$$L_x^2 = \hbar^2 [a_y^\dagger a_y a_z a_z^\dagger + a_z^\dagger a_z a_y a_y^\dagger - a_y^\dagger a_y^\dagger a_z a_z - a_z^\dagger a_z^\dagger a_y a_y],$$

$$L_y^2 = \hbar^2 [a_z^\dagger a_z a_x a_x^\dagger + a_x^\dagger a_x a_z a_z^\dagger - a_z^\dagger a_z^\dagger a_x a_x - a_x^\dagger a_x^\dagger a_z a_z].$$

The sum of all these components: $L^2 = L_x^2 + L_y^2 + L_z^2$ can be expressed through the operator n of the total number of excitations, $n = \sum_j a_j^\dagger a_j$, if one adds and subtracts the sum of the "diagonal" terms $\sum_j (a_j^\dagger a_j)^2$. Then, one finds directly that

$$L^2 = \hbar^2 [n(n+1) - Q^\dagger Q], \quad Q = \sum_j a_j^2.$$

(d) For states with energy $E = (7/2)\hbar\omega$, there are two excitations: $n = 2$, and in total, there are $g_2 = 6$ such states:

$$|2, 0, 0\rangle, \quad |0, 2, 0\rangle, \quad |0, 0, 2\rangle, \quad |1, 1, 0\rangle, \quad |1, 0, 1\rangle, \quad |0, 1, 1\rangle.$$

Operator Q annihilates all "1-1" states, since it destroys two excitations in each coordinate:

$$Q|1, 1, 0\rangle = Q|1, 0, 1\rangle = Q|0, 1, 1\rangle = 0.$$

Expression derived in part (c) means then that these states correspond to $L^2 = 6\hbar^2$ and can not participate in the $L^2 = 0$ state. The part $Q^\dagger Q$ of the L^2 operator has the following matrix form in the basis of the remaining 3 states, $|2, 0, 0\rangle, |0, 2, 0\rangle, |0, 0, 2\rangle$:

$$Q^\dagger Q = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Characteristic equation for this matrix:

$$\lambda^2(\lambda - 6) = 0$$

gives the eigenvalues 0, 0, 6. Expression for L^2 obtained in (c) shows that the $L^2 = 0$ state is the eigenvector of this matrix with the eigenvalue 6. The corresponding vector is found immediately from the form of the matrix as

$$|L^2 = 0\rangle = \frac{1}{\sqrt{3}}[|2, 0, 0\rangle + |0, 2, 0\rangle + |0, 0, 2\rangle].$$

Statistical Mechanics 1

Specific heat, magnetic susceptibility, and the Stoner instability

Consider electrons in a band with a density of states (for a single spin) of the form

$$g(\epsilon) = g_0 \left(1 - \frac{\epsilon^2}{\epsilon_0^2} \right), \quad (1)$$

for $-\epsilon_0 < \epsilon < \epsilon_0$ and zero otherwise. The band is half-filled⁵ at $T = \mu = 0$. Assuming first that the electrons do not interact, the grand potential is

$$\Omega(T, V, \mu, H) = -VT \sum_{\sigma} \int_{-\epsilon_0}^{\epsilon_0} d\epsilon g(\epsilon) \ln[1 + e^{-\beta(\epsilon + 2\mu_B \sigma H - \mu)}]. \quad (2)$$

Here $\beta = 1/T$, V is the volume, μ_B is the Bohr magneton, H is the magnetic field and $\sigma = \pm 1/2$ is the electron spin.

We will start with zero magnetic field, $H = 0$.

- (3 points) Derive an expression from Ω for the number of electrons N and the energy of the electrons E in terms of $g(\epsilon)$, T , V and μ (you may leave any integrals unevaluated).
- (1 point) Compute N at half filling, i.e. the value of N at $T=\mu=0$.
- (2 points) Show that the chemical potential is independent of temperature (and hence zero) if the number of electrons is kept at its half filling value of part (b).
- (5 points) For low temperatures and N at half filling, compute the specific heat, $C = \frac{1}{V} \frac{\partial E}{\partial T}$, keeping the lowest term in the expansion $C = c_0 + c_1 T + c_2 T^2 + \dots$.
Formulate your answer in terms of ϵ_0 , g_0 and numerical coefficients. Some helpful integrals are given at the end.

Now we look at non-zero magnetic field, assuming $\mu_B H \ll T$.

- (5 points) For low temperature and N at half filling, compute the magnetization M (where M is proportional to H) and the susceptibility, $\chi = \frac{1}{V} \frac{\partial M}{\partial H} = \frac{1}{V} \frac{M}{h}$, keeping the lowest term in the expansion $\chi = b_0 + b_1 T + b_2 T^2 + \dots$.
Formulate your answer in terms of ϵ_0 , g_0 and numerical coefficients. Some helpful integrals are given at the end.
- (1 point) What is the value of the ratio $\frac{C/T}{\chi/\mu_B^2}$? (Note: If you do this correctly, this quantity is independent of the density of states.)

⁵This description is relevant in transition metals, where the Fermi energy lies in a narrow band originating from the d electrons.

Now we turn to interacting electrons. We represent the interaction by a “mean field” contribution to the energy:

$$E_{int} = \gamma \frac{N_+ N_-}{V} \quad (3)$$

where γ is the so-called Stoner coupling constant and N_+ and N_- are the number of electron in the spin up and spin down state, respectively. This interaction (the exchange interaction) describes the fact that Coulomb repulsion between the electrons is reduced for electrons with parallel spins, when the spacial component of the wave function has to be anti-symmetric.

(g) (3 points) At zero temperature the system develops spontaneous magnetization if $\gamma > \gamma_{crit}$. What is the value of γ_{crit} ?

Integrals you may need:

We define

$$\alpha_n \equiv \int_{-\infty}^{\infty} \frac{x^n e^x}{(e^x + 1)^2} \cdot \quad (4)$$

The first few coefficients are

$$\alpha_0 = 1, \quad (5)$$

$$\alpha_1 = 0, \quad (6)$$

$$\alpha_2 = \frac{\pi^2}{3}, \quad (7)$$

$$\alpha_3 = 0, \quad (8)$$

$$\alpha_4 = \frac{7\pi^4}{15}. \quad (9)$$

Solution

We define the Fermi function

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \quad (10)$$

Here is a summary of some basic properties of the Fermi function:

$$f' = \frac{df}{d\epsilon} = \frac{-\beta e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)} + 1)^2} = \frac{-\beta}{(e^{\beta(\epsilon-\mu)/2} + e^{-\beta(\epsilon-\mu)/2})^2} \quad (11)$$

The function $-f'$ is symmetric around $\epsilon = \mu$ and it approaches a Dirac delta function at low temperatures. Other derivatives of f can be expressed with this function

$$\frac{df}{d\beta} = \frac{-(\epsilon - \mu)}{(e^{\beta(\epsilon-\mu)/2} + e^{-\beta(\epsilon-\mu)/2})^2} = \frac{\epsilon - \mu}{\beta} f' \quad (12)$$

$$\frac{df}{dT} = -\frac{\epsilon - \mu}{T} f' \quad (13)$$

See the Appendix for a quick guide on how to handle integrals involving $f'(\epsilon)$.

(a) At zero field the \sum_{σ} brings in a factor 2. The particle number is

$$N = -\frac{\partial \Omega}{\partial \mu} = -2TV \int_{-\infty}^{\infty} g(\epsilon) \frac{-\beta e^{-\beta(\epsilon-\mu)}}{1 + e^{-\beta(\epsilon-\mu)}} d\epsilon = 2V \int_{-\infty}^{\infty} g(\epsilon) f(\epsilon) d\epsilon \quad (14)$$

For the energy:

$$E = \Omega - TS - \mu N \quad (15)$$

Here the entropy is

$$S = -\frac{\partial \Omega}{\partial T} = 2V \int_{-\infty}^{\infty} g(\epsilon) \left(\ln[1 + e^{-\beta(\epsilon-\mu)}] + \frac{d}{dT} \ln[1 + e^{-\beta(\epsilon-\mu)}] \right) d\epsilon \quad (16)$$

The second term is

$$\frac{d}{dT} \ln[1 + e^{-\beta(\epsilon-\mu)}] = \frac{e^{-\beta(\epsilon-\mu)}}{1 + e^{-\beta(\epsilon-\mu)}} [-(\epsilon - \mu)] \left(-\frac{1}{T^2} \right) \quad (17)$$

and therefore

$$S = 2V \int_{-\infty}^{\infty} g(\epsilon) \left(\ln[1 + e^{-\beta(\epsilon-\mu)}] + \frac{T}{1 + e^{\beta(\epsilon-\mu)}} \frac{\epsilon - \mu}{T^2} \right) d\epsilon \quad (18)$$

When inserted to $E = \Omega - TS - \mu N$ the first (logarithmic) term is canceled with Ω and in the second term the part that contains μ is canceled with μN . The remaining part is

$$E = 2V \int_{-\infty}^{\infty} \epsilon g(\epsilon) f(\epsilon) d\epsilon \quad (19)$$

(b) At $T = 0$, the fermi function is unity below μ and zero above μ here $\mu = 0$. The number therefore reduces to

$$N = 2V \int_{-\epsilon}^0 g(\epsilon) d\epsilon \quad (20)$$

$$= \frac{4Vg_0\epsilon_0}{3} \quad (21)$$

(c) It is easy to show that using $\mu = 0$ yields the same particle number, independent of temperature:

$$\frac{dN}{d\beta} = 2V \int_{-\infty}^{\infty} g(\epsilon) \frac{df(\epsilon)}{d\beta} d\epsilon = 0 \quad (22)$$

because $g(\epsilon)$ is an even function of ϵ and $\frac{df(\epsilon)}{d\beta} = \frac{\epsilon}{\beta} f'$ is an odd function. Therefore the chemical potential is entirely independent of the temperature

(d) Now we will work out the specific heat. From now on we will use $\mu = 0$. Since $CV = \partial E / \partial T$ we find

$$CV = 2V \frac{d}{dT} \int_{-\infty}^{\infty} \epsilon g(\epsilon) f(\epsilon) d\epsilon = 2V \int_{-\infty}^{\infty} \epsilon g(\epsilon) \frac{df(\epsilon)}{dT} d\epsilon = 2V \int_{-\infty}^{\infty} \epsilon g(\epsilon) [-f'(\epsilon)] \frac{\epsilon}{T} d\epsilon \quad (23)$$

We use the low temperature expansion developed in the appendix. We take $K = \epsilon^2 g(\epsilon)$ (see Appendix), with $K' = 2\epsilon g + \epsilon^2 g'$, $K'' = 2g + 4\epsilon g' + \epsilon^2 g''$, all evaluated at the chemical potential $\epsilon = \mu = 0$ where $K'' = 2g(\epsilon = 0) = g_0$. The expansion yields:

$$CV = \frac{2V}{T} \left(\frac{\alpha_2}{2} T^2 2g_0 \right) + \text{higher} \quad (24)$$

The final result is

$$C = 2\alpha_2 g_0 T + \dots \quad (25)$$

It turns out that the leading term is proportional to the density of states.

(e) Now we will work out the magnetic susceptibility. The magnetization reads

$$M = -\frac{\partial \Omega}{\partial H} = -TV \sum_{\sigma} \int_{-\infty}^{\infty} g(\epsilon) \frac{2\mu_B \sigma \beta e^{-\beta(\epsilon + 2\mu_B \sigma H - \mu)}}{1 + e^{-\beta(\epsilon + 2\mu_B \sigma H - \mu)}} d\epsilon \quad (26)$$

We introduce $\Delta = \mu_B H$ and write out the two terms of the sum

$$M = -V\mu_B \int_{-\infty}^{\infty} [g(\epsilon)[f(\epsilon + \Delta) - f(\epsilon - \Delta)]] d\epsilon \quad (27)$$

We expand the Fermi function, $f(\epsilon + \Delta) = f + f'\Delta$

$$M = 2V\mu_B^2 H \int_{-\infty}^{\infty} g(\epsilon) [-f'(\epsilon)] d\epsilon \quad (28)$$

Here the leading term is in the 0^{th} order and

$$M = 2V\mu_B^2 H[g(\epsilon = 0) + \frac{\alpha_2}{2}T^2 g''(\epsilon = 0)] + \dots \quad (29)$$

The leading order susceptibility is

$$\chi = 2\mu_B^2 g_0 \quad (30)$$

(f) Using parts (d) and (e) we find that $\frac{C/T}{\chi/\mu_B^2} = \alpha_2$.

(g) Now we will analyze the Stoner-instability. The energy and the particle number with non-zero field are

$$\frac{E}{V} = \int_{-\infty}^{\infty} \epsilon g(\epsilon)[f(\epsilon + \Delta) + f(\epsilon - \Delta)]d\epsilon = 2 \int_{-\infty}^{\infty} \epsilon g(\epsilon)f(\epsilon)d\epsilon + \Delta^2 \int_{-\infty}^{\infty} \epsilon g(\epsilon)f''(\epsilon)d\epsilon \quad (31)$$

$$\frac{N}{V} = \int_{-\infty}^{\infty} g(\epsilon)[f(\epsilon + \Delta) + f(\epsilon - \Delta)]d\epsilon = 2 \int_{-\infty}^{\infty} g(\epsilon)f(\epsilon)d\epsilon + \Delta^2 \int_{-\infty}^{\infty} g(\epsilon)f''(\epsilon)d\epsilon \quad (32)$$

where we used the expansion $f(\epsilon + \Delta) + f(\epsilon - \Delta) = 2(f + \frac{1}{2}f''\Delta^2)$. By partial integration we can turn the second term in the particle number into

$$\Delta^2 \int_{-\infty}^{\infty} g(\epsilon)f''(\epsilon)d\epsilon = -\Delta^2 \int_{-\infty}^{\infty} g'(\epsilon)f'(\epsilon)d\epsilon \quad (33)$$

because $gf'|_{\pm\infty} = 0$. This integral is zero, because g' and all higher order odd derivatives of g are zero. That is good news, the chemical potential remains $\mu = 0$, even in magnetic field.

For the energy, the temperature dependence of the first term can be reconstructed from the specific heat:

$$2 \int_{-\infty}^{\infty} \epsilon g(\epsilon)f(\epsilon)d\epsilon = \frac{E_0}{V} + \int C dT = \frac{E_0}{V} + \alpha_2 g_0 T^2 \quad (34)$$

where E_0 is the ground state energy. For the second term the partial integration yields

$$\Delta^2 \int_{-\infty}^{\infty} \epsilon g(\epsilon)f''(\epsilon)d\epsilon = -\Delta^2 \int_{-\infty}^{\infty} (g + \epsilon g')f'd\epsilon \quad (35)$$

We do not need to care about temperature dependence (because of the Δ^2 pre-factor), and we can just keep the leading term, $-\int_{-\infty}^{\infty} (g + \epsilon g')f'd\epsilon = g_0$. Finally we get

$$\frac{E}{V} = \frac{E_0}{V} + \alpha_2 g_0 T^2 + \Delta^2 g_0 \quad (36)$$

This expression has an energy minimum at $\Delta = 0$. Let us take a look at the interaction term. First we express N_+ and N_- :

$$\frac{N_{\pm}}{V} = \int_{-\infty}^{\infty} g(\epsilon)f(\epsilon \pm \Delta) = \int_{-\infty}^{\infty} gf d\epsilon \pm \Delta \int_{-\infty}^{\infty} g(\epsilon)f'(\epsilon)d\epsilon = \frac{N}{2V} \pm \Delta \int_{-\infty}^{\infty} g(\epsilon)f'(\epsilon)d\epsilon \quad (37)$$

Again, $\pm \int_{-\infty}^{\infty} g(\epsilon)f'(\epsilon)d\epsilon = \mp g_0$, and the product entering into the interaction term is

$$\frac{N_+ N_-}{V^2} = \frac{N^2}{4V^2} - \Delta^2 g_0^2 \quad (38)$$

The energy of the interacting system is

$$\frac{E}{V} = \frac{E_0}{V} + \alpha_2 g_0 T^2 + \Delta^2 g_0 + \frac{N^2}{4V^2} - \Delta^2 g_0^2 = \frac{E_0}{V} + \frac{N^2}{4V^2} + \alpha_2 g_0 T^2 + \Delta^2 g_0 (1 - \gamma g_0) \quad (39)$$

The critical value of the coupling is reached when there is no longer an energy minimum at $\Delta = 0$. At zero temperature this happens when $\gamma_{crit} = 1/g_0$.

At zero temperature, the Stoner instability can be solved without using the Bethe-Sommerfeld expansion, simply taking the Fermi function as a step function and evaluating the integrals with constant density of states, g_0 .

Appendix: The Bethe - Sommerfeld expansion.

Assume we want to calculate an integral of type

$$F = \int_{-\infty}^{\infty} K(\epsilon) [-f'(\epsilon)] d\epsilon \quad (40)$$

We expand K into a Taylor series around $\epsilon = \mu$:

$$K = \sum_{n=0}^{\infty} \frac{(\epsilon - \mu)^n}{n!} \frac{d^n K}{d\epsilon^n} d\epsilon \quad (41)$$

and we get

$$F = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} [-f'(\epsilon)] \frac{(\epsilon - \mu)^n}{n!} \frac{d^n K}{d\epsilon^n} d\epsilon \quad (42)$$

We substitute $x = \beta(\epsilon - \mu)$ to get

$$F = \sum_{n=1}^{\infty} \frac{\alpha_n}{n!} \frac{d^n K}{d\epsilon^n} \quad (43)$$

The coefficients are

$$\alpha_n = \int_{-\infty}^{\infty} x^n [-f'(x)] dx \quad (44)$$

Since f' is an even function, $\alpha_n = 0$ for odd n . The first few non-vanishing coefficients are

$$\alpha_0 = 1 \quad (45)$$

$$\alpha_2 = \int_{-\infty}^{\infty} \frac{x^2 e^x}{(e^x + 1)^2} dx = \frac{\pi^2}{3} \quad (46)$$

$$\alpha_4 = \int_{-\infty}^{\infty} \frac{x^4 e^x}{(e^x + 1)^2} dx = \frac{7\pi^4}{15} \quad (47)$$

Statistical Mechanics 2

Thermodynamics of weak solutions

This problem explores thermodynamics of a weak solution consisting N_s molecules of solvent and N molecules of solute (minor component) with small relative concentration $c = N/N_s \ll 1$. The Gibbs potential of such *dilute* solution can be approximated as

$$\Phi(p, T, N_s, N) \approx N_s \mu_{s0} + N \phi(p, T) + NT \log \frac{N}{N_s} \quad (1)$$

where μ_{s0} is the chemical potential of pure solvent ($N = 0$), $\phi(p, T)$ is the contribution of a single solute molecule to the Gibbs potential, and the third term describes the entropy correction due to indistinguishable nature of solute molecules and is closely related to the “Boltzmann’s factor” in the ideal gas. Note that this Gibbs potential is extensive, $\Phi(p, T, N_s, N) = N_s f(p, T, c)$, $c = N/N_s$.

(A) [3pt] How much mechanical work can be extracted in a process where such solution is further diluted by a factor of 2 (assuming constant temperature and pressure)?

(B) [3pt] Compute the chemical potential of **solvent** molecules for solute concentration $c > 0$.

(C) [5pt] Consider a volume of solution ($c_1 = c$) and a volume of pure solvent ($c_2 = 0$) separated by a heat-conducting membrane impermeable to the solute molecules. What is the condition for their equilibrium? Find the osmotic pressure acting on the membrane $\Delta p = p_1 - p_2$.

(D) [5pt] Now consider equilibrium of salty water and ice (which cannot contain any significant amount of salt). How does the melting temperature change for fixed $c = \text{const}$?

Hint: examine how the derivation of Clausius-Clapeyron equation $\frac{dP}{dT} = \frac{s_1 - s_2}{v_1 - v_2}$ is modified if one of the phases can contain solute.

(E) [4pt] Estimate the shift in the ice melting temperature in equilibrium with 1g NaCl/liter salt water.

Some useful physical quantities (at 1 atm) are given on the next page:

Water specific heat	$\approx 4.2 \text{ kJ}/(\text{kg} \cdot \text{K})$
Ice specific heat	$\approx 2.1 \text{ kJ}/(\text{kg} \cdot \text{K})$
Ice melting latent heat (1atm)	$L = 334 \text{ kJ}/\text{kg}$
Standard atomic weight, Na	$\mu_{Na} \approx 22.99 \text{ g}/\text{mol}$
Standard atomic weight, Cl	$\mu_{Cl} \approx 35.45 \text{ g}/\text{mol}$

Solution

Introduction: For pure solvent ($N = 0$), only the first term contributes, so it represents the chemical potential of pure solvent molecules. The 2nd and 3rd terms in Eq. (1) can be thought of the effect of adding solute molecules one by one. As long as the relative concentration $N/N_s \ll 1$ is small, their contributions to the enthalpy are independent from each other and are represented by the 2nd term. This results in total contribution $\propto N$, where $\phi(p, T)$ does not depend on the concentration N/N_s reflecting the fact that solute molecules are too dilute to interact with each other. The 3rd term represents correction to the entropy similar to the Boltzmann factor in the canonical statistical sum of an ideal gas. Since all the solute molecules are identical, the statistical weight of their any particular configuration in the solution is reduced by factor $1/N!$, which amounts to *negative* shift in entropy $\Delta s_{(B)} = -\log N!$ and the corresponding *positive* shift in the Gibbs potential $\Delta G_{(B)} = -T\Delta s_{(B)}$. The Gibbs potential is an extensive quantity, so this correction must scale as $\propto N$ if the concentration $N/N_s = \text{const}$; therefore,

$$\Delta G_{(B)} = T \cdot N \log \frac{N}{N_s}. \quad (2)$$

In other words, the 3rd term compensates for over-estimation of solute entropy that would happen if solute molecules are considered distinguishable. The actual entropy is smaller than that by $\log \frac{N}{cN_s}$ because each new molecule can be permuted with all the previously added identical solute molecules. The $-\log(cN_s)$ term is necessary to ensure that the entropy is extensive, where c is an arbitrary constant, amounts to an inconsequential additive constant shift in $\phi(p, T)$.

(A) The maximal work that can be performed by an out-of-equilibrium system is equal to the change of its Gibbs potential.

$$(W_{\max})_{p,T=\text{const}} = G_{\text{initial}} - G_{\text{final}}. \quad (3)$$

The initial system consists of N solute and N_s pure solvent, and the final system consists of the same amount of solute and $2N_s$ solvent, so

$$\begin{aligned} (W_{\max})_{p,T=\text{const}} &\approx [N_s\mu_{s0} + N\phi(p, T) + NT \log \frac{N}{N_s}] + N_s\mu_{s0} \\ &\quad - [2N_s\mu_{s0} + N\phi(p, T) + NT \log \frac{N}{2N_s}] \\ &= NT \log 2. \end{aligned} \quad (4)$$

(B) [3pt] The chemical potential is Gibbs energy per particle,

$$\mu_s(p, T) = \left(\frac{\partial G}{\partial N_s} \right)_{p,T,N} = \mu_{s0} + NT \left(-\frac{1}{N_s} \right) = \mu_{s0} - cT, \quad (5)$$

where $c = N/N_s$. The second term appears due to reduced entropy of the solute molecules as they are diluted by adding one molecule of solvent.

(C) Note that pressures $p_{1,2}$ can be different because the two parts are not in direct mechanical contact. Since the membrane is impenetrable to the solute molecules, the condition of equilibrium is the equality of chemical potentials of the solvent on both sides of the membrane.

$$\begin{aligned}\mu_s(p_1, T, c) &= \mu_s(p_2, T, c = 0) \\ \Leftrightarrow \mu_{s0}(p_1, T) - cT &= \mu_{s0}(p_2, T) \approx \mu_{s0}(p_1, T) + \left(\frac{\partial \mu_{s0}}{\partial p}\right)_T (p_2 - p_1) \\ \Leftrightarrow \Delta p = p_1 - p_2 &= \left(\frac{\partial \mu_{s0}}{\partial p}\right)_T^{-1} cT\end{aligned}\quad (6)$$

The partial derivative $\left(\frac{\partial \mu_{s0}}{\partial p}\right)_T = v_s$ is easily found from the identity for the chemical (Gibbs) potential $d\mu = -sdT + vdp$, where v_s is the volume per solvent molecule, and the final result is

$$\Delta p = \frac{cT}{v_s} = \frac{NT}{N_s v_s} = \frac{NT}{V}. \quad (7)$$

(D) The Clapeyron-Clausius equation follows from the fact that chemical potentials of the two phases must have equal changes $\Delta\mu_I = \Delta\mu_s$ for equal changes in temperature $T \rightarrow T + \Delta T$ and pressure $p \rightarrow p + \Delta p$, because the the phases remain in thermal and mechanical contact. If one of the phases has a small relative concentration c of solute, the solvent (water) chemical potential is modified by $(-cT)$. If the pressure does not change ($\Delta p = 0$), the phase equation $\Delta\mu_I = \Delta\mu_W$ is

$$\left(\frac{\partial \mu_I}{\partial T}\right)_p \Delta T = \Delta\mu_I = \Delta\mu_W = \left(\frac{\partial \mu_W}{\partial T}\right)_p \Delta T - cT, \quad (8)$$

With relation $\left(\frac{\partial \mu}{\partial T}\right)_p = s$ (entropy per molecule) and $T(s_W - s_I) = l$ (latent heat per molecule), one arrives at

$$\Delta T = -\frac{cT}{s_W - s_I} = \frac{cT^2}{l}. \quad (9)$$

(E) The molecular weight of water is $\mu_W = 18 \text{ mol/g}$, so the latent heat of melting ice per molecule is

$$l = \frac{L\mu_W}{N_A} \approx \frac{334 \text{ J/g} \cdot 18\text{g/mol}}{N_A} \approx 6.01 \text{ kJ/mol} \cdot N_A^{-1} \quad (10)$$

The relative salt concentration is

$$c = \frac{N_{\text{salt}}}{N_W} = \frac{m_{\text{salt}}/\mu_{\text{salt}}}{m_W/\mu_W} \approx 3.08 \cdot 10^{-4} \ll 1, \quad (11)$$

and the relative shift in temperature is

$$\frac{\Delta T}{T} = \frac{cT[\text{J}]}{l[\text{J}]} = \frac{ck_B T[\text{K}]}{l} = \frac{c(k_B N_A)T}{L\mu_W} \approx \frac{3.08 \cdot 10^{-4} \cdot 8.31 \text{ J}/(\text{mol K}) \cdot 273 \text{ K}}{6.01 \cdot 10^3 \text{ J/mol}} \approx 1.16 \cdot 10^{-4} \quad (12)$$

so that the absolute temperature shift is

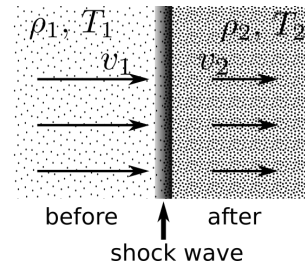
$$\Delta T \approx 0.0317 \text{ K}. \quad (13)$$

Since the problem only asks for an estimate, answers with one-digit precision are acceptable.

Statistical Mechanics 3

Shock wave in ideal gas

Consider a shock wave steadily propagating through an ideal gas from *right to left*: the gas in front of the shock is initially in the state 1, and behind the shock wave it is in the state 2. The density and temperature change from (ρ_1, T_1) to (ρ_2, T_2) only within the thin wave front. The gas before and after the shock wave is in equilibrium.



It is convenient to consider the reference frame (see figure) in which the shock wave is stationary while the density, temperature, and velocity of the gas change as the gas passes from left (state 1) to right (state 2) through the front. Assume also that the constant-volume heat capacity per molecule $C_V/N = c$ of the gas is independent of density and temperature.

(A) [3pt] What is the entropy of such gas in equilibrium at temperature T and pressure P ?

Even though the density, temperature, and velocity change discontinuously at the front, the following quantities are conserved (neglect any vertical motion of the gas as a whole):

$$\rho_1 v_1 = \rho_2 v_2, \quad (1)$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2, \quad (2)$$

$$\frac{\varepsilon_1 + p_1}{\rho_1} + \frac{1}{2}v_1^2 = \frac{\varepsilon_2 + p_2}{\rho_2} + \frac{1}{2}v_2^2, \quad (3)$$

where $\varepsilon_{1,2} = E_{1,2}/V_{1,2}$ are the *volume* densities of the gas internal energy.

(B) [3pt] Explain the physical origin and derive the conservation laws (1,2,3).

(C) [4pt] Using these conservation laws, find the change of the temperature $y = T_2/T_1$ if the density is increased by factor $x = \rho_2/\rho_1 > 1$ behind the shock wave.

(D) [5pt] How does the entropy of the gas change? Compute the change of entropy per particle $\Delta s = s_2 - s_1$ and provide qualitative explanation for its sign (you may expand in $(x - 1) \ll 1$).

(E) [5pt] Compute the speed of the shock wave $u = v_1$ in the reference frame of the initial gas and compare it to the adiabatic speed of sound u_1 . What about v_2 and u_2 (the same behind the shock wave)?

Shock wave in ideal gas: Solutions

(A) Solutions from either microscopic description of the gas (with reasonable assumptions) or integrating thermodynamic identities are accepted for the full grade.

From the thermodynamic point of view, one can integrate the constant-volume heat capacity

$$C = T \left(\frac{\partial S}{\partial T} \right)_V, \quad S(T_2, V) - S(T_1, V) = \int_{T_1}^{T_2} \frac{Nc dT}{T} = Nc \log \frac{T_2}{T_1}. \quad (4)$$

If the gas expands reversibly at constant temperature, its internal energy $dU = \delta Q - p dV$ does not change (since c does not depend on the volume) and its entropy changes due to heat exchange equal to the work done by the gas,

$$dS = \frac{\delta Q}{T} = \frac{p dV}{T}, \quad S(V_2, T) - S(V_1, T) = \int_{V_1}^{V_2} \frac{1}{T} \frac{NT}{V} dV = N \log \frac{V_2}{V_1} \quad (5)$$

Finally, the entropy of the gas is equal to

$$S(T, V) = N \left(c \log \frac{T}{T_0} + \log \frac{V}{V_0} \right) + S_0, \quad (6)$$

or, expressing it through pressure and temperature,

$$S(T, P) = N \left(c \log \frac{T}{T_0} - \log \frac{P}{P_0} \right) + S_0, \quad (7)$$

where the constant is determined by the entropy at the reference point $S_0 = S(T_0, P_0)$.

(B) The gas passing through area A of the wave front in time Δt is compressed from volume $\Delta V_1 = Av_1 \Delta t$ to volume $\Delta V_2 = Av_2 \Delta t$, while its mass is conserved,

$$\rho_1 \Delta V_1 = \Delta M = \rho_2 \Delta V_2 \quad \Leftrightarrow \quad \rho_1 v_1 = \rho_2 v_2. \quad (8)$$

The momentum of the gas passing through the wave front is changed because of the pressure difference before and after the shock wave $p_1 \neq p_2$,

$$\Delta M v_2 - \Delta M v_1 = (p_1 - p_2) A \Delta t \quad \Leftrightarrow \quad p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2. \quad (9)$$

Similarly, the total kinetic + internal energy of the gas is increased due to work done by the pressures on both sides of the shock wave front. Denoting the density of the internal energy as $\varepsilon_{1,2}$,

$$\begin{aligned} (\varepsilon_2 + \frac{1}{2} \rho_2 v_2^2) \Delta V_2 - (\varepsilon_1 + \frac{1}{2} \rho_1 v_1^2) \Delta V_1 &= p_1 \Delta V_1 - p_2 \Delta V_2 \\ \Leftrightarrow v_1 (\varepsilon_1 + p_1 + \frac{1}{2} \rho_1 v_1^2) &= v_2 (\varepsilon_2 + p_2 + \frac{1}{2} \rho_2 v_2^2) \\ \Leftrightarrow \frac{\varepsilon_1 + p_1}{\rho_1} + \frac{1}{2} v_1^2 &= \frac{\varepsilon_2 + p_2}{\rho_2} + \frac{1}{2} v_2^2 \end{aligned} \quad (10)$$

In the last line, we used mass conservation condition $v \propto 1/\rho$ (8) and the combination $(\varepsilon + p)/\rho$ represents enthalpy per unit mass.

(C) Using mass and momentum conservation (1,2) from above, it is straightforward to find the mass flux $j = \rho_1 v_1 = \rho_2 v_2$ as

$$j^2 = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)^{-1} (p_2 - p_1), \quad (11)$$

from which it follows that the pressure will also increase. Combining this with the energy conservation condition (3), one obtains

$$\frac{\varepsilon_2}{\rho_2} - \frac{\varepsilon_1}{\rho_1} = \frac{1}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) (p_1 + p_2). \quad (12)$$

Using then the ideal gas equation of state $p = nT = \frac{\rho}{m}T$ and internal energy density $\varepsilon = cnT = \frac{c\rho T}{m}$, where m is the molecular mass, the equation is easily expressed in terms of $x = \rho_2/\rho_1$ and $y = T_2/T_1$:

$$c(y - 1) = \frac{1}{2} \left(1 - \frac{1}{x} \right) (1 + xy) \quad (13)$$

and its solution yields

$$\frac{T_2}{T_1} = y = \frac{2c + 1 - 1/x}{2c + 1 - x} \quad (14)$$

Note that increase in density leads on increase in temperature.

(D) The entropy of ideal gas with heat capacity $C_V = Nc$ is

$$S = N \left[c \log T - \log \frac{N}{V} + \text{const} \right] \quad (15)$$

so that its change per molecule is

$$\Delta s = \frac{S_2}{N} - \frac{S_1}{N} = c \log \frac{T_2}{T_1} - \log \frac{\rho_2}{\rho_1} = c \log \frac{2c + 1 - 1/x}{2c + 1 - x} - \log x \quad (16)$$

In order to find the direction of the change of entropy, one can compute its derivative with respect to $x = \rho_2/\rho_1$:

$$\frac{\partial s}{\partial x} = \frac{(c + 1)(2c + 1)(x - 1)^2}{x(2c + 1 - x)(x(2c + 1) - 1)} > 0, \quad (17)$$

for $(2c + 1)^{-1} < x < (2c + 1)$, i.e., the entropy per particle increases.

Since the shock wave propagation is a spontaneous process, the entropy of the gas must increase as it passes through the shock wave front.

(E) The speed of sound is governed by adiabatic compressibility,

$$u_1^2 = \left(\frac{\partial p}{\partial \rho} \right)_S = \gamma \frac{p}{\rho} = \frac{c + 1}{c} \frac{T_1}{m}, \quad (18)$$

where $\gamma = C_p/C_V = (c + 1)/c$. The speed of the shock wave in the rest frame of the initial gas is equal to v_1 , which is easily found from the mass flux j above:

$$v_1 = \frac{j}{\rho_1} = \frac{1}{\rho_1} \sqrt{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^{-1} (p_2 - p_1)} = \sqrt{\frac{T_1}{m} \left(1 - \frac{1}{x}\right)^{-1} (xy - 1)} \quad (19)$$

The ratio of the speed of shock wave and sound wave is

$$\frac{v_1}{u_1} = \sqrt{\frac{c}{c+1} \left(1 - \frac{1}{x}\right)^{-1} (xy - 1)}, \quad (20)$$

and substituting the temperature ratio y ,

$$\frac{v_1}{u_1} = \sqrt{\frac{2cx}{2c+1-x}}, \quad (21)$$

which is larger than 1 for $x > 1$, i.e. the shock wave travels faster than sound in the initial gas.

The ratio v_2/u_2 for the gas behind the shock wave can be found by simply replacing $x \rightarrow 1/x$. Alternatively,

$$\frac{v_2}{u_2} = \frac{v_2}{v_1} \cdot \frac{u_1}{u_2} \cdot \frac{v_1}{u_1} = \frac{\rho_1}{\rho_2} \cdot \sqrt{\frac{T_1}{T_2}} \cdot \frac{v_1}{u_1} = \sqrt{\frac{1}{x^2 y} \frac{2cx}{2c+1-x}} = \sqrt{\frac{2c/x}{2c+1-1/x}} < 1, \quad (22)$$

i.e., behind the shock wave, the sound propagates faster than the former. Note also that in the initial gas reference frame, the gas behind the shock wave travels with speed $(v_1 - v_2) > 0$ to the left.

