

Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

Spring 2024 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

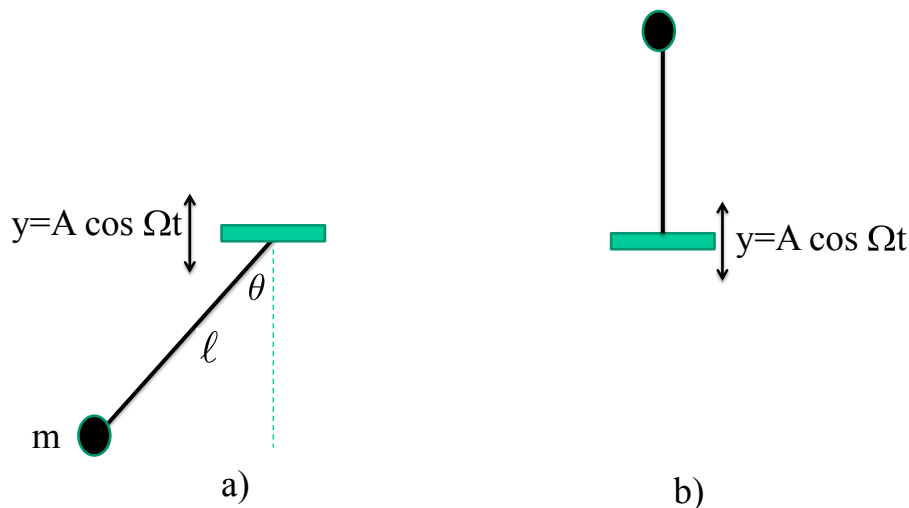
Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used.**

Classical Mechanics 1

A driven pendulum

Consider a simple pendulum consisting of a massless rod of length ℓ and an endpoint mass m . The pendulum's point of support oscillates vertically according to $y = A \cos \Omega t$ with $A \ll \ell$, see figure (a) below.



- (a) (6 points) Derive the Lagrangian of the system and determine the equations of motion. It is convenient (but not necessary) to omit all total time derivatives.
- (b) (4 points) Consider the limit where $\Omega^2 \gg \omega_0^2 \equiv g/\ell$. Write the angle of the pendulum as $\theta(t) = \Theta(t) + \xi(t)$ where $\Theta(t)$ is a slow (approximately constant) variable, and $\xi(t)$ is a rapid variable with a small amplitude of order $\xi_0 \equiv A/\ell \ll 1$. Approximate the equations of motion to order $\Omega^2 \xi_0 \gg \omega_0^2$ to determine the rapid angular variation $\xi(t)$ at specified Θ . Carefully explain your approximations.
- (c) (6 points) Now approximate the equations of motion for $\theta(t) = \Theta(t) + \xi(t)$ to order ω_0^2 and $\Omega^2 \xi_0^2$, which are the same order of magnitude. Derive an effective (time-averaged) equation of motion for the angle $\Theta(t)$ by averaging over a single period of the fast oscillations. Carefully explain your steps.
- (d) (4 points) Consider the vertical inverted position of the pendulum, as shown in figure (b) above. Determine the condition for stability of this position.

Solution

(a) Taking the coordinates

$$x = \ell \sin \theta \quad (1)$$

$$y = -\ell \cos \theta + A \cos \Omega t \quad (2)$$

Then we have

$$\dot{x} = \ell \cos \theta \dot{\theta} \quad (3)$$

$$\dot{y} = \ell \sin \theta \dot{\theta} - A\Omega \sin(\Omega t) \quad (4)$$

The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \quad (5)$$

Substituting in the expressions for the coordinates and dropping all terms that only depend on time (and not θ and $\dot{\theta}$) we find

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell \cos \theta - mA\ell\Omega \sin(\Omega t) \sin(\theta)\dot{\theta} \quad (6)$$

Integrating the last term by parts

$$-\Omega \sin(\Omega t) \sin(\theta)\dot{\theta} = \Omega \sin(\Omega t) \partial_t \cos(\theta) \quad (7)$$

$$= \partial_t [\Omega \sin(\Omega t) \cos(\theta)] - \Omega^2 \cos(\Omega t) \cos \theta \quad (8)$$

and dropping the total derivative yields the Lagrangian in its final form

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell \cos \theta - mA\ell\Omega^2 \cos(\Omega t) \cos(\theta). \quad (9)$$

Evaluating the Euler-Lagrange equations, we find

$$m\ell^2\ddot{\theta} = -mg\ell \sin(\theta) + mA\ell\Omega^2 \cos(\Omega t) \sin \theta \quad (10)$$

Dividing by $m\ell^2$ we have

$$\ddot{\theta} = -\omega_0^2 \sin(\theta) + \xi_0\Omega^2 \cos(\Omega t) \sin \theta \quad (11)$$

where we defined

$$\omega_0^2 = \frac{g}{\ell} \quad (12)$$

$$\xi_0 = \frac{A}{\ell} \quad (13)$$

(b) We then write

$$\theta(t) = \Theta(t) + \xi(t) \quad (14)$$

Substituting and expanding

$$\ddot{\theta} + \ddot{\xi} = -\omega_0^2 \sin(\Theta + \xi) + \xi_0 \Omega^2 \cos(\Omega t) \sin(\Theta + \xi) \quad (15)$$

Now we expand the sin

$$\sin(\Theta + \xi) = \sin \Theta \cos \xi + \cos \Theta \sin \xi \quad (16)$$

$$= \sin(\Theta) \left(1 - \frac{\xi^2}{2}\right) + \cos(\Theta)\xi \quad (17)$$

Using this expansion we find

$$\ddot{\theta} + \ddot{\xi} = -\omega_0^2 [\sin \Theta + \mathcal{O}(\xi_0)] + \xi_0 \Omega^2 \cos(\Omega t) [\sin \Theta + \cos \Theta \xi + \mathcal{O}(\xi_0^2)] \quad (18)$$

The terms in this expression are of order

$$\text{RHS} \sim [\omega_0^2 + \mathcal{O}(\omega_0^2 \xi_0)] + [\Omega^2 \xi_0 + \Omega^2 \xi_0^2 + \mathcal{O}(\Omega^2 \xi_0^3)] \quad (19)$$

which indicates the overall consistency of the approximation scheme.

Then the equation of motion for terms of order $\Omega^2 \xi$ is

$$\ddot{\xi} = \xi_0 \Omega^2 \cos(\Omega t) \sin \Theta \quad (20)$$

which yields $\xi(t)$

$$\xi(t) = -\xi_0 \cos(\Omega t) \sin \Theta \quad (21)$$

(c) With this expression for ξ , we then have an equation of motion for Θ . First we note

$$\ddot{\xi}(t) - \xi_0 \Omega^2 \cos(\Omega t) \sin \Theta = -2\xi_0 \Omega \sin(\Omega t) \cos(\Theta) \dot{\Theta} \quad (22)$$

Then

$$\ddot{\Theta} = -\omega_0^2 \sin \Theta + \xi_0 \Omega^2 \cos(\Omega t) \cos \Theta \xi + 2\xi_0 \Omega \sin(\Omega t) \cos(\Theta) \dot{\Theta} \quad (23)$$

Using that

$$\overline{\ddot{\xi}} = 0 \quad (24)$$

$$\overline{\sin(\Omega t)} = 0 \quad (25)$$

$$\overline{\cos(\Omega t)\xi} = -\frac{1}{2}\xi_0 \sin \Theta \quad (26)$$

We average over time yielding

$$\ddot{\Theta} = -\omega_0^2 \sin \Theta - \frac{1}{2}\xi_0^2 \Omega^2 \cos \Theta \sin \Theta \quad (27)$$

(d) At $\Theta \simeq \pi + \varphi$ we have

$$\sin \Theta \simeq -\varphi \quad (28)$$

$$\cos \Theta \simeq -1 \quad (29)$$

Leading to

$$\ddot{\varphi} = \omega_0^2 \varphi - \frac{1}{2} \xi_0^2 \Omega^2 \varphi \quad (30)$$

This will correspond to a stable equilibrium when for

$$\frac{1}{2} \xi_0^2 \Omega^2 > \omega_0^2 \quad (31)$$

Or

$$\Omega > \sqrt{\frac{2g}{\ell} \frac{\ell}{A}} \quad (32)$$

Classical Mechanics 2

An oscillating log

Consider a uniform log of length L , cross-sectional area A and mass M floating vertically in water of density ρ_0 . The log is attached by a spring of spring constant k to a uniform beam, which is pivoted at its center as shown below. The beam also has mass M but has length $2L$. In equilibrium the beam is horizontal and the spring is uncompressed with natural length x_0 . Assume the log moves vertically when the beam-log system is perturbed.

- (a) (4 points) First consider an isolated log, unattached to the spring, that is partially submerged and in static equilibrium. Determine the work required to increase its depth by a distance x .
- (b) (12 points) Find the natural frequencies for small displacements of the beam-log system.

Hint: To simplify algebra, introduce appropriate definitions and dimensionless variables.

- (c) (3 points) Consider the limit of a very stiff spring. Determine the natural frequencies and normal modes in this limit. Sketch the normal modes and describe the physics of the modes qualitatively in the stiff spring limit.
- (d) (1 points) Suppose that the liquid has a small viscous coefficient so that the log experiences a small damping force, $F = -\eta \dot{x}$. For each normal mode, determine the (fractional) rate of energy dissipation per cycle, \dot{E}/E , in limit of a stiff spring constant.

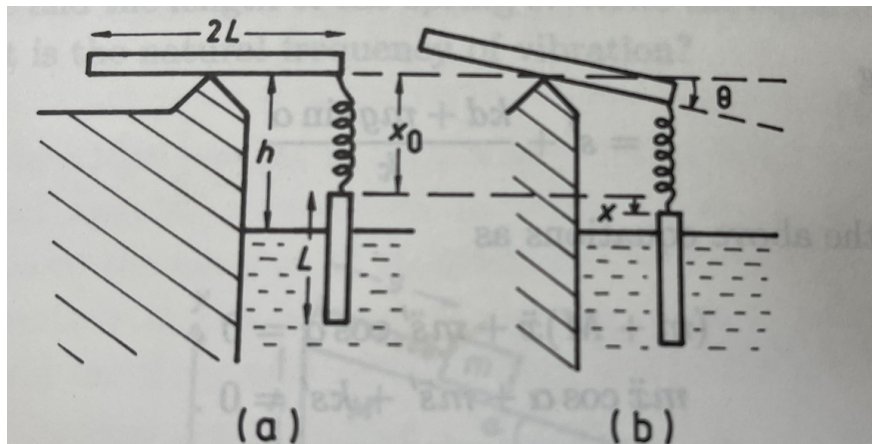


Figure 1: (a) Log and beam in equilibrium (b) Log and beam displaced from equilibrium

Solution

(a) The force of gravity is Mg . The depth of the log is $d = L - (h - x_0)$ where the increase in pressure relative to the atmosphere is $\rho g d$. Thus we have

$$Mg = \rho g A d. \quad (1)$$

When the rod is displaced by an additional distance x (downward), the upward force is:

$$F_{\text{net}} = -Mg + \rho A g(d + x) = \frac{Mg}{d} \equiv Kx, \quad (2)$$

where $k_0 \equiv Mg/d$. So the work done is

$$W = \frac{1}{2} K x^2. \quad (3)$$

(b) The Lagrangian of the system consists of the kinetic energy of the rod and log

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2, \quad (4)$$

where

$$I = \int_{-L}^L dM y^2 = \frac{1}{3} M L^2. \quad (5)$$

The change in length of the spring is

$$\ell = L\theta - x \quad (6)$$

So the potential energy associated with increasing the depth by x and rotating the rod by θ is

$$U = \frac{1}{2} k (L\theta - x)^2 + \frac{1}{2} K x^2 \quad (7)$$

The last term incorporates the gravitational potential energy and buoyancy forces.

The Lagrangian then is

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{6} M L^2 \dot{\theta}^2 - \frac{1}{2} k (L\theta - x)^2 - \frac{1}{2} K x^2. \quad (8)$$

We will actually parametrize the motion by $y = L\theta$ yielding our Lagrangian

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{6} M \dot{y}^2 - \frac{1}{2} k (y - x)^2 - \frac{1}{2} K x^2. \quad (9)$$

Computing the equation of motion we find

$$M \ddot{x} = k(y - x) - Kx, \quad (10)$$

$$\frac{1}{3} M \ddot{y} = -k(y - x). \quad (11)$$

Dividing by M we have

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = - \begin{pmatrix} \omega_0^2 + \Omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (12)$$

where we introduced $\omega_0^2 = \frac{k}{m}$ and $\Omega^2 = \frac{K}{m}$.

We from now on set $\omega_0 = 1$ and define $\kappa = \Omega^2/\omega_0^2$. The system of equations to be solved is

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = - \begin{pmatrix} 1 + \kappa & -1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (13)$$

Looking for the eigen solutions of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-i\omega t} \vec{X}, \quad (14)$$

we can find non-trivial solutions if

$$\det \begin{pmatrix} -\lambda + (1 + \kappa) & -1 \\ -3 & 3 - \lambda \end{pmatrix} = 0, \quad (15)$$

where $\lambda = \omega^2$.

Thus

$$(3 - \lambda)(-\lambda + (1 + \kappa)) + 3 = 0 \quad (16)$$

Expanding out the quadratic and solving the equation (carefully).

$$\lambda_{\pm} = \frac{1}{2} \left((4 + \kappa) \pm \sqrt{(4 + \kappa)^2 - 12\kappa} \right) \quad (17)$$

Restoring units

$$\lambda_{\pm} = \omega_{\pm}^2 = \frac{1}{2} \left((4\omega_0^2 + \Omega^2) \pm \sqrt{(4\omega_0^2 + \Omega^2)^2 - 12\Omega^2\omega_0^2} \right) \quad (18)$$

(c) In the stiff spring limit, we take $\kappa \rightarrow 0$. Intuitively this means that the length of the spring is fixed. It corresponds to the equation of motion

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = - \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (19)$$

We find

$$\lambda_+ \simeq 4 \quad (20)$$

$$\lambda_- \simeq 0 \quad (21)$$

Of course the zero mode is an artifact of the limit $\kappa \rightarrow 0$. More generally we can expand the result of part (b) yielding

$$\lambda_- \simeq \frac{3}{4}\kappa \quad (22)$$

Or restoring units

$$\lambda_- \simeq \frac{3}{4}\frac{K}{M} \quad (23)$$

The eigen-modes are the zero mode

$$\vec{X}_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (24)$$

which leaves the length of the spring unchanged.

There is also a non-zero mode where the spring is oscillating

$$\vec{X}_+ = \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \quad (25)$$

The non-zero mode can be inferred from the orthogonality condition. Specifically, the eignemodes are orthogonal with respect to the mass matrix given in eq. 12 which forms a generalized eigenvalue problem, i.e.

$$\vec{X}_-^T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \vec{X}_+ = 0 \quad (26)$$

(d) The time averaged rate of dissipation

$$\frac{dE}{dt} = \langle \dot{x} F_D \rangle = \eta \langle \dot{x}^2 \rangle \quad (27)$$

The kinetic energy of the system is

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{6}M\dot{y}^2 \quad (28)$$

The average kinetic energy and potential energies are equal leading to

$$E = M\langle \dot{x}^2 \rangle + \frac{1}{3}M\langle \dot{y}^2 \rangle \quad (29)$$

So the fraction rate of energy loss is

$$\frac{1}{E} \frac{dE}{dt} = \frac{\eta}{M} \frac{\langle \dot{x}^2 \rangle}{\langle \dot{x}^2 + \frac{1}{3}\dot{y}^2 \rangle} \quad (30)$$

So for the zero mode

$$\left(\frac{1}{E} \frac{dE}{dt} \right)_- = \frac{\eta}{M} \frac{1}{1 + \frac{1}{3}} = \frac{\eta}{M} \frac{3}{4}. \quad (31)$$

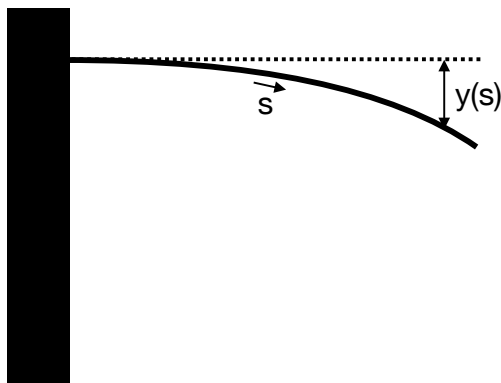
For the non-zero mode

$$\left(\frac{1}{E} \frac{dE}{dt} \right)_+ = \frac{\eta}{M} \frac{\frac{1}{9}}{\frac{1}{9} + \frac{1}{3}} = \frac{\eta}{M} \frac{1}{4} \quad (32)$$

Classical Mechanics 3

Elastic Rod

Consider a thin elastic rod of length L with one end clamped horizontally and the other end free. Let s denote the length along the rod and let $y(s)$ denote the position of the rod below the horizontal at position s , as shown in the figure.



There are two contributions to the potential energy, from the curvature of the rod and from gravity, so that:

$$U[y(s)] = \int_0^L ds \left[\frac{1}{2} k (\partial_s^2 y(s))^2 - g y(s) \rho(s) \right], \quad (1)$$

where k is related to the Young's modulus of the rod material, $g > 0$ is the acceleration due to gravity, and $\rho(s)$ is the linear mass density of the rod at position s . Note, as usual, $\partial_s^2 y(s)$ denotes $\frac{\partial^2 y(s)}{\partial s^2}$.

- [8pts]** Using the calculus of variations, derive a fourth-order differential equation for the stationary configuration of the rod, $y(s)$, that minimizes the potential energy, in terms of the parameters in Eq. (1).
- [8pts]** State four boundary conditions can be used to integrate the equation you found in part (a) and explain their origin. Your explanation can be mathematical or physical.
- [4pts]** Consider the special case of a rod with uniform density, $\rho(s) = \rho_0$. Solve the equation you found in part (a) using your specified boundary conditions from part (b).

Solution

(a) **[8pts]** Using the calculus of variations, derive a fourth-order differential equation for the shape of the rod, $y(s)$, that minimizes the potential energy, in terms of the parameters in Eq. (1).

If $y(s)$ minimizes $U[y(s)]$, then for any small deviation $y(s) \rightarrow y(s) + \delta(s)$, $U[y(s) + \delta(s)] - U[y(s)] = 0$. From Eq. (1),

$$\begin{aligned} U[y(s) + \delta(s)] &= \int_0^L ds \left[\frac{1}{2}k (y''(s) + \delta''(s))^2 - g(y(s) + \delta(s))\rho(s) \right] \\ &= U[y(s)] + \int_0^L ds [ky''(s)\delta''(s) - g\delta(s)\rho(s)] + \mathcal{O}(\delta''^2), \end{aligned} \quad (2)$$

where the prime notation indicates a derivative with respect to s , i.e., $y''(s) = \partial_s^2 y(s)$. Thus, when $y(s)$ minimizes $U[y(s)]$, the integral must vanish. Integrating by parts twice:

$$\int_0^L ds [ky''(s)\delta''(s) - g\delta(s)\rho(s)] = [ky''(s)\delta'(s)]_0^L + \int_0^L ds [-ky'''(s)\delta'(s) - g\delta(s)\rho(s)] \quad (3)$$

$$= [ky''(s)\delta'(s)]_0^L - [ky'''(s)\delta(s)]_0^L + \int_0^L ds [ky''''(s) - g\rho(s)]\delta(s) \quad (4)$$

Since the above integral must vanish for any small deviation $\delta(s)$, it must be that

$$\boxed{ky''''(s) = g\rho(s)} \quad (5)$$

(b) **[8pts]** State four boundary conditions can be used to integrate the equation you found in part (a) and explain their origin. Your explanation can be mathematical or physical.

For Eq. (4) to vanish also requires:

$$y''(L)\delta'(L) = 0 \quad (6)$$

$$y''(0)\delta'(0) = 0 \quad (7)$$

$$y'''(L)\delta(L) = 0 \quad (8)$$

$$y'''(0)\delta(0) = 0 \quad (9)$$

Physically, since the rod is fixed to the wall, $y(0) = 0$, which requires $\delta(0) = 0$; hence, the last equation vanishes. Further, since the rod must emerge perpendicular to the wall, $y'(0) = 0$, which requires $\delta'(0) = 0$; hence, the second equation also vanishes. Since the first and third equations must hold for any small variation $\delta(s)$, we find the following four boundary conditions:

$$\boxed{y(0) = y'(0) = y''(L) = y'''(L) = 0} \quad (10)$$

(c) **[4pts]** Consider the special case of a rod with uniform density, $\rho(s) = \rho_0$. Solve the equation you found in part (a) using your specified boundary conditions from part (b).

From Eq. (5) with uniform density, we are faced with the fourth-order differential equation:

$$y''''(s) = \frac{g}{k}\rho_0 \quad (11)$$

The solution to this equation takes the form

$$y(s) = \frac{g}{24k}\rho_0 [s^4 + c_3s^3 + c_2s^2 + c_1s + c_0], \quad (12)$$

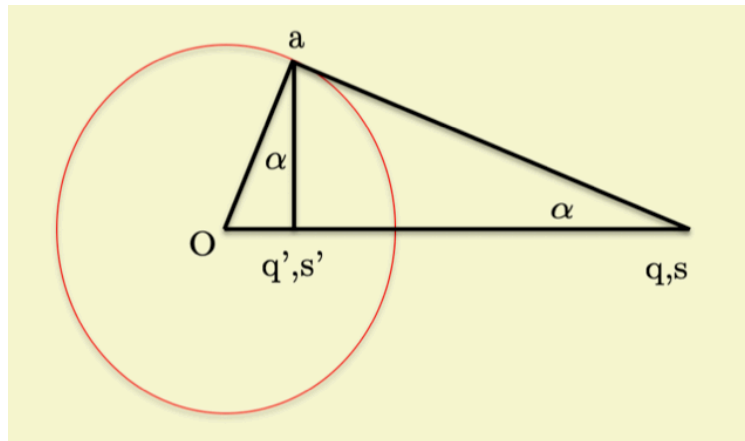
where $c_{3,2,1,0}$ are determined by the boundary conditions in Eq. (10), yielding

$$\boxed{y(s) = \frac{g}{24k}\rho_0 [s^4 - 4Ls^3 + 6L^2s^2]} \quad (13)$$

Electromagnetism 1

Charge inside a conducting sphere

Consider a point charge q **outside** a hollow, conducting and grounded sphere of radius a , as illustrated in the figure below



(a) (7 points)

Use the image construction shown in the figure to show that the triangles Oas' and Oas are similar triangles and determine the appropriate image charge. Derive the potential everywhere around the charge q .

(b) (5 points)

Find the induced surface charge density on the surface of the sphere.

(c) (5 points)

Find the magnitude and direction of the force acting on the charge.

(d) (3 points)

How does the answer change if the sphere is at fixed potential V or fixed total charge Q ?

Solution

a. If q at position \vec{s} is outside, q' at position \vec{s}' its image is inside, for otherwise the potential is twice singular outside. With this in mind, the potential is

$$\varphi(\vec{r}) = \frac{q}{|\vec{r} - \vec{s}|} + \frac{q'}{|\vec{r} - \vec{s}'|} \quad (1)$$

The hollow conductor is grounded, so the surface is at zero potential, i.e

$$\varphi(\vec{a}) = \frac{q}{|\vec{a} - \vec{s}|} + \frac{q'}{|\vec{a} - \vec{s}'|} = 0 \quad (2)$$

Since the triangle Oas' and Oas are similar (share two same angles) it follows that the sides are similar,

$$\frac{|\vec{a} - \vec{s}'|}{|\vec{a} - \vec{s}|} = \frac{a}{s} = -\frac{q'}{q} \quad (3)$$

When the points are lined up, $ss' = a^2$ (harmonic) which implies for the charges $aq + sq' = 0$.

b. The electric field is given by

$$\vec{E}(\vec{r}) = \frac{q(\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^3} + \frac{q'(\vec{r} - \vec{s}')}{|\vec{r} - \vec{s}'|^3} \quad (4)$$

The induced charge density is

$$\sigma(\vec{a}) = \frac{1}{4\pi} \vec{a} \cdot \vec{E}(\vec{a}) = \frac{q}{4\pi a} \frac{(a^2 - s^2)}{|\vec{a} - \vec{s}|^3} \quad (5)$$

which is of opposite sign to the charge q . The total induced charge is

$$Q(a) = \frac{q}{4\pi a} \int_0^\pi 2\pi \sin\theta d\theta \frac{a^2 - s^2}{(a^2 + s^2 - 2a\cos\theta)^{\frac{3}{2}}} = -\frac{qa}{s} = q' \quad (6)$$

which is the image charge as expected.

c. The Force on the charge due to the induced charge on the hollow surface is that of the image charge

$$\vec{F} = -q^2 \frac{a\vec{s}}{(a^2 - s^2)^2} \quad (7)$$

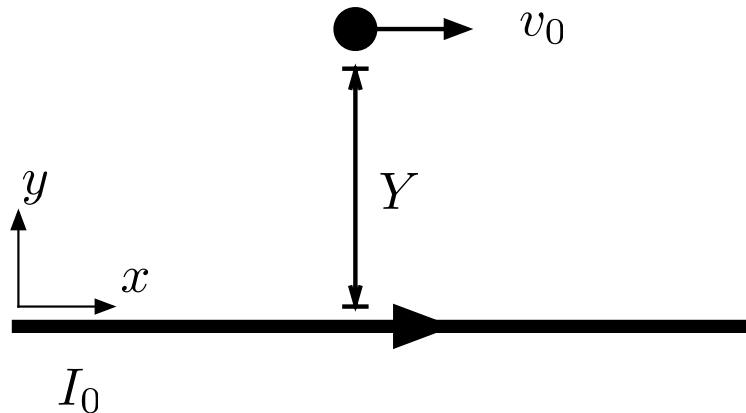
d. If we add a second image charge Q at the center of the sphere, the surface of the sphere is still an equipotential, and the ensuing electric potential satisfies Laplace equation everywhere outside. By choosing this charge to be $Q = aV$, this amounts to solving the problem for a fixed potential V on the hollow sphere. By choosing this charge to be $Q - q'$, this amounts to solving the problem for a hollow sphere with a net surface charge Q .

Electromagnetism 2

A small sphere and a wire

- (2 points) Determine the Lorentz invariants quadratic in the field strength tensor, $F_{\mu\nu}$. Evaluate these invariants in terms of \mathbf{E} and \mathbf{B} .
- (8 points) Consider a Lorentz frame \mathcal{O} where a long straight wire on the x -axis carries a steady current I_0 .
 - (1 point) Determine the field strength tensor $F_{\mu\nu}$ in the xy plane in \mathcal{O} .
 - (3 points) Using the covariant form of the transformations law of $F_{\mu\nu}$ and the four-current J^μ , determine the electromagnetic field and four-current in the xy plane of a frame $\underline{\mathcal{O}}$ moving with velocity v along the x -axis relative to \mathcal{O} .
 - (2 points) Working entirely in $\underline{\mathcal{O}}$ show explicitly that the electric field is consistent with the Maxwell equations.
 - (2 points) Using notions of Lorentz invariance explain why the electric and magnetic fields are orthogonal in $\underline{\mathcal{O}}$.

Now consider a small, solid and neutral metal sphere of radius a , moving non-relativistically with velocity v_0 in a Lorentz frame \mathcal{O} . The trajectory of the sphere is parallel to the long straight wire with current I_0 of part (b), but displaced by a distance Y (see below).



- (7 points) Determine the force (magnitude and direction) between the sphere and the wire.
Hint: Analyze the situation in the rest frame of the sphere. Express the force in terms of the induced electric dipole moment $\mathbf{p} = \alpha_E \mathbf{E}$, and then compute the susceptibility, α_E .
- (3 points) Without changing frames, qualitatively explain the direction of the force between sphere and the wire in the frame \mathcal{O} .

Solution

The solution uses Heaviside-Lorentz units.

(a) We have $F_{\mu\nu}F^{\mu\nu} = -2(E^2 - B^2)$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} = 4\mathbf{E} \cdot \mathbf{B}$. We have used the metric

$$\eta_{\mu\nu} = (-1, 1, 1, 1) \quad (1)$$

and followed the convention that

$$-F_{0i} = F^{0i} = E^i \quad F^{ij} = \epsilon^{ijk}B_k \quad (2)$$

(b) (i) The magnetic field is

$$\mathbf{B} = \frac{I/c}{2\pi\rho}\mathbf{e}_\phi \quad (3)$$

and the electric field vanishes. For definiteness we examine \mathbf{B} in the x, y plane where we find

$$B^z = \frac{I/c}{2\pi y} \quad (4)$$

So the only nonzero components are

$$F^{xy} = -F^{yx} = \frac{I/c}{2\pi y} \quad (5)$$

(ii) The Lorentz transformation is

$$\underline{F}^{\mu\nu}(\underline{x}) = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma} \quad (6)$$

$$E^i = \Lambda_\ell^0 \Lambda^i_m F^{\ell m} \quad (7)$$

The only nonzero components is

$$E^y = \Lambda_x^0 \Lambda_y^y F^{xy} = -\frac{\gamma v I_0/c}{2\pi y} \quad (8)$$

The transverse coordinates are unchanged by the boost and we have

$$E^y = \Lambda_x^0 \Lambda_y^y F^{xy} = -\frac{\gamma\beta I_0/c}{2\pi y} \quad (9)$$

The current is

$$\underline{J}^\mu = \Lambda^\mu_\nu J^\nu \quad (10)$$

So

$$\underline{\rho}c = \Lambda_x^0 J^x \quad (11)$$

$$\underline{J}^x = \Lambda_x^x J^x \quad (12)$$

$$(13)$$

Here $J^x = I_0\delta(y)\delta(z)$ so the charge per length is

$$\underline{\lambda} = -\gamma\beta I_0/c \quad (14)$$

and the current is

$$\underline{I}_0 = \gamma I_0 \quad (15)$$

(iii) Clearly the electric field is consistent with the maxwell equations. It can be written

$$E^y = \frac{\lambda}{2\pi y} \quad (16)$$

This is just the electric field from a line of charge

(iv) According to part (a), $\mathbf{E} \cdot \mathbf{B}$ is a lorentz invariant. In the frame \mathcal{O} the electric field is zero and $\mathbf{E} \cdot \mathbf{B} = 0$. Thus in any other frame the electric field maybe non-zero, but it will always be orthogonal to \mathbf{B} .

(c) In the frame of the sphere there is an electric field

$$\underline{E} = \frac{\lambda}{2\pi y} \quad (17)$$

and a small stationary metal sphere. The electric field is nearly constant over the dimensions of the sphere

$$\underline{\mathbf{E}} = \frac{\lambda}{2\pi R} \hat{\mathbf{y}} \quad (18)$$

Then according to a familiar problem of electrostatics (see Griffith's chapter 2) which will not be reproduced here, the induced dipole moment of a metal sphere in a constant electric field is

$$\mathbf{p} = 4\pi a^3 E_0 \hat{\mathbf{y}} \quad (19)$$

So $\alpha_E = 4\pi a^3$. We then use the fact that the potential energy of the induced dipole moment is

$$U_{\text{dip}} = -\frac{1}{2}\alpha_E E^2 \quad (20)$$

The force is $F^i = -\partial^i U_{\text{dip}}$

$$F^y = -\partial^y \left[-\frac{1}{2}\alpha_E \left(\frac{\lambda}{2\pi y} \right)^2 \right] \quad (21)$$

$$= -\left(\frac{\lambda}{2\pi} \right)^2 \frac{4\pi a^3}{y^3} \quad (22)$$

Expressing the result in terms of \mathcal{O} frame quantities we have in the non-relativistic limit

$$F^y = -\left(\frac{v}{c} \right)^2 \left(\frac{I}{2\pi c} \right)^2 \frac{4\pi a^3}{y^3} \quad (23)$$

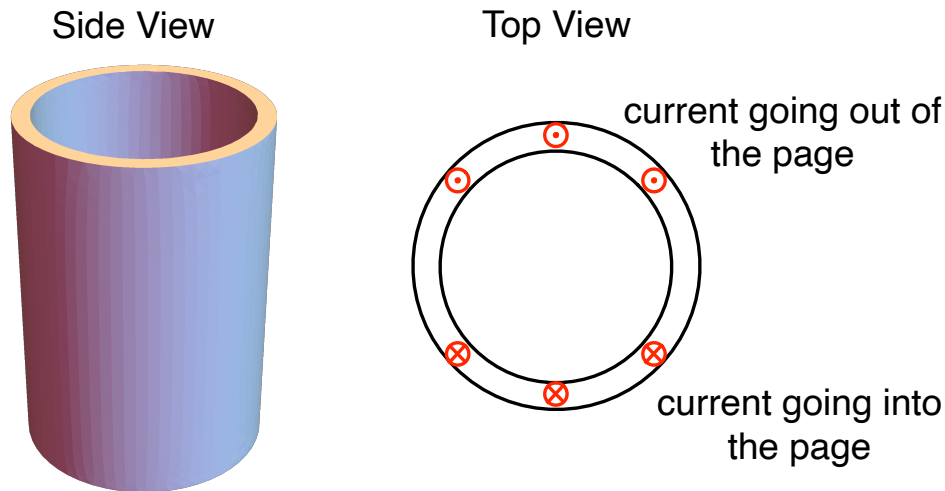
The force is attractive.

(d) The Lorentz force $F = qvB$ causes the charges in the metal to separate, with the positive charges in the sphere moving closer to the wire and the negative charges moving farther from the wire. This continues until the electrostatic forces in the metal balance the magnetic Lorentz force (see figure). Then these separated positive charges are closer to the wire, and feel an attractive Lorentz force. The negative charges are farther from the wire, and feel a repulsive, and smaller, Lorentz force. The force is smaller since the magnetic field from the wire decreases with distance. The net force on the sphere is thus attractive.

Electromagnetism 3

Decay of a surface current:

A long cylindrical shell of radius a and thickness $\Delta \ll a$ has conductivity σ as shown below. Inside and outside of the shell is vacuum. The shell carries a surface current $\mathbf{K}(\phi) = \hat{\mathbf{z}} K_o \sin \phi$.



- (a) (6 points) Determine the Coulomb gauge vector potential everywhere. Possibly useful formulas are given below.
- (b) (4 points) Determine the magnetic field everywhere and sketch the magnetic field lines.
- (c) (2 points) What is the electric field in the shell? You may consider the current to be uniform across the shell.

At time $t = 0$ the battery driving the surface current is switched off.

- (d) (6 points) Determine $\mathbf{K}(t, \phi)$ at subsequent times using a quasi-static approximation.
Hint: Determine the induced electric field in the shell due to a time dependent current of the form $\mathbf{K}(t, \phi) = \hat{\mathbf{z}} K(t) \sin \phi$.
- (e) (2 points) Estimate the decay time numerically for a centimeter sized copper shell with $\Delta/a \sim 0.1$. The resistivity of copper is $\rho \equiv 1/\sigma = 1.7 \times 10^{-8} \Omega \cdot \text{m}$.

Cylindrical Coordinates

$$\rho = \sqrt{x^2 + y^2} \quad \phi = \tan^{-1}(y/x) \quad (1)$$

Gradient, Divergence, Curl, and Scalar Laplacian

$$\nabla\psi = \frac{\partial\psi}{\partial\rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial\psi}{\partial\phi} \hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z} \hat{\boldsymbol{z}} \quad (2)$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \quad (3)$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho} \right) \hat{\boldsymbol{\phi}} + \left(\frac{1}{\rho} \frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{1}{\rho} \frac{\partial A_\rho}{\partial\phi} \right) \hat{\boldsymbol{z}} \quad (4)$$

$$\nabla^2\psi = \frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial\psi}{\partial\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \quad (5)$$

Vector Laplacian

∇^2 acting on components of \mathbf{A} (e.g. $\nabla^2 A_z$) indicates the scalar Laplacian in the cylindrical coordinate system.

$$\nabla^2 \mathbf{A} = \left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_\phi}{\partial\phi} \right) \hat{\boldsymbol{\rho}} + \left(\nabla^2 A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial\phi} \right) \hat{\boldsymbol{\phi}} + \nabla^2 A_z \hat{\boldsymbol{z}}$$

Solution

The solution is presented in Heaviside-Lorenz units

(a) In the Coulomb gauge we are to solve

$$-\nabla^2 \mathbf{A} = \frac{\mathbf{J}}{c} \quad (6)$$

Given the symmetry of the problem we try a solution of the form $A^z(\rho, \phi)$. Then inside and outside of the sheet we have

$$-\nabla^2 A^z = 0 \quad (7)$$

So

$$\left[\frac{-1}{\rho} \partial_\rho (\rho \partial_\rho) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] A^z = 0 \quad (8)$$

The separated solutions take the form

$$A^z(\rho, \phi) = \sum_m C_m(\rho) \cos(m\phi) + D_m(\rho) \sin(m\phi) \quad (9)$$

Since the current only is $\propto \sin(\phi)$, we will make a trial ansatz

$$A^z = D(\rho) \sin(\phi) \quad (10)$$

Leading to an equation for D

$$\frac{-1}{\rho} \partial_\rho (\rho \partial_\rho D) + \frac{D}{\rho^2} = 0 \quad (11)$$

This is easily solved

$$B(\rho) = \frac{C_1}{\rho} + C_2 \rho \quad (12)$$

Demanding regularity as $\rho \rightarrow 0$ and $\rho \rightarrow \infty$ gives

$$D(\rho) = \begin{cases} C_2 \rho & \rho < a \\ \frac{C_1}{\rho} & \rho > a \end{cases} \quad (13)$$

Demanding continuity we get

$$D(\rho) = \begin{cases} C_1 \frac{\rho}{a} & \rho < a \\ \frac{C_1 a}{\rho} & \rho > a \end{cases} \quad (14)$$

We compute from this solution

$$\mathbf{B} = \frac{1}{\rho} \frac{\partial A^z}{\partial \phi} \mathbf{e}_\rho - \frac{\partial A_z}{\partial \rho} \mathbf{e}_\phi \quad (15)$$

$$= \frac{D}{\rho} \cos \phi \mathbf{e}_\rho - D'(\rho) \sin \phi \mathbf{e}_\phi \quad (16)$$

where away from $\rho = a$ we have

$$-D'(\rho) = \begin{cases} -\frac{C_1}{a} & \rho < a \\ \frac{C_1 a}{\rho^2} & \rho > a \end{cases} \quad (17)$$

From the boundary conditions

$$B_\phi|_{\rho=a^+} - B_\phi|_{\rho=a^-} = \frac{K_z}{c} \quad (18)$$

We find

$$\frac{2C_1}{a} = \frac{K_0}{c} \quad C_1 = \frac{K_0 a}{2c} \quad (19)$$

Leading to the final form of the solution

$$A^z = \begin{cases} \frac{K_0 \rho}{2c} \sin(\phi) & \rho < a \\ \frac{K_0 a^2}{2c \rho} \sin(\phi) & \rho > a \end{cases} \quad (20)$$

(b) Using the solution of the previous parts we find

$$B^\rho = \begin{cases} \frac{K_0}{2c} \cos(\phi) & \rho < a \\ \frac{K_0 a}{2c \rho^2} \cos(\phi) & \rho > a \end{cases} \quad (21)$$

$$B^\phi = \begin{cases} -\frac{K_0}{2c} \sin(\phi) & \rho < a \\ \frac{K_0 a}{2c \rho^2} \sin(\phi) & \rho > a \end{cases} \quad (22)$$

Since

$$\mathbf{e}_\rho = \cos \phi \mathbf{e}_x + \sin \phi \mathbf{e}_y \quad (23)$$

$$\mathbf{e}_\phi = -\sin \phi \mathbf{e}_x + \cos \phi \mathbf{e}_y \quad (24)$$

we see that

$$\mathbf{B} = \frac{K_0}{2c} \mathbf{e}_x \quad (25)$$

i.e. the magnetic field is constant in the shell. Outside of the shell, the magnetic field takes a simple dipole form. Putting together these ingredients leads to Fig. 1

(c) The electric field is given by Ohm's Law, $\mathbf{J} = \sigma \mathbf{E}$. So we have

$$\mathbf{E} = \frac{\mathbf{K}(t, \phi)}{\sigma \Delta} \quad (26)$$

(d) After the field is current is switched off the current begins to decay. Working with the ansatz

$$K^z = \hat{\mathbf{z}} K(t) \sin(\phi) \quad (27)$$

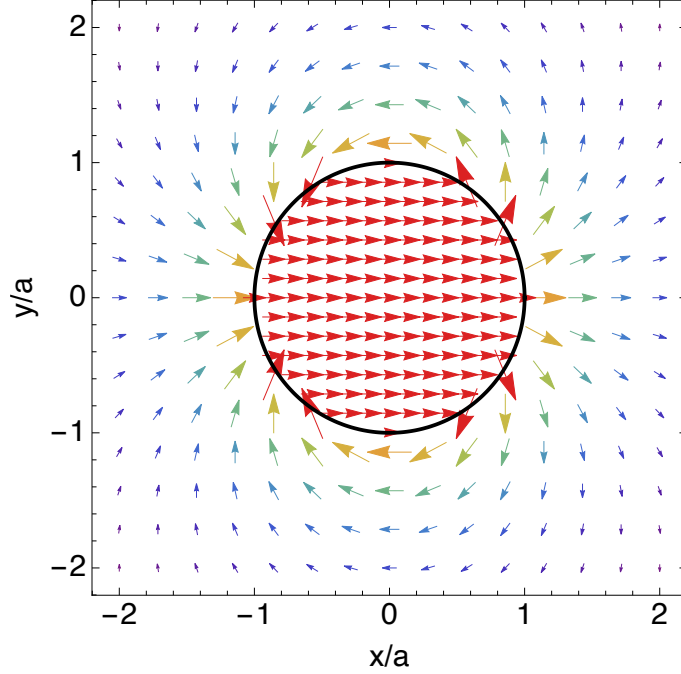


Figure 1: Magnetic field lines from a surface current

Then the vector is the same as before

$$A^z = \begin{cases} \frac{K(t)\rho}{2c} \sin(\phi) & \rho < a \\ \frac{K(t)a^2}{2c\rho} \sin(\phi) & \rho > a \end{cases} \quad (28)$$

The induced electric field is

$$E^z(t, \rho, \phi) = -\frac{1}{c} \partial_t A^z(t, \rho, \phi), \quad (29)$$

This electric field supports the current through Ohm's Law. The current in the sheet

$$\frac{K^z(t, \phi)}{\Delta\sigma} = E^z(t, \rho, \phi)|_{\rho=a} \quad (30)$$

Leading to an equation for $K(t)$

$$\frac{K(t)}{\Delta\sigma} = -\frac{a\dot{K}}{2c^2} \quad (31)$$

with solution

$$K(t) = K_0 e^{-2c^2 t / (\Delta a \sigma)} \quad (32)$$

with a characteristic decay time

$$\tau = \frac{\Delta a \sigma}{2c^2} \quad (33)$$

(e) The quantity $D_0 \equiv c^2/\sigma$ is known as the magnetic diffusion coefficient, as the magnetic fields in a conductor obey a diffusion equation in a quasi-static approximation. For

copper this is of order $D_0 \sim \text{cm}^2/\text{millisec}$ In MKS units we have

$$\frac{\sigma_{\text{MKS}}}{\epsilon} = \sigma \quad (34)$$

We evaluate

$$D_0 = \frac{c^2}{\sigma} = c^2 \epsilon_0 \rho_{\text{MKS}} = \frac{c}{Z_0} \rho_{\text{MKS}} \quad (35)$$

where $Z_0 = 376. \Omega = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Plugging in the numbers

$$D_0 = 0.13 \frac{\text{cm}^2}{\text{ms}} \quad (36)$$

Taking a centimeter device and $\Delta = 0.1$ we find

$$\tau \sim 0.4 \text{ ms} \quad (37)$$

Quantum Mechanics 1

Coherent States

Consider a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2.$$

A coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad (1)$$

with α a complex number. Coherent states are the closest quantum analog of classical states with well-defined amplitudes and phases. They are extremely important in, e.g., laser physics, quantum optics, radio cavities, Bose-Einstein condensates, and just about everything else that's both macroscopic and quantum.

- (a) (4 points) Show that a coherent state $|\alpha\rangle$ can be written in the form

$$|\alpha\rangle = C e^{\alpha \hat{a}^\dagger} |0\rangle, \quad (2)$$

where C is a (complex) normalization constant.

- (b) (2 point) Calculate C .
- (c) (2 point) Calculate the probability of finding a coherent state $|\alpha\rangle$ in the quantum harmonic oscillator's $|n\rangle$ state
- (d) (2 points) Calculate the average energy of the coherent state $|\alpha\rangle$.
- (e) (5 points) Calculate Δx and Δp . What is the total uncertainty of a coherent state? Comment on the physical consequences of the result obtained.
- (f) (5 points) By examining eq. (1) in coordinate space, determine the wave function of the coherent state in a coordinate representation, $\psi_\alpha(x) \equiv \langle x|\alpha\rangle$. Use this wave function to determine the Wigner transform of the coherent state

$$W(x_0, p_0) \equiv \int_{-\infty}^{\infty} dy e^{iy p_0} \psi_\alpha^*(x_0 + y/2) \psi_\alpha(x_0 - y/2).$$

What is the interpretation of $W(x_0, p_0)$ and the parameter α ?

Quantum Mechanics 2

Identical Particles

Two quantum particles of masses m_1 and m_2 interact with each other via the three-dimensional harmonic potential $V(\mathbf{r}_1, \mathbf{r}_2) = k(\mathbf{r}_1 - \mathbf{r}_2)^2/2$.

- (a) [3 points] Write down the Hamiltonian of the system and change variables to $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$.
- (b) [4 points] Find the energy spectrum and the eigenfunctions of the system with degeneracy of the different energy levels.
- (c) [3 points] Find the energy spectrum and the eigenfunctions of the system with degeneracies, when the particles are identical bosons each of mass m .
- (d) [3 points] Find the energy spectrum and the eigenfunctions of the system with degeneracies, when the particles are identical fermions each of mass m . Assume that fermions are “spinless” (i.e., spin-polarized).
- (e) [3 points] Argue what the total angular momentum of the ground state is in (c) and (d)?
- (4) [4 points] Now assume that the fermions have spin 1/2 and are not spin-polarized. Moreover, they have nonvanishing spin-spin interaction $V = u\vec{\sigma}_1 \cdot \vec{\sigma}_2$, where $\vec{\sigma}$ is the vector of Pauli matrices. Find the energy spectrum and the eigenfunctions in this case.

Solution

(a) The Hamiltonian of the system is given by

$$H = -\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial \mathbf{r}_1^2} - \frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial \mathbf{r}_2^2} + \frac{k(\mathbf{r}_1 - \mathbf{r}_2)^2}{2}, \quad (1)$$

We consider the center of mass frame, i.e.

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M}, \quad (2)$$

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \quad (3)$$

where $M = m_1 + m_2$, so that $\partial/\partial \mathbf{r}_{1,2} = (m_{1,2}/M)\partial/\partial \mathbf{R} \mp \partial/\partial \mathbf{r}$.

The Hamiltonian in these coordinates is given by

$$H = -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial \mathbf{R}^2} - \frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial \mathbf{r}^2} + \frac{k\mathbf{r}^2}{2}, \quad (4)$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

(b) The eigenstates of the Hamiltonian are given by

$$\psi_{n_x, n_y, n_z, \mathbf{K}}(\mathbf{r}, \mathbf{R}) = e^{i\mathbf{K} \cdot \mathbf{R}} \psi_{n_x, n_y, n_z}^{HO}(\mathbf{r}), \quad (5)$$

where the oscillator wave functions (not normalized) are given by

$$\psi_{n_x, n_y, n_z}^{HO}(\mathbf{r}) = H_{n_x}(x/\lambda) H_{n_y}(y/\lambda) H_{n_z}(z/\lambda) e^{-\frac{r^2}{2\lambda^2}}, \quad (6)$$

and $\lambda = \left(\frac{\hbar^2}{k\mu}\right)^{1/4}$ is the oscillator length. The corresponding eigenenergies

$$E_{n_x, n_y, n_z, K} = \frac{\hbar^2 K^2}{2M} + \hbar\omega \left(N + \frac{3}{2}\right), \quad (7)$$

where

$$N = n_x + n_y + n_z, \quad n_{x,y,z} = 0, 1, 2, \dots \quad (8)$$

where $\omega = \sqrt{k/\mu}$. It is easy to see that the degeneracy of the state with quantum number N is $(N+1)(N+2)/2$

(c) However if particles are spinless bosons, then the wavefunction has to be symmetric with respect to the interchange of two particles, i.e., $\mathbf{r} \rightarrow -\mathbf{r}$, which means that either all values of n_x, n_y, n_z are even, or two are odd, which yields $n_x + n_y + n_z = 2k$, where k is a non-negative integer. So for bosons the energy spectrum is

$$E_{k,K}^{(\text{bosons})} = \frac{\hbar^2 K^2}{4m} + \hbar\omega \left(2k + \frac{3}{2}\right), \quad (9)$$

where $K \in (-\infty, +\infty)$ and $k = 0, 1, 2, \dots$ and the degeneracy of the eigenenergy with given \mathbf{K}, k is $(2k + 1)(k + 1)$.

(d) Similarly for fermions either all three integers are odd or there is only one of them is odd and $n_x + n_y + n_z = 2k + 1$

$$E_{k,K}^{(\text{fermions})} = \frac{\hbar^2 K^2}{4m} + \hbar\omega \left(2k + \frac{5}{2} \right), \quad (10)$$

where $K \in (-\infty, +\infty)$ and $k = 0, 1, 2, \dots$. The degeneracy of the eigenenergy with given \mathbf{K}, k is $(2k + 3)(k + 1)$.

(e) The total angular momentum can only be even for bosons and odd for fermions. Therefore, the total angular momentum of the ground state is 0, 1 in (c), (d), respectively.

(f) Rewriting the spin scalar product of the spin operators in terms of squares of these operators:

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{1}{2} [(\vec{\sigma}_1 + \vec{\sigma}_2)^2 - \vec{\sigma}_1^2 - \vec{\sigma}_2^2]$$

and using the properties of the spin-1/2 operators we express the spin-spin interaction like this:

$$V = \frac{u}{2} [(\vec{\sigma}_1 + \vec{\sigma}_2)^2 - 6]$$

. This means that the eigenstates of the spin-spin interaction are the states with the definite values of the total spin of the two fermions. If the total spin is 1, the eigenstate $|\psi_1\rangle$ is an arbitrary superposition of the “triplet” states symmetric with respect to exchange of the two spins:

$$|\psi_1\rangle = c_1 |\uparrow\uparrow\rangle + c_2 |\downarrow\downarrow\rangle + c_3 (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle).$$

If the total spin is 0, the eigenstate $|\psi_0\rangle$ is the “singlet” state antisymmetric with respect to exchange of the two spins:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

The corresponding eigenvalues are u and $-3u$.

Thus, if the spin part of the wavefunction is a triplet state, the orbital part of the wavefunction should be antisymmetric and the energies are

$$E_{k,K}^{(\text{triplet})} = E_{k,K}^{(\text{fermions})} + u,$$

with the orbital part of the wavefunction the same as in Sec. (d). If the spin part of the wavefunction is the singlet, the orbital part of the wavefunction should be symmetric and the energies are

$$E_{k,K}^{(\text{singlet})} = E_{k,K}^{(\text{bosons})} - 3u,$$

with the orbital part of the wavefunction the same as in Sec. (c).

Quantum Mechanics 3

Decay rates in a 3D spherical-shell potential

A quantum particle of mass m moves in a 3D “spherical-shell” potential $V(r)$ which has the delta-functional dependence on the distance r to the center O of a spherical coordinate system:

$$V(r) = U_0\delta(r - a),$$

Assume that the particle has vanishing angular momentum, $l = 0$, relative to O , and energy E .

(a) (4 pts) What is the dependence of the particle wavefunction $\psi(\vec{r})$ on the angular coordinates θ and ϕ of the spherical coordinate system with the center O ? Write down the radial part of the stationary Schrödinger equation for the radial wavefunction $\psi(r)$ of the particle, and the general form of $\psi(r)$ that satisfies (1) the Schrödinger equation for $r < a$ and $r > a$, and (2) the appropriate boundary conditions at $r = 0$ and $r \rightarrow \infty$, assuming that there are no particles in the problem that are incident on the potential from infinity.

(b) (4 pts) What are the boundary conditions the wavefunction should satisfy at $r = a$? Transform them to obtain the equation for the wavevector p that determines the energy $E = \hbar^2 p^2 / 2m$ of the particle.

(c) (4 pts) Derive the form the boundary conditions from part (b) reduce to in the limit of the infinitely strong potential $U_0 \rightarrow \infty$. In this limit, find the wavevectors p_n , the energies $E_n^{(0)}$, and normalized wavefunctions $\psi_n(r, \theta, \phi)$ of the stationary states of the particle inside the shell.

(d) (6 pts) For finite U_0 , the states in the shell are not completely stationary. This fact can be accounted for by allowing the wavevector p to be complex:

$$p = k - i\kappa, \quad \kappa > 0.$$

Solve the equations derived in part (b) for such p and large but finite potential strength U_0 by perturbation expansion in $1/U_0$. Keep the terms up to (and including) the second order in $1/U_0$. Find the decay rates γ_n and energies E_n , $E = E_n - i\hbar\gamma_n/2$, of the quasistationary states inside the shell.

(e) (2 pts) When the wavevector p is complex, the wavefunctions diverge at $r \rightarrow \infty$: $\psi(r) \propto e^{\kappa r}$. Provide a brief physical explanation why this feature of the wavefunction is natural and does not disqualify it.

Solution

(a) The states with vanishing angular momentum, $l = 0$, are spherically symmetric, and the particle wavefunction $\psi(\vec{r})$ is independent of the angular coordinates θ and ϕ , i.e., reduces to the function $\psi(r)$ of the radial coordinate r only. The stationary Schrödinger equation for this function has the standard form

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + V(r)\psi = E\psi(r).$$

Introducing an auxiliary function $u(r)$ by the relation $\psi(r) = u(r)/r$, one reduces this equation to the 1D Schrödinger equation:

$$-\frac{\hbar^2}{2m} u'' + V(r)u = Eu(r).$$

If there are no particles incident on the potential from infinity, $u(r)$ should contain only the wave that propagates towards infinity at $r > a$, i.e., $u(r) \propto e^{ipr}$ at $r \rightarrow \infty$. Next, to keep $\psi(r)$ sufficiently well-behaved at $r = 0$, the function $u(r)$ should vanish at $r = 0$. With these boundary conditions, the usual solutions of the 1D Schrödinger equation for $u(r)$ give for $\psi(r)$:

$$\psi(r) = \frac{1}{r} \begin{cases} A \sin(pr), & r < a, \\ B e^{ipr}, & r > a, \end{cases}$$

where A and B are some constants.

(b) At $r = a$ the function $u(r)$ should satisfy the standard boundary condition that describes the delta-functional potential. Since the factor $1/r$ is continuous at $r = a$, this condition is not changed by the transition between $u(r)$ and $\psi(r)$. This means that the wavefunction $\psi(r)$ satisfies the following conditions at $r = a$:

$$\psi(a+0) = \psi(a-0), \quad \psi'(a+0) - \psi'(a-0) = \frac{2mu}{\hbar^2} \psi(a).$$

Applying these conditions to the wavefunction from part (a) we get:

$$A \sin pa = B e^{ipa}, \quad ipB e^{ipa} - pA \cos pa = \frac{2mu}{\hbar^2} B e^{ipa}.$$

Dividing the first equation by the second one, we get the following equation for the wavevector p :

$$\tan pa = \frac{p}{ip - \lambda}, \quad \lambda \equiv \frac{2mu}{\hbar^2}.$$

(c) In the limit of the infinitely strong potential $u \rightarrow \infty$, the boundary condition at $r = a$ reduces to:

$$\psi(a) = \frac{\hbar^2}{2mu} [\psi'(a+0) - \psi'(a-0)]|_{u \rightarrow \infty} = 0.$$

This means that $B = 0$, i.e., the wavefunctions vanish outside of the shell. Inside the shell:

$$\sin pa = 0.$$

This gives:

$$p = p_n = \frac{\pi}{a}n, \quad E_n^{(0)} = \frac{\hbar^2 p_n^2}{2ma}, \quad \psi_n(r, \theta, \phi) = \frac{1}{\sqrt{2\pi a}} \frac{\sin pr}{r}.$$

(d) For strong potential, $\lambda \gg 1$, and the equation for the wavevector p derived in part (b) can be expanded in $1/\lambda$ with the required accuracy like this:

$$\tan pa = -\frac{p}{\lambda} - i\frac{p^2}{\lambda^2}.$$

Solving this equation by iterations, we get:

$$p = p_n - \frac{p_n}{\lambda a} + \frac{p_n}{\lambda^2 a^2} - i\frac{p_n^2}{\lambda^2 a}.$$

From this, we get the energy $E = \hbar^2 p^2/2m$, and therefore the decay rates:

$$\gamma_n = \frac{2\hbar p_n^3}{m\lambda^2 a}$$

and the energies of the quasistationary states:

$$E_n = E_n^{(0)} \left(1 - \frac{2}{\lambda a} + \frac{3}{\lambda^2 a^2}\right).$$

The obtained expression for the decay rate can be interpreted qualitatively as the transmission probability $D_n = |t_n|^2$ of the weakly-transparent delta-functional barrier with the transmission amplitude $|t_n| = 2p_n/\lambda$ times the attempt frequency $f = \hbar p_n/(2ma) = v_n/(2a)$.

(e) The outgoing wave towards infinity carries the probability density away from the potential region, thus reducing the probability inside the shell. Consistent with this, increasing wavefunction amplitude at larger distances r represents the larger wavefunction amplitudes of the quasistationary state at the earlier times in the decay process.

Statistical Mechanics 1

N electrons and N sites

A system contains N independent sites and N electrons at temperature T . At a given site there is one accessible orbital, but that orbital can be empty, occupied by one electron of either spin, or occupied by two electrons of opposite spin. The electrons can hop from site to site.

The site energy is zero if the site is either empty or singly occupied, and Δ if it is doubly occupied.

- (a) (4 points) List all states and determine the grand partition function of the system.
- (b) (8 points) Determine the chemical potential as a function of temperature. Check your work.
- (c) (8 points) Calculate the mean energy and heat capacity of the sites.

Solution

(a) For every site there are four states e labelled $s = 0$, $s = 1u$, $s = 1d$, and $s = \text{“double”}$.

state s	E_s	N_s
0	0	0
1u	0	1
1d	0	1
double	Δ	2

The grand partition function of a site is

$$Z_G = 1 + e^{\beta\mu} + e^{\beta\mu} + e^{2\beta\mu} e^{-\beta\Delta}, \quad (1)$$

$$= 1 + 2z + z^2 e^{-\beta\Delta}. \quad (2)$$

where $z \equiv e^{\beta\mu}$ is the fugacity.

In the grand partition function the sites are all independent so the states are labelled by an independent s_i for every site (i labels the site). The states are simply $S = \{s_i\}$, where each s_i runs over the possibilities given above. The energy of a state S is $E_S = \sum_i E_{s_i}$ and the number of electrons in a state S is $N_S = \sum_i N_{s_i}$. The grand partition function of M sites is

$$\mathcal{Z}_G = \sum_{\{s_i\}} e^{-\beta(E_S - \mu N_S)} = Z_G^M. \quad (3)$$

We will work with a single site from now on.

(b) From the problem statement there is on average one electron per site $\langle n \rangle = 1$. The chemical potential is such that this condition (one per site) is fulfilled.

We evaluate the mean number of particles in a site as follows:

$$\langle n \rangle = \sum_{states} P_s N_s, \quad (4)$$

The probability of being in a state is

$$P_s = \frac{e^{\beta\mu N_s} e^{-\beta E_s}}{Z_G}, \quad (5)$$

leading to

$$\langle n \rangle = \frac{2z + 2z^2 e^{-\beta\Delta}}{1 + 2z + z^2 e^{-\beta\Delta}}. \quad (6)$$

Requiring that $\langle n \rangle = 1$ yields

$$\langle n \rangle = \frac{2z + 2z^2 e^{-\beta\Delta}}{1 + 2z + z^2 e^{-\beta\Delta}} = 1, \quad (7)$$

or

$$1 = z^2 e^{-\beta\Delta}. \quad (8)$$

So, we find

$$z = e^{\beta\Delta/2} \quad \text{or} \quad \mu = \Delta/2, . \quad (9)$$

(c) The energy is the derivative of the grand partition function at fixed z . Here we proceed directly

$$\langle \epsilon \rangle = \sum_s P_s E_s. \quad (10)$$

Thus

$$\langle \epsilon \rangle = \sum_s P_s E_s = \frac{\Delta}{1 + 2e^{\beta\mu} + e^{-\beta(\Delta-2\mu)}} \rightarrow \frac{\Delta}{2 + 2e^{\beta\Delta/2}}, \quad (11)$$

where in the last step we replace μ with the result of part (b), $\mu = \mu_*(T, N)$. The total mean energy of the N sites is

$$U(T, N) = \langle E \rangle = \frac{N\Delta}{2 + 2e^{\beta\Delta/2}} \quad (12)$$

The heat capacity of N sites is the derivative of this at fixed N , leading to

$$C_V = \left(\frac{\partial U}{\partial T} \right)_N = -Nk_B\beta^2 \left(\frac{\partial U}{\partial \beta} \right)_N = Nk_B \frac{(\beta\Delta)^2 e^{\beta\Delta/2}}{(2 + 2e^{\beta\Delta/2})^2}. \quad (13)$$

Discussion: Generally for a fixed number of electrons N_e , the chemical potential of the electrons is a function of temperature and N_e . It happens, somewhat artificially, that μ is independent of T in this problem. Thus, it is generally incorrect with fixed N_e to write, $c_v = \partial \langle \epsilon(T, \mu) \rangle / \partial T$. The correct procedure is to replace μ with the result of part (b), i.e. $\mu_*(T, N_e)$, before differentiating, $\epsilon(T, N_e) = \epsilon(T, \mu_*(T, N_e))$.

Statistical Mechanics 2

Quark/baryon phase transition

Nucleons (protons and neutrons) are bound states of three quarks. Each nucleon carries one unit of conserved baryon number, while each quark carries $(1/3)$. Quarks and nucleons have spin- $1/2$ and obey Fermi statistics.

When nucleons are compressed to high density, some may dissociate into a gas of quarks. In this problem, we will construct a simple model of such phase transition neglecting all internal quantum numbers of the nucleons and quarks, except their $1/2$ -spins. We will assume that nucleons with mass m are nonrelativistic and do not interact, while quarks are relativistic with kinetic energies $\varepsilon_q^{\text{kin}} = c|\vec{p}|$, where c is the speed of light.

(a) (6 points) We will consider only zero temperature, where the nucleon and quark gases are degenerate.

- i) Calculate the number density $n = N/V$ of spin- $\frac{1}{2}$ fermions from the Fermi momentum p_F .
- ii) Find the mean kinetic energy of non-relativistic nucleons $\varepsilon_n^{\text{kin}}$ and express it using n_n .
- iii) Find the mean kinetic energy of relativistic quarks $\varepsilon_q^{\text{kin}}$ and express it using n_q .

(b) (6 points) We will need chemical potential $\mu = \varepsilon + P/n$ to study the phase transition and equilibrium. Here $P = -\left(\frac{\partial E}{\partial V}\right)_{S=0} \equiv -\left(\frac{\partial E}{\partial V}\right)_{T=0}$ is the pressure at zero temperature and entropy.

- i) Show that the pressure can be calculated as $P = n^2 \left(\frac{\partial \varepsilon}{\partial n}\right)_{S=T=0}$.
- ii) Using this formula, calculate the pressure of the degenerate nucleon gas.
- iii) Show that the chemical potential for the nucleon gas is $\mu_n = \varepsilon_{nF} = p_{nF}^2/(2m)$.

(c) (8 points) In our model, quarks are nucleon-bound at low density n_q because the quark gas has additional energy $B > 0$ *per unit volume*, or $\Delta\varepsilon_q = BV/N_q = B/n_q$ *per quark*. Since the temperature is zero, at any density n_B the system will prefer the state with the lowest total energy (assume that the system is maintained at constant volume).

- i) Calculate the pressure P_q of the quark gas. Can the pressure P_q be zero or negative? Explain, qualitatively, the meaning of $P_q = 0$ point.
- ii) Sketch the energies of both gases as functions of baryon density $n_B = n_n = 3n_q$. Show that at low density the nucleon gas has lower energy, whereas the quark gas has lower energy at high density.

- iii) Write the equation determining the transition density n_T between the two phases. Describe and sketch the condition at which nucleons dissociate into a gas of quarks.

Solution

(a)

The relation between the Fermi momentum and the fermion density at zero temperature is

$$n = g_s \int_{|\vec{p}| \leq p_F} \frac{d^3 \vec{p}}{(2\pi\hbar)^3} = \frac{g_s p_F^3}{6\pi^2 \hbar^3} \quad (1)$$

where $g_s = 2$ is the spin degeneracy. Applying the same to the nonrelativistic fermion gas to find the kinetic energy density of the nucleons

$$\frac{E_n^{\text{kin}}}{V} = g_s \int_{|\vec{p}| \leq p_{nF}} \frac{d^3 \vec{p}}{(2\pi\hbar)^3} \frac{p^2}{2m} = \frac{g_s p_{nF}^5}{10\pi^2 \hbar^3 m} = \frac{3}{5} n_n \frac{p_{nF}^2}{2m} = \frac{3}{5} n_n \varepsilon_{nF}, \quad (2)$$

so that the mean energy per nucleon is

$$\varepsilon_n^{\text{kin}} = \frac{E_n^{\text{kin}}/V}{n_n} = \frac{3}{5} \varepsilon_{nF} = \frac{3}{5} \frac{p_{nF}^2}{2m} = \frac{3}{10m} \left(\frac{6\pi^2 \hbar^3 n_n}{g_s} \right)^{2/3} \propto n_n^{2/3}. \quad (3)$$

For a relativistic gas of quarks, the kinetic energy density is

$$\frac{E_q^{\text{kin}}}{V} = g_s \int_{|\vec{p}| \leq p_{qF}} \frac{d^3 \vec{p}}{(2\pi\hbar)^3} c|p| = \frac{g_s c p_{qF}^4}{8\pi^2 \hbar^3} = \frac{3}{4} n_q c p_{qF} = \frac{3}{4} n_q \varepsilon_{qF}, \quad (4)$$

and the mean quark kinetic energy is thus

$$\varepsilon_q^{\text{kin}} = \frac{E_q^{\text{kin}}/V}{n_q} = \frac{3}{4} \varepsilon_{qF} = \frac{3}{4} c p_{qF} = \frac{3c}{4} \left(\frac{6\pi^2 \hbar^3 n_q}{g_s} \right)^{1/3} \propto n_q^{1/3}. \quad (5)$$

The relation between the density and the Fermi momentum $n \propto p_F^3$ is universal, so if $n_T = n_n = \frac{1}{3} n_q$ then

$$\frac{p_{nF}}{p_{qF}} = \left(\frac{n_n}{n_q} \right)^{1/3} = (1/3)^{1/3} \approx 0.693. \quad (6)$$

(b)

The pressure can be computed from the per-particle energy using the chain rule and switching the variable $V \rightarrow n$ using $nV = N = \text{const}$, hence $N \left(\frac{\partial n}{\partial V} \right)_N = -N^2/V^2 = -n^2$:

$$P = - \left(\frac{\partial E}{\partial V} \right)_{S,N} = -N \left(\frac{\partial \varepsilon}{\partial V} \right)_{S,N} = -N \left(\frac{\partial n}{\partial V} \right)_N \left(\frac{\partial \varepsilon}{\partial n} \right)_{S,N} = n^2 \left(\frac{\partial \varepsilon}{\partial n} \right)_S \quad (7)$$

Differentiation of the mean energy is greatly simplified by observing that $\varepsilon_n \propto n_n^{2/3}$, hence

$$\frac{\partial \varepsilon_n}{\partial n_n} = \frac{2}{3n_n} \varepsilon_n = \frac{2}{5n_n} \varepsilon_{nF}, \quad \text{and } P_n = \frac{2}{5} n_n \varepsilon_{nF}. \quad (8)$$

and the nucleon chemical potential, as expected, is

$$\mu_n = \varepsilon_n + \frac{P_n}{n_n} = \frac{3}{5} \varepsilon_{nF} + \frac{2}{5} \varepsilon_{nF} = \varepsilon_{nF}. \quad (9)$$

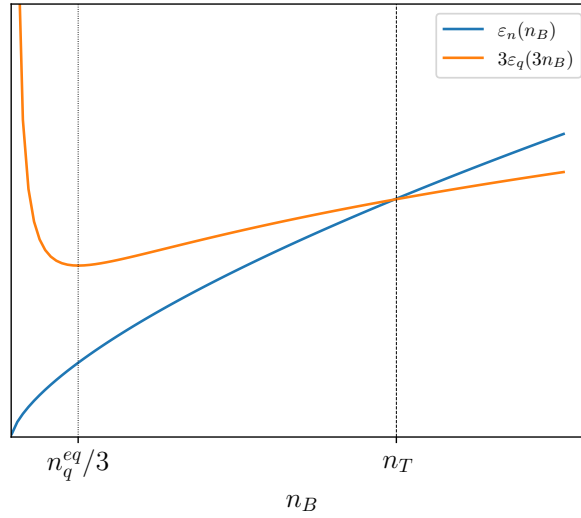
(c)

Using the per-quark mean energy

$$\varepsilon_q = \frac{3}{4} \varepsilon_{qF} + \frac{B}{n_q}, \quad (10)$$

and observing that $\varepsilon_q^{\text{kin}} \propto n_q^{1/3}$ one obtains the pressure

$$P_q = n_q^2 \left(\frac{\partial \varepsilon_q}{\partial n_q} \right)_{S=T=0} = \frac{1}{4} n_q \varepsilon_{qF} - B. \quad (11)$$



The negative pressure contribution results from the quark gas interaction energy proportional to the volume, irrespective of the number of particles. As the gas with smaller volume (larger density) will have lower interaction energy, this interaction leads to gas compression, somewhat similar to van der Waals gas. On the other hand, higher density leads to larger mean kinetic energy $\varepsilon_q^{\text{kin}} = \frac{3}{4} \varepsilon_{qF} \propto n_q^{1/3}$. The equilibrium point at which $P_q = 0$

$$P_q = 0 \quad \Leftrightarrow \quad n_q^{\text{eq}} \propto B^{3/4} \quad (12)$$

corresponds to the minimal energy of the quark gas, where the pressure of degenerate Fermi gas equalizes the tension of bulk energy density.

The sketches of nucleon and quark energies per unit baryon number are shown in Fig(left). The general behavior of the total energy of the nucleon gas and the quark gas is

$$\begin{aligned}\varepsilon_n &= Cn_n^{2/3}, \\ \varepsilon_q &= An_q^{1/3} + Bn_q^{-1} = A'n_n^{1/3} + B'n_n^{-1},\end{aligned}$$

where A, C are some constants. In the second line, we have assumed that $n_q = 3n_n$ and A', B' are scaled constants. For $n_n \rightarrow 0$, the quark gas energy will grow $\propto 1/n_n$ while the nucleon gas energy will decrease $\propto n_n$. For $n_n \rightarrow \infty$, the leading term in $E_q \propto n_n^{1/3}$ while in $E_n \propto n_q^{2/3}$ and $E_n \geq E_q$ for $n \geq n_T$ for some n_T .

The equation for the equilibrium corresponds to the intersection of the curves on the plot,

$$\varepsilon_n(n_T) = 3\varepsilon(3n_T). \tag{13}$$

Statistical Mechanics 3

Electron bands in two-dimensional semiconductor

Consider a non-interacting electron gas in a 2d semiconductor. In crystalline solids, the electrons can be described as fermions with electronic charge but effective “band mass”, with energy dispersion following:

$$E(p) = \begin{cases} p^2/2m_c + \Delta & \text{if } E > 0 \\ -p^2/2m_v & \text{if } E \leq 0 \end{cases}$$

Here m_c and m_v are the effective masses of the electron in the corresponding conduction and valence bands. Δ is the energy gap between the two bands.

- (a) (4 points) What is the energy dependent density of states of this 2d electron gas?
- (b) (6 points) Assume that at zero temperature $T = 0$ the Fermi energy is E_D above the gap: $E_F = \Delta + E_D$, and the total number of electrons in the system is fixed. Find out the equation which allows determination of the temperature dependence of the chemical potential $\mu(T)$. (The equation should not contain any integral).
- (c) (4 points) Assume the simple case of “intrinsic semiconductor” where $E_D = 0$ (i.e., the valence band ($E < 0$) is completely filled and the conduction band is completely empty at $T = 0$). Qualitatively how does the chemical potential change with increasing temperature for $m_c > m_v$?
- (d) (6 points) Now still with $E_D = 0$ and further simplify the problem by assuming electron-hole symmetry (i.e. $m_c = m_v$), calculate the temperature dependence of the density of the electrons in the conduction band ($E > \Delta$) and holes in the valence band ($E < 0$). Calculate the specific heat of the entire electron gas (electrons and holes) in the 2d semiconductor at low temperature $T \ll \Delta$.

Hint: You may find the following integral useful:

$$\int \frac{1}{1 + e^{\beta(\epsilon - \mu)}} d\epsilon = -\frac{1}{\beta} \ln [1 + e^{-\beta(\epsilon - \mu)}] + \text{const}$$

Solution (a) The density of states D is such that:

$$\frac{2}{4\pi^2\hbar^2}d^2p = D(E)dE \quad (1)$$

From the dispersion of the conduction band, this is:

$$\frac{1}{2\pi^2\hbar^2}2\pi p_r dp_r = D(E)\frac{p_r}{m_c}dp_r \quad (2)$$

which leads to

$$D(E > \Delta) = \frac{m_c}{\pi\hbar^2} \quad (3)$$

Similarly, for the valence band:

$$D(E \leq 0) = \frac{m_v}{\pi\hbar^2} \quad (4)$$

In the band gap, there is no state so

$$D(\Delta > E > 0) = 0 \quad (5)$$

(b) The total number of electrons is fixed, so the number of electrons excited should be equal to the number of holes at finite temperatures.

$$\int_{E_F}^{\infty} f(E)D(E)dE = \int_{-\infty}^{E_F} (1 - f(E))D(E)dE \quad (6)$$

$$\frac{m_c}{\pi\hbar^2} \int_{\Delta+E_D}^{\infty} \frac{1}{1 + e^{\beta(E-\mu)}} dE = \frac{m_c}{\pi\hbar^2} \int_{\Delta}^{\Delta+E_D} \frac{1}{1 + e^{-\beta(E-\mu)}} dE + \frac{m_v}{\pi\hbar^2} \int_{-\infty}^0 \frac{1}{1 + e^{-\beta(E-\mu)}} dE \quad (7)$$

Using the given integral this is:

$$\frac{m_c}{\pi\hbar^2} \frac{1}{\beta} \ln [1 + e^{-\beta(\Delta+E_D-\mu)}] = \frac{m_c}{\pi\hbar^2} \frac{1}{\beta} \ln \left[\frac{1 + e^{\beta(\Delta+E_D-\mu)}}{1 + e^{\beta(\Delta-\mu)}} \right] + \frac{m_v}{\pi\hbar^2} \frac{1}{\beta} \ln [1 + e^{-\beta\mu}] \quad (8)$$

The equation for the chemical potential is then, after simplifications:

$$\begin{aligned} m_c \ln [e^{-\beta(\Delta+E_D-\mu)}(1 + e^{\beta(\Delta-\mu)})] &= m_v \ln [1 + e^{-\beta\mu}], \\ \text{or } -\beta m_c E_D + m_c \ln [1 + e^{\beta(\mu-\Delta)}] &= m_v \ln [1 + e^{-\beta\mu}] \end{aligned}$$

(c) A purely qualitative argument can be made as follows. If $m_c > m_v$, then the conduction band has larger density of states than the valence band, i.e. there is more phase space for

electrons than for holes. Therefore, at any finite temperature the holes must be spread wider in energy compared to electrons. This necessarily means that with increasing temperature, the half-filled state (equally probable to be an electron or a hole) must shift down in energy so that the ‘‘Boltzmann tail’’ of the distribution ($|E - mu| \gg T$) has larger overlap with the valence band.

Alternatively, the equations above with $E_D = 0$ take the form

$$m_c \ln [1 + e^{\beta(\mu-\Delta)}] = m_v \ln [1 + e^{-\beta\mu}] \quad (9)$$

For $\beta \rightarrow \infty$ (low temperature), the chemical potential must be $0 < \mu < \Delta$ so that $e^{\beta(\mu-\Delta)}$, $e^{-\beta\mu} \ll 1$ and one can expand the logarithms to get

$$m_c e^{\beta(\mu-\Delta)} \approx m_v e^{-\beta\mu} \implies (\mu - \Delta/2) \approx \frac{1}{2\beta} \ln \frac{m_v}{m_c} \rightarrow 0 \quad (10)$$

thus $\mu \rightarrow (\Delta/2)$ as $T \rightarrow 0$. Note that at $T > 0$ the chemical potential is below $(\Delta/2)$, which can be shown using inequality

$$\frac{\ln [1 + e^{\beta(\mu-\Delta)}]}{\ln [1 + e^{-\beta\mu}]} = \frac{m_v}{m_c} < 1 \implies e^{\beta(\mu-\Delta)} < e^{-\beta\mu} \implies \mu < \frac{1}{2}\Delta, \quad (11)$$

meaning that with increasing temperature the chemical potential will decrease.

(d) For $m_c = m_v$, this equation for the chemical potential further simplifies to

$$\beta(\mu - \Delta) = -\beta\mu \quad (12)$$

and the chemical potential is independent from the temperature:

$$\mu = \frac{\Delta}{2} \quad (13)$$

The electron density is then

$$\int_{\Delta}^{\infty} f(E)D(E)dE = \frac{m_c}{\beta\pi\hbar^2} \ln [1 + e^{-\beta(\Delta-\mu)}] = \frac{m_c T}{\pi\hbar^2} \ln [1 + e^{-\Delta/(2T)}] \quad (14)$$

and it is equal to the density of holes. The change of total energy of the electrons in the

system from zero temperature to a finite temperature is:

$$\begin{aligned}
\Delta U &= \int_{-\infty}^0 E[f(E, T) - 1]D(E)dE + \int_{\Delta}^{\infty} Ef(E, T)D(E)dE \\
&= \frac{m}{\pi\hbar^2} \left[- \int_{-\infty}^0 \frac{E}{1 + e^{-\beta(E-\Delta/2)}} dE + \int_{\Delta}^{\infty} \frac{E}{1 + e^{\beta(E-\Delta/2)}} dE \right] \\
&= \frac{m}{\pi\hbar^2} \left[\int_0^{\infty} \frac{E}{1 + e^{\beta(E+\Delta/2)}} dE + \int_0^{\infty} \frac{E + \Delta}{1 + e^{\beta(E+\Delta/2)}} dE \right] \\
&= \frac{m}{\pi\hbar^2} \int_0^{\infty} \frac{2E + \Delta}{1 + e^{\beta(E+\Delta/2)}} dE
\end{aligned}$$

Here $m = m_c = m_v$. Effectively, this is equivalent to equilibrium of electron and hole Fermi gases. Electron-hole pairs can be created or recombine with energy Δ per pair, or $\Delta/2$ per each. At low temperature, $e^{\beta(E+\Delta/2)} \gg 1$, and one can neglect the unity in the denominator

$$\frac{m}{\pi\hbar^2} \int_0^{\infty} \frac{2E + \Delta}{1 + e^{\beta(E+\Delta/2)}} dE \approx \frac{m}{\pi\hbar^2} \int_0^{\infty} (2E + \Delta) e^{-\beta(E+\Delta/2)} dE = \frac{m}{\pi\hbar^2} e^{-\beta\Delta/2} \left(\frac{\Delta}{\beta} + \frac{2}{\beta^2} \right)$$

and the specific heat (per unit area) is

$$c = \frac{d\Delta U}{dT} \approx \frac{m}{\pi\hbar^2} \frac{d}{dT} \left[e^{-\frac{\Delta}{2T}} (T\Delta + 2T^2) \right] = \frac{m}{\pi\hbar^2} e^{-\frac{\Delta}{2T}} \left[2\Delta + 4T + \frac{\Delta^2}{2T} \right], \quad (15)$$

and has dimension [Area⁻¹].