Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

Fall 2022 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

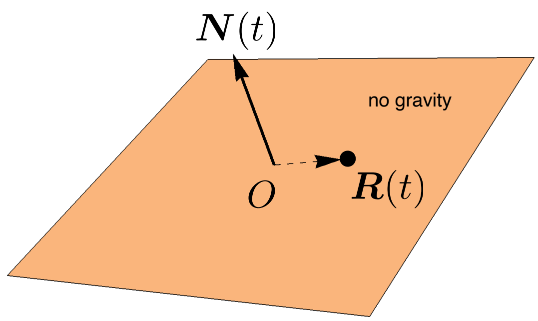
Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

A constrained oscillator

A particle of mass m moving in three dimensions is bound to the origin O by a harmonic spring of spring constant κ. The particle is constrained to lie in a plane having normal vector $\boldsymbol{N}(t)$, as shown below, which may vary in time. Neglect gravity throughout. Let $\boldsymbol{R}(t)$ denote the position vector of the particle as a function of time.



(a) (6 points) Write down a Lagrangian that governs the dynamics of $\boldsymbol{R}$, using a Lagrange multiplier Λ to enforce the geometric constraint $\boldsymbol{R}(t) \cdot \boldsymbol{N}(t) = 0$. Thus determine the equation of motion satisfied by $\boldsymbol{R}(t)$, from which Λ has been eliminated.

(b) (6 points) Assume that $\boldsymbol{R}(t)$ takes the form $\boldsymbol{R}(t) = \boldsymbol{A}(t)\sin(\omega t)$. Show that under a *slow* variation of $\boldsymbol{N}(t)$ the amplitude vector $\boldsymbol{A}(t)$ evolves according to

$$\frac{1}{G}\frac{d\boldsymbol{A}(t)}{dt} = (\boldsymbol{N}(t) \times \dot{\boldsymbol{N}}(t)) \times \boldsymbol{A}(t)\,, \tag{1}$$

where the overdot indicates a time derivative, and determine the numerical constant G. State the criterion for the variation of $\boldsymbol{N}$ to be slow.

Hint: The vector identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$ may be useful.

During the time interval $0 \le t \le 3\tau$, the plane's orientation $\boldsymbol{N}(t)$ evolves according to the following formula and is also indicated in the figure below:

$$\boldsymbol{N}(t) = \begin{cases} \boldsymbol{e}_z \cos(\pi t/2\tau) + \boldsymbol{e}_x \sin(\pi t/2\tau)\,, & 0 \le t \le \tau, \\ \boldsymbol{e}_x \cos(\pi(t-\tau)/2\tau) + \boldsymbol{e}_y \sin(\pi(t-\tau)/2\tau)\,, & \tau \le t \le 2\tau, \\ \boldsymbol{e}_y \cos(\pi(t-2\tau)/2\tau) + \boldsymbol{e}_z \sin(\pi(t-2\tau)/2\tau)\,, & 2\tau \le t \le 3\tau\,, \end{cases} \tag{2}$$

where $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ are the Cartesian basis vectors.

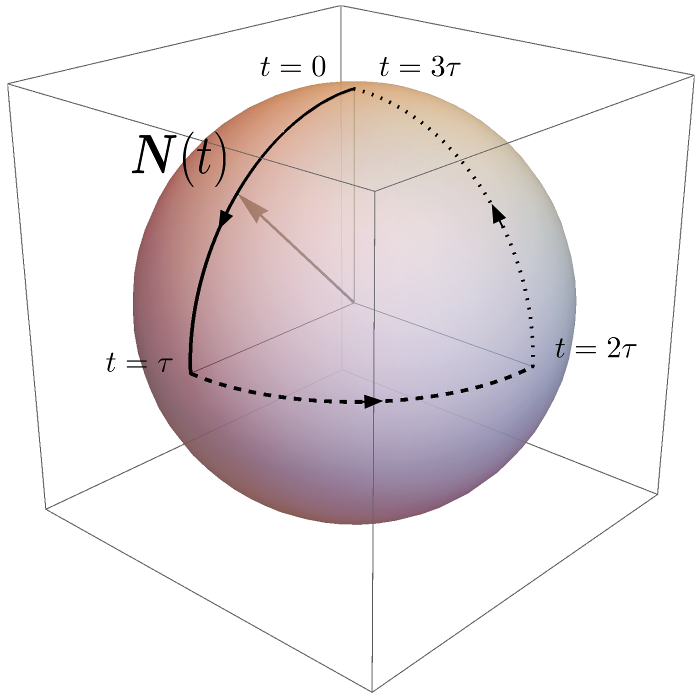


Figure 1: Direction of the unit vector $\boldsymbol{N}(t)$ as a function of time for part (d).

(c) (5 points) Suppose that $\boldsymbol{A}(t)|_{t=0} = \boldsymbol{e}_x$. Determine $\boldsymbol{A}(t)|_{t=3\tau}$. Compare the angle through which $\boldsymbol{A}(t)$ has rotated after the time interval 3τ to the solid angle swept out by $\boldsymbol{N}(t)$ over the same interval.

(d) (3 points) How would your answer to part (c) change if $\boldsymbol{N}(t)$ were to follow the same *path* on the sphere, but at a slow but otherwise arbitrarily varying rate? Explain why.

Solution

1. The Lagrangian, including the Lagrange multiplier term to enforce the constraint $\boldsymbol{R}(t)\cdot\boldsymbol{N}(t)=0$, is:
$$L=\frac{m}{2}|\dot{\boldsymbol{R}}|^2-\frac{\kappa}{2}|\boldsymbol{R}|^2+\Lambda\boldsymbol{R}\cdot\boldsymbol{N}. \tag{3}$$
The corresponding Euler-Lagrange equation is
$$\mathbf{0}=\frac{d}{dt}\frac{\partial L}{\partial\dot{\boldsymbol{R}}}-\frac{\partial L}{\partial\boldsymbol{R}}=m\ddot{\boldsymbol{R}}+\kappa\boldsymbol{R}-\Lambda\boldsymbol{N}. \tag{4}$$
To determine the Lagrange multiplier Λ, take the scalar product of the Euler-Lagrange equation with $\boldsymbol{N}$ and use the constraint $\boldsymbol{N}\cdot\boldsymbol{R}=0$ and the normalization $\boldsymbol{N}\cdot\boldsymbol{N}=1$ to obtain
$$0=m\boldsymbol{N}\cdot\ddot{\boldsymbol{R}}+\kappa\boldsymbol{N}\cdot\boldsymbol{R}-\Lambda\boldsymbol{N}\cdot\boldsymbol{N}=m\boldsymbol{N}\cdot\ddot{\boldsymbol{R}}-\Lambda. \tag{5}$$
Solve to obtain $\Lambda=m\boldsymbol{N}\cdot\ddot{\boldsymbol{R}}$, and insert for Λ into the Euler-Lagrange equation to obtain
$$m\ddot{\boldsymbol{R}}+\kappa\boldsymbol{R}=\boldsymbol{N}\big(\boldsymbol{N}\cdot\ddot{\boldsymbol{R}}\big). \tag{6}$$

2. Assume $\boldsymbol{R}(t)=\boldsymbol{A}(t)\sin\omega t$, note that
$$\begin{aligned}\dot{\boldsymbol{R}}(t) &= \dot{\boldsymbol{A}}(t)\sin\omega t+\omega\boldsymbol{A}(t)\cos\omega t,\\ \ddot{\boldsymbol{R}}(t) &= \ddot{\boldsymbol{A}}(t)\sin\omega t+2\omega\dot{\boldsymbol{A}}(t)\cos\omega t-\omega^2\boldsymbol{A}(t)\sin\omega t,\end{aligned}$$
and insert this form into the Euler-Lagrange equation to obtain (writing s for $\sin\omega t$ and c for $\cos\omega t$)
$$m\Big(\ddot{\boldsymbol{A}}\,s+2\omega\dot{\boldsymbol{A}}\,c-\omega^2\boldsymbol{A}\,s\Big)+\kappa\boldsymbol{A}\,s=\boldsymbol{N}\cdot\Big(\ddot{\boldsymbol{A}}\,s+2\omega\dot{\boldsymbol{A}}\,c-\omega^2\boldsymbol{A}\,s\Big) \tag{7}$$
Choosing ω to be the natural oscillator frequency $\sqrt{\kappa/m}$, we see that terms 3 and 4 cancel identically and term 7 vanishes via the constraint. Furthermore, because the variation of $\boldsymbol{A}$ is slow, being induced by the slow variation of $\boldsymbol{N}$, to leading order we may assume that the terms proportional to $\ddot{\boldsymbol{A}}$ (i.e., terms 1 and 5) are small, relative to the terms of order $\dot{\boldsymbol{A}}$ (i.e., terms 2 and 6) and may therefore be omitted. Hence we arrive at the equation of motion for $\boldsymbol{A}$, accurate for slow variations of $\boldsymbol{N}$:
$$\dot{\boldsymbol{A}}=\big(\boldsymbol{N}\cdot\dot{\boldsymbol{A}}\big)\boldsymbol{N}. \tag{8}$$
To obtain the desired form of the equation of motion, we note that the time-derivative of the constraint equation gives $\dot{\boldsymbol{N}}\cdot\boldsymbol{A}+\boldsymbol{N}\cdot\dot{\boldsymbol{A}}=0$, which enables us to rewrite the equation of motion as
$$\dot{\boldsymbol{A}}=-\big(\dot{\boldsymbol{N}}\cdot\boldsymbol{A}\big)\boldsymbol{N}. \tag{9}$$
Then adding $\dot{\boldsymbol{N}}\big(\boldsymbol{N}\cdot\boldsymbol{A}\big)$ (which is a vanishing contribution owing to the constraint), and using the given vector triple product identity, we arrive at the required form:
$$\frac{1}{G}\frac{d\boldsymbol{A}}{dt}=\big(\boldsymbol{N}\times\dot{\boldsymbol{N}}\big)\times\boldsymbol{A}, \tag{10}$$
with $G=1$.

3. We are given $\boldsymbol{N}(t)$ during three time-intervals. Computing $\dot{\boldsymbol{N}}$ in each interval and forming $\boldsymbol{N} \times \dot{\boldsymbol{N}}$ for each interval, we find:

$$\boldsymbol{N} \times \dot{\boldsymbol{N}} = \begin{cases} (\pi/2\tau)\, \boldsymbol{e}_y\,, & 0 \le t \le \tau\,, \\ (\pi/2\tau)\, \boldsymbol{e}_z\,, & \tau \le t \le 2\tau\,, \\ (\pi/2\tau)\, \boldsymbol{e}_x\,, & 2\tau \le t \le 3\tau\,, \end{cases}$$

where $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ are the Cartesian basis vectors. We are given that $\boldsymbol{A}(t)|_{t=0} = \boldsymbol{e}_x$, i.e., initially $\boldsymbol{A}$ is initially aligned along $\boldsymbol{e}_x$. Observing that the equation of motion for $\boldsymbol{A}$ describes precession about $\boldsymbol{N} \times \dot{\boldsymbol{N}}$, which is constant in each of the three time-intervals (as our computation has shown), we see that:
(i) during $0 \le t \le \tau$, $\boldsymbol{A}$ precesses around $\boldsymbol{e}_y$, progressing from $\boldsymbol{e}_x$ to $-\boldsymbol{e}_z$;
(ii) during $\tau \le t \le 2\tau$, $\boldsymbol{A}$ would precess around $\boldsymbol{e}_z$, so it remains at $-\boldsymbol{e}_z$; and
(iii) during $2\tau \le t \le 3\tau$, $\boldsymbol{A}$ precesses around $\boldsymbol{e}_x$, progressing from $-\boldsymbol{e}_z$ to $\boldsymbol{e}_y$.
Thus, we find that $\boldsymbol{A}(t)|_{t=3\tau} = \boldsymbol{e}_y$.

Over the course of $0 \le t \le 3\tau$, $\boldsymbol{A}$ has rotated from $\boldsymbol{e}_x$ to $\boldsymbol{e}_y$, i.e., through an angle $\pi/2$. During that time-interval, the solid angle swept out by $\boldsymbol{N}$ is $4\pi/8$, i.e., also $\pi/2$, so the two are numerically equal.

4. The equation of motion for $\boldsymbol{A}$ is readily seen to be invariant under reparametrizations of time, i.e., $t \to T(t)$. The factors of dT/dt arising from the chain rule cancel. (Do, however, recall that the equation was derived assuming slow variations of $\boldsymbol{N}$, relative to the oscillator natural period $2\pi\sqrt{m/\kappa}$.) Therefore, $\boldsymbol{A}$ would evolve along the same path (i.e., *where* $\boldsymbol{A}$ goes would be unchanged) and it would end at the same location, but its motion along that path would change (i.e., its *when* would change). So, the end-point would remain $\boldsymbol{A}(t)|_{t=3\tau} = \boldsymbol{e}_y$ (i.e., it would continue to equal the area swept out by the trajectory of $\boldsymbol{N}$ on the unit sphere.

Classical Mechanics 2

Masses on a rotating thin rod

Consider two point particles, each of mass m, attached to a central rod via massless rods of length L, arranged as shown in Fig. 1. The upper mass is constrained to rotate around the central rod with frequency ω; it is attached at the top via a hinge so that the angle $\theta(t)$ is variable. The lower mass can move up and down on the central rod without friction. Consider the effect of gravity unless explicitly mentioned. All answers should be expressed in terms of only m, L, ω and θ.

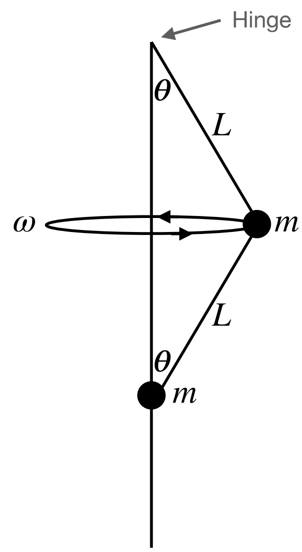


Figure 1: Rotating balls attached via massless rods.

(a) **[5pts]** Write the Lagrangian and derive the Euler-Lagrange equation for θ.

(b) **[4pts]** If the system starts at $\theta \simeq 0$ with initial angular velocity $\dot{\theta} = \Omega$, what is $\dot{\theta}$ when $\theta = \pi/2$? Neglect gravity.

Hint: are there any constants of motion?

(c) **[2pts]** In the absence of gravity, find a solution where the masses rotate at a fixed angle θ with $0 < \theta < \pi$. What is the value of θ?

(d) **[4pts]** Find the critical frequency at which a solution exists where θ is fixed. What is the value of θ? Check that your result agrees with part (c).

(e) **[5pts]** Now consider small oscillations about the equilibrium angle. What is the frequency of oscillations?

(a) Start the problem by finding the Lagrangian. Let the origin be at the top of the central rod. Then the upper mass has coordinates:

$$x_1 = L \sin\theta \cos\phi \tag{1}$$
$$y_1 = L \sin\theta \sin\phi \tag{2}$$
$$z_1 = -L \cos\theta, \tag{3}$$

where ϕ is the azimuthal angle of the masses. The lower mass has coordinates:

$$x_2 = y_2 = 0 \tag{4}$$
$$z_2 = -2L \cos\theta \tag{5}$$

Thus, the kinetic energy is given by:

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right) = \frac{mL^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta + 4\dot{\theta}^2 \sin^2\theta\right) \tag{6}$$

The potential energy is due to the force of gravity and given by:

$$U = -3mgL\cos\theta \tag{7}$$

The Lagrangian is given by

$$L = T - U = \frac{mL^2}{2}\left(\dot{\theta}^2 + \omega^2 \sin^2\theta + 4\dot{\theta}^2 \sin^2\theta\right) + 3mgL\cos\theta, \tag{8}$$

where we have made the substitution from the problem statement, $\dot{\phi} = \omega$.

The Euler-Lagrange equation is then:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = \frac{\partial L}{\partial \theta} \tag{9}$$
$$\frac{d}{dt}\left[mL^2\dot{\theta} + 4mL^2\dot{\theta}\sin^2\theta\right] = mL^2\omega^2 \sin\theta\cos\theta + 4mL^2\dot{\theta}^2 \sin\theta\cos\theta - 3mgL\sin\theta \tag{10}$$

These equations work out to

$$mL^2\ddot{\theta} + 4mL^2\ddot{\theta}\sin^2\theta + 8mL^2\dot{\theta}^2\sin\theta\cos\theta = mL^2\omega^2\sin\theta\cos\theta 4mL^2\dot{\theta}^2\sin\theta\cos\theta - 3mgL\sin\theta \tag{11}$$

Or finally

$$\ddot{\theta}\left(1 + 4\sin^2\theta\right) = \left(\omega^2 - 4\dot{\theta}^2\right)\sin\theta\cos\theta - \frac{3g}{L}\sin\theta \tag{12}$$

(b) The system has a constant of motion

$$h = \frac{\partial L}{\partial \dot{q}}\dot{q} - L = \frac{1}{2}mL^2(1 + 4\sin^2\theta)\dot{\theta}^2 - \frac{mL^2\omega^2}{2}\sin^2\theta \tag{13}$$

From the initial conditions $h = \frac{1}{2}mL^2\Omega^2$. When $\theta = \pi/2$ we have

$$\frac{5}{2}mL^2\dot{\theta}^2 - \frac{1}{2}mL^2\omega^2 = \frac{1}{2}mL^2\Omega^2\,, \tag{14}$$

leading to

$$\dot{\theta} = \sqrt{\frac{\Omega^2+\omega^2}{5}}\,. \tag{15}$$

We note that if $\Omega = \omega/2$, then $\dot{\theta} = \Omega/2$, i.e. $\dot{\theta}$ is constant. This is also evident from the equation of motion.

(c) In the absence of gravity, $g = 0$. At equilibrium $\ddot{\theta} = 0$. Since the problem states that θ is fixed, $\dot{\theta} = 0$ as well. Thus, Eq. (12) yields $\sin\theta\cos\theta = 0$, which implies $\theta = 0, \pi/2$, or π. Only one of these, $\theta = \pi/2$, satisfies $0 < \theta < \pi$.

(d) We now return to Eq. (12) with $\ddot{\theta} = \dot{\theta} = 0$ but $g \neq 0$, yielding $\omega^2\cos\theta = \frac{3g}{L}$. This equation only has a solution when $\frac{3g}{L\omega^2} < 1$. Thus, the critical value of angular frequency is $\omega = \sqrt{3g/L}$. When ω is above this frequency, $\theta = \cos^{-1}\frac{3g}{L\omega^2}$. This is consistent with part (c) because when $g \to 0$, $\theta \to \cos^{-1} 0 = \pi/2$.

(e) For small oscillations, $\theta = \theta_c + \delta$, where $\theta_c = \cos^{-1}\frac{3g}{L\omega^2}$ (from part (d)) and $\delta \ll 1$. To find the oscillation frequency, we will need to plug back into Eq. (12). To do so, we first compute:

$$\sin\theta = \sin(\theta_c+\delta) = \sin\theta_c\cos\delta + \cos\theta_c\sin\delta \approx \sin\theta_c + \delta\cos\theta_c \tag{16}$$
$$\cos\theta = \cos(\theta_c+\delta) = \cos\theta_c\cos\delta - \sin\theta_c\sin\delta \approx \cos\theta_c - \delta\sin\theta_c \tag{17}$$

Thus, to linear order in δ,

$$\sin\theta\cos\theta \approx \sin\theta_c\cos\theta_c + \delta(\cos^2\theta_c - \sin^2\theta_c) \tag{18}$$

and

$$\sin^2\theta \approx \sin^2\theta_c + 2\delta\cos\theta_c\sin\theta_c \tag{19}$$

Plugging into Eq. (12) and only keeping terms linear in δ yields:

$$\ddot{\delta}\left(1+4\sin^2\theta_c\right) = \omega^2\left[\sin\theta_c\cos\theta_c + \delta(\cos^2\theta_c - \sin^2\theta_c)\right] - \frac{3g}{L}\left[\sin\theta_c + \delta\cos\theta_c\right] \tag{20}$$

Plugging in $\cos\theta_c = \frac{3g}{L\omega^2}$ yields:

$$\ddot{\delta}\left(1+4\sin^2\theta_c\right) = -\omega^2\delta\sin^2\theta_c \tag{21}$$

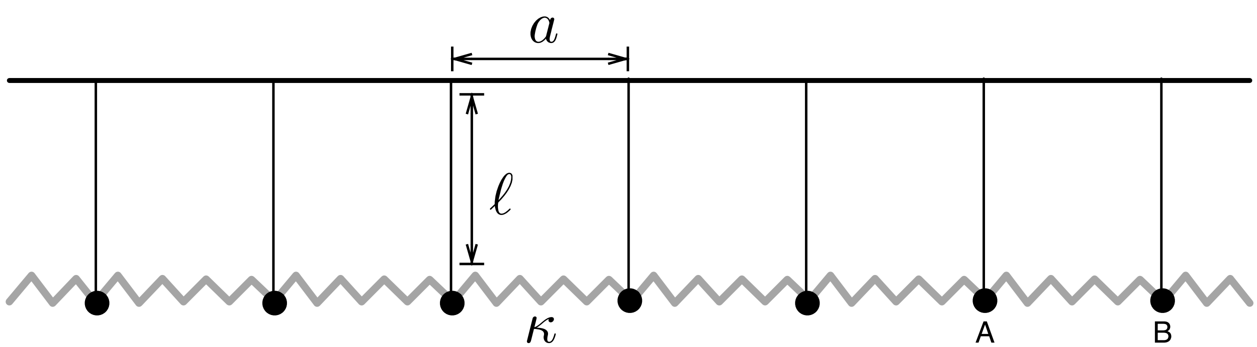
Thus, δ oscillates with frequency

$$\omega\sqrt{\frac{\sin^2\theta_c}{1+4\sin^2\theta_c}} = \omega\sqrt{\frac{1-\frac{9g^2}{L^2\omega^4}}{5-\frac{36g^2}{L^2\omega^4}}} \tag{22}$$

Classical Mechanics 3

A coupled chain of pendulums

Consider an infinite chain of coupled pendulums in the earth's gravitational field. The pendulums are separated by a distance a, and consist of massless rods of length ℓ which may pivot freely from their suspension points (see figure). The masses at the ends of the pendulums have mass m and are connected by springs of spring constant κ, which are unstretched when the system is at rest. All rods and springs may be considered massless.



(a) (5 points) Write down the Lagrangian of the system for small angular oscillations and find the equations of motion.

(b) (6 points) Determine the oscillation frequency $\omega(k)$ for eigenmodes of wavenumber k. Determine the group velocity for $ka \ll 1$, and sketch the result versus k.

(c) (5 points) Consider the continuum action

$$S[q(t,x)] = \int dt dx \, \frac{1}{2}\mu(\partial_t q(t,x))^2 - \frac{1}{2}Y(\partial_x q(t,x))^2 - \frac{1}{2}\gamma^2 q^2(t,x)\,.$$

where μ, Y, and γ are constants:

(i) Determine the equations of motion.

(ii) Find the dispersion curve $\omega(k)$ for the plane wave solutions $Ae^{ikx - i\omega(k)t}$ to the continuum equations of part (c). What should the continuum parameters μ, Y, and γ be to reproduce the discrete results of part (b) at small k.

(d) (4 points) Now return to the discrete setup of parts (a) and (b). Consider a rightmoving plane wave of wavenumber k and amplitude $\mathcal{A}_0$ (i.e. an eigenmode). For this wave, compute the time averaged work done per unit time by pendulum A on pendulum B (see figure).

Solution

(a) The equilibrium position of the j-th oscillator is $(x_j, y_j) = (ja, 0)$. The angles all fluctuating by small amounts. The change in postitions

$$\delta x_j = \ell\, \theta_j \tag{1}$$

$$\delta y_j = \frac{1}{2}\ell\, \theta_j^2 \tag{2}$$

The Lagrangian is

$$L = \sum_j \frac{1}{2}m\left(\frac{\delta x_j}{dt}\right)^2 - mg\delta y_j - \frac{1}{2}\kappa(\delta x_j - \delta x_{j-1})^2 \tag{3}$$

This expands to

$$L = \sum_j \frac{1}{2}m\ell^2\dot{\theta}_j^2 - \frac{1}{2}mg\ell\theta_j^2 - \frac{1}{2}\kappa\ell^2(\theta_j - \theta_{j-1})^2 \tag{4}$$

Writing out the equation of motion we find

$$m\ell^2\ddot{\theta}_j = -mg\ell\theta_j - \kappa\ell^2(\theta_j - \theta_{j-1}) + \kappa\ell^2(\theta_{j+1} - \theta_j) \tag{5}$$

Dividing by $m\ell^2$ we find

$$\ddot{\theta}_j = -\Omega^2\theta_j + \omega_0^2\left(\theta_{j+1} - 2\theta_j + \theta_j\right) \tag{6}$$

where $\Omega^2 = g/\ell$ and $\omega_0^2 = \kappa/m$.

(b) Now we substitute $\theta_j = Ae^{ikx_j - i\omega t}$ into Eq. (6) . Note that

$$\theta_{j+1} = Ae^{ik(x_j+a)-i\omega t} = e^{ika}\, Ae^{i(kx_j - i\omega t)}\,. \tag{7}$$

Thus minor manipulations lead to

$$-\,\omega^2 = -\Omega^2 + \omega_0^2(e^{ika} - 2 + e^{-ika})\,. \tag{8}$$

And so, using $4\sin^2(ka/2) = 2 - 2\cos(ka)$, we find that

$$\omega^2 = \Omega^2 + 4\omega_0^2\sin^2(ka/2)\,. \tag{9}$$

For small k we find

$$\omega(k) = \pm\sqrt{\Omega^2 + v_0^2k^2}\,, \tag{10}$$

where $v_0 \equiv \omega_0 a$. The group velocity is

$$\frac{d\omega}{dk} = \pm\frac{v_0^2k}{\sqrt{\Omega^2 + v_0^2k^2}} \tag{11}$$

This is the dispersion curve of massive relativistic particle of mass m and momentum p if one identifies $\Omega = (mc^2)$, $v_0 = c$, and $p = k$.

(i) From the Euler Lagrange equations

$$-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \right) + \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{12}$$

We find

$$\partial_t(\mu\partial_t q) - \partial_x(Y\partial_x q) + \gamma^2 q = 0\,. \tag{13}$$

(ii) Substiting the ansatz $Ae^{ikx-i\omega t}$ we find

$$-\mu\omega^2 + Yk^2 + \gamma^2 q^2 = 0 \tag{14}$$

and the dispersion curve is

$$\omega = \pm\sqrt{\gamma^2 + \frac{Y}{\mu}k^2} \tag{15}$$

So we want to take

$$\frac{Y}{\mu} \Rightarrow v_0^2 = \frac{\kappa a^2}{m}\,, \tag{16}$$

$$\gamma^2 \Rightarrow \Omega^2 = \frac{g}{\ell}\,, \tag{17}$$

in order that the dispersion curves match. Finally one would (if needed) set

$$\mu \Rightarrow \frac{m\ell^2}{a}\,, \tag{18}$$

so that

$$\int dx\mu\,(\partial_t q)^2 \simeq \sum_j m\ell^2\,(\partial_t\theta_j)^2\,. \tag{19}$$

(c) From the equation of motion the torque on the j-th oscillator by the $j-1$-th oscillator is

$$\tau = -\kappa\ell^2(\theta_j - \theta_{j-1}) \tag{20}$$

The power is

$$\frac{dW}{dt} = \tau\dot{\theta}_j \tag{21}$$

Inserting the eigen functions we have

$$\tau = -\kappa\ell^2 A_0 e^{-i\omega t+ikx_j}\left(1 - e^{-ika}\right) \tag{22}$$

$$\dot{\theta} = A_0 e^{-i\omega t+ikx_j}(-i\omega) \tag{23}$$

Using that

$$\overline{A(t)B(t)} = \frac{1}{2}\mathrm{Re}\left[A_\omega B_\omega^*\right] \tag{24}$$

So we find

$$
\begin{aligned}
\overline{\dot{W}} =& -\tfrac{1}{2}\kappa\ell^2|A_0|^2\omega \mathrm{Re}\left[(1-e^{-ika})i\right] && (25)\\
=&\tfrac{1}{2}\kappa\ell^2|A_0|^2\omega\sin(ka) && (26)\\
=&\tfrac{1}{2}m\omega_0^2\ell^2|A_0|^2\omega\sin(ka) && (27)
\end{aligned}
$$

Discussion:

The work done per time is the energy density times the group velocity. To show this compute the mean energy density in the eigen mode (the energy in a site divided by a):

$$
\begin{aligned}
\mathcal{E} =& \frac{1}{a}\left\langle \frac{1}{2}m\ell^2\dot{\theta}_j^2 + \frac{1}{2}m\Omega\theta_j^2 + \frac{1}{2}m\omega_0^2(\theta_j-\theta_{j-1})^2 \right\rangle && (28)\\
=&\frac{1}{4a}m\ell^2\left(\omega^2+\Omega^2+4\omega_0^2\sin^2(ka/2)\right) && (29)\\
=&\frac{1}{2a}m\ell^2\omega^2 && (30)
\end{aligned}
$$

The group velocity is

$$
\frac{d\omega}{dk} = \frac{d}{dk}\sqrt{\Omega^2+4\omega_0^2\sin^2(ka/2)} = \frac{a\omega_0^2\sin(ka)}{\omega} \tag{31}
$$

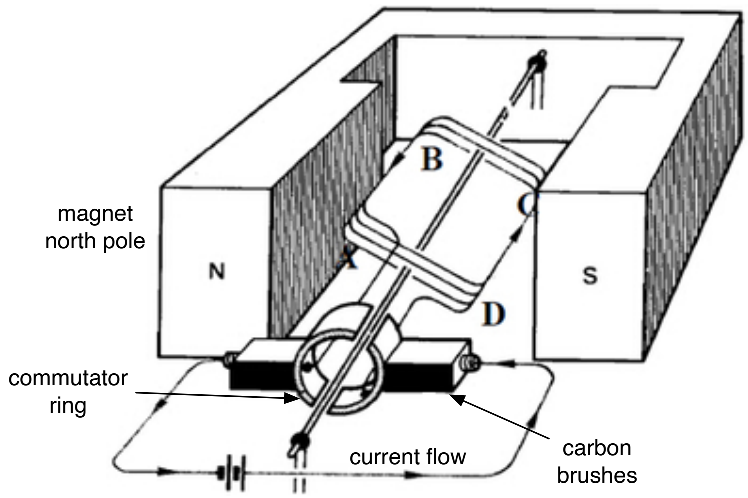
Multiplying these two we see as claimed that

$$
\mathcal{E}\left(\frac{d\omega}{dk}\right) = \overline{\dot{W}} \tag{32}
$$

Electromagnetism 1

A primitive motor

A primitive motor is constructed with a battery of voltage V, a uniform magnetic field of magnitude B, and insulated wire of resistance R. The coil has negligible mass, consists of n turns and the face of the coil is approximately square with area $A = L^2$. The commutator ring rotates with the coil, and, at the instant illustrated below, the current in the loop flows along the path DCBA.



(a) (6 points) Clearly explain how the motor works addressing the following:

(i) the direction of rotation,

(ii) the role of the carbon brushes and the split commutator ring.

By means of external regulating torques, the motor is constrained to rotate with constant operating angular velocity ω.

(b) (2 points) What is the maximum instantaneous torque the motor can deliver?

(c) (6 points) If the power delivered by the motor is positive throughout its cycle, determine the coil's maximum angular velocity, ω_{max}. Include the "back emf" induced by the rotation of the coil.

(i) Estimate ω_{max} in Hz for a centimeter sized device with 100 turns, a typical battery voltage, and a typical magnetic field.

(d) At what operating frequency ω is the time averaged power delivered by the motor a maximum.

When the motor is operated at this frequency, determine the (time-average) power dissipated as heat, the (time averaged) power delivered as mechanical work, and (time averaged) the power delivered by the battery – interpret your results using energy considerations. What is the efficiency of the engine operating at this frequency?

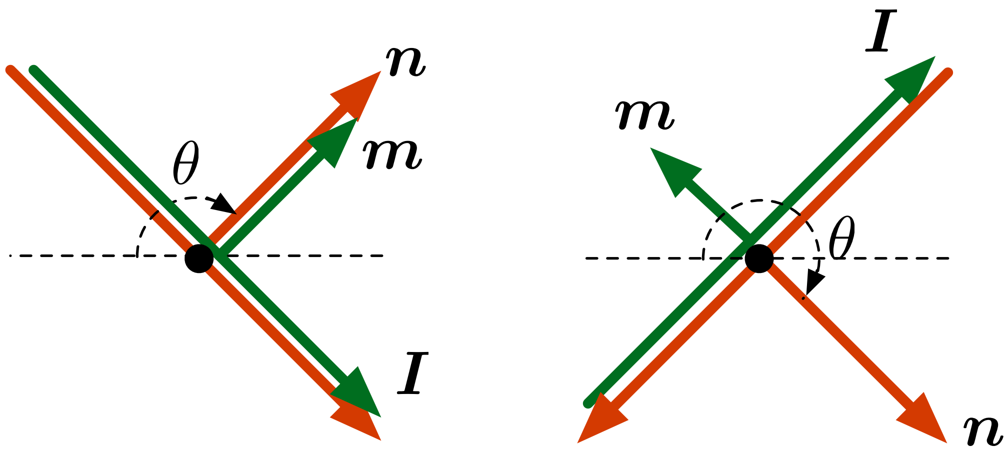


Figure 1: Schematic of the motor as viewed from the front as illustrated in the problem statement. The plane of the coil and its normal $\boldsymbol{n}$ are illustrated by the red arrows. The red arrows define what is positive circulation (viewed from the front). The green arrows indicate the current flow and the associated magnetic moment $\boldsymbol{m}$. The current flow is positive for $\theta \in [0, \pi]$ and negative for $\theta \in [\pi, 2\pi]$.

Solution

(a) A motor works since magnetic forces try to align the magnetic moment of the current loop with the magnetic field. The red arrows in the figure define the geometric orientation of the loop when viewed from the front. The green arrows indicate the direction of the current and the magnetic moment. Thus the torque on the loop is clockwise in figure (a), with the current as drawn. When the angle θ becomes greater than π the current switches directions, due to the split ring. This mechanical construction alternates the positive terminal of the battery between the two ends of the coil (see picture in picture – the commutator ring rotates with the coil). For θ greater π (figure (b)) the green magnetic moment still will want to align with the magnetic field, leading to a continuously clockwise torque.

Our conventions are standardized in the figure. The DC voltage in the loop is

$$V(\theta) = V_0 \, \mathrm{sign}(\sin(\theta)) \tag{1}$$

which is a clever way of writing

$$V(\theta) = V_0 \, \mathrm{sign}(\sin(\theta)) \tag{2}$$

(b) We assume that $\theta \in [0, \pi]$. The torque is $\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B}$ and where $\boldsymbol{m} = IA/c\boldsymbol{n}$. The magnitude of the torque is

$$\tau = \frac{IAB \sin\theta}{c} \tag{3}$$

and is clockwise (into the page as drawn in Fig. 1). The current is V/R (since the emf) is neglected and $\sin\theta$ is less than unit leading to

$$\tau = \frac{VAB\sin\theta}{cR} \tag{4}$$

(c) There are two emfs in the loop, from the battery and the induced emf.

The magnetic flux through the loop is (see Fig. 1(a))

$$\Phi_B = -AB\cos(\theta) \tag{5}$$

and the induced EMF is

$$\mathcal{E}_{\rm ind} = -\frac{1}{c}\partial_t \Phi_B = -\frac{AB\omega}{c}\sin\theta \tag{6}$$

Taking $\theta \in [0, \pi]$ we have

$$\mathcal{E} = V + \mathcal{E}_{\rm ind} = V - \frac{AB\omega}{c}\sin\theta \tag{7}$$

So the current in the loop is

$$I = \frac{1}{R}\left(V - AB(\omega/c)\sin\theta\right) \tag{8}$$

This needs to be positive for the motor to deliver positive torque and hence positive power. Thus the the maximum frequency

$$\omega_{\rm max} = \frac{cV}{AB} \tag{9}$$

The torque is zero at $\omega = \omega_{\rm max}$. To make an estimate we switch to SI units, setting $V \sim 1\,\mathrm{V}$ and $B \sim 1\,\mathrm{mT}$. the area is $A \sim 100\,\mathrm{cm}^2$ with $n = 100$ the number of turns. We estimate

$$\omega \sim 100\,s^{-1}\left(\frac{1Volt}{V}\right)\left(\frac{0.01T}{B}\right)\left(\frac{10^4 cm^2}{nA}\right) \tag{10}$$

(d) The work done per time is

$$P_{\rm mech} = \tau\omega = \frac{IAB\sin\theta}{c}\omega = \frac{1}{R}\left[V - AB(\omega/c)\sin\theta\right]AB\sin\theta(\omega/c) \tag{11}$$

$$= \frac{V^2}{R}\left[1 - u\sin(\theta)\right]u\sin\theta \tag{12}$$

wherew $u = \omega/\omega_{\rm max}$. Averaging over a half cylce

$$\langle\sin\theta\rangle = \frac{1}{\pi}\int d\theta\sin\theta = \frac{2}{\pi} = c_1 \tag{13}$$

$$\langle\sin^2\theta\rangle = \frac{1}{\pi}\int d\theta\sin^2\theta = \frac{1}{2} \tag{14}$$

So

$$\langle P_{\rm mech}\rangle = \frac{V^2}{R}u(c_1 - \frac{1}{2}u) \tag{15}$$

where $u = \omega/\omega_{\text{max}}$. The maximum of this expression instantaneous when

$$u_{\text{max}} = c_1 \tag{16}$$

yielding the maximum mechanical work

$$\langle P_{\text{mech}} \rangle = \frac{V^2}{R} \frac{c_1^2}{2} \tag{17}$$

For later use, We find that the current instantaneous current is

$$I = \frac{V}{R}(1 - u \sin\theta) = \frac{V}{R}(1 - c_1 \sin\theta) \tag{18}$$

The instantaneous takes the form

$$P_{\text{mech}} = I(AB(\omega/c)\sin\theta) = I(V - IR) = P_{\text{batt}} - P_{\text{ohm}} \tag{19}$$

which shows that the power delivered by the battery P_{batt} is a sum of the useful mechanical power P_{mech} and the rate of ohmic dissipation.

We have

$$\langle P_{\text{batt}} \rangle_\theta = \langle I \rangle_\theta V = \frac{V^2}{R}(1 - c_1^2) \tag{20}$$

The work dissipated as heat is

$$\langle P_{\text{ohm}} \rangle_\theta = \left\langle I^2 \right\rangle_\theta R = \frac{V^2}{R} \left\langle (1 - c_1 \sin\theta)^2 \right\rangle_\theta = \frac{V^2}{R} \left(1 - \frac{3}{2} c_1^2\right) \tag{21}$$

So we have

$$P_{\text{batt}} - P_{\text{ohm}} = \frac{V^2}{R} \frac{c_1^2}{2} \tag{22}$$

in agreement with eq. 17

Electromagnetism 2

Rotating sphere in a magnetic field

A uniform conducting sphere of radius a, with electric and magnetic permeabilities $\epsilon = \mu = 1$ and conductivity σ, rotates with constant angular velocity ω around the z-axis. A uniform magnetic field of magnitude B is applied along the axis of rotation. The initial charge on the sphere is zero. Ignoring the magnetic field due to the rotating sphere, evaluate in the steady state:

(a) (4 points) The electric field in the sphere.

Hint: What is the Lorentz force in the sphere?

(b) (4 points) The volume charge density inside the sphere

(c) (8 points) The electric potential and field everywhere

(d) (4 points) The charge density on the surface of the sphere.

Legendre Polynomials

$P_0(x)$	1
$P_1(x)$	x
$P_2(x)$	$\frac{1}{2}(3x^2 - 1)$
$P_3(x)$	$\frac{1}{2}(5x^3 - 3x)$

Solution

a. The rotating interior charges are subject to a magnetic force $e(\vec{v}\times\vec{B})$ with $c=1$. In equilibrium, the free charges redistribute themselves in bulk, to generate an electric force $e\vec{E}$ to balance locally the magnetic force.

Hence, in the interiorr with $r<a$:

$$\vec{E}=-\vec{v}\times\vec{B}=-(\omega\hat{z}\times\vec{r})\times B\hat{z}=-\omega B(\vec{r}-(\hat{z}\cdot\vec{r})-\vec{r})=-\omega B\vec{r}_\perp \tag{1}$$

For $r>a$ we will work it out directly from the potential below (see Eq. 10).

b. The volume charge distribution in the interior of the sphere follows from Gauss law using (1). In spherical coordinates

$$\vec{r}_\perp=(r\sin\theta)(\hat{r}\sin\theta+\hat{\theta}\cos\theta) \tag{2}$$

and the charge density is then

$$\rho=\frac{1}{4\pi}\vec{\nabla}\cdot\vec{E}=-\frac{\omega B}{4\pi}\left(\frac{1}{r^2}\frac{\partial r^3\sin\theta}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial r\sin^2\theta\cos\theta}{\partial\theta}\right)=-\frac{\omega B}{2\pi} \tag{3}$$

The associated total charge is then

$$Q=\rho V_3=-\frac{2}{3}\omega Ba^3 \tag{4}$$

c. The potential inside follows from $\vec{E}=-\vec{\nabla}\varphi$,

$$\vec{E}=-\omega Br(\hat{r}\sin^2\theta+\hat{\theta}\sin\theta\cos\theta)=-\hat{r}\frac{\partial\varphi}{\partial r}-\frac{\hat{\theta}}{r}\frac{\partial\varphi}{\partial\theta} \tag{5}$$

in spherical coordinates for $r<a$, hence

$$\varphi(r<a)=\varphi(0)+\omega Br^2\sin^2\theta=\varphi(0)+\frac{\omega Br^2}{3}(1-P_2(\cos\theta)) \tag{6}$$

In the rightmost result we used the Legendre polynomials

$$P_0=1\qquad P_2=\frac{1}{2}(3\cos^2\theta-1) \tag{7}$$

For $r>a$, the potential for the extended and spherical but uniform charge distribution, can be extended in Legendre polynomials

$$\varphi(r>a)=\sum_{n=0}^{\infty}C_n\frac{a^n}{r^{n+1}}P_n(\cos\theta) \tag{8}$$

Continuity at $r = a$ suggests that (6) matches (8), only if the $n = 0, 2$ multipoles contribute in (8), hence

$$\varphi(r > a) = \frac{a}{r}\varphi(0) + \frac{\omega B a^2}{3}\left(\frac{a}{r} - \frac{a^3}{r^3}P_2(\cos\theta)\right) \tag{9}$$

The electric field outside is readily found from $\vec{E} = -\vec{\nabla}\varphi$ or

$$\begin{aligned} E_r(r > a) &= \frac{a}{r^2}\varphi(0) + \frac{\omega B a^2}{3}\left(\frac{a}{r^2} - \frac{3a^3}{r^4}P_2(\cos\theta)\right) \\ E_\theta(r > a) &= -\frac{\omega B a^5}{r^4}\sin\theta\cos\theta \end{aligned} \tag{10}$$

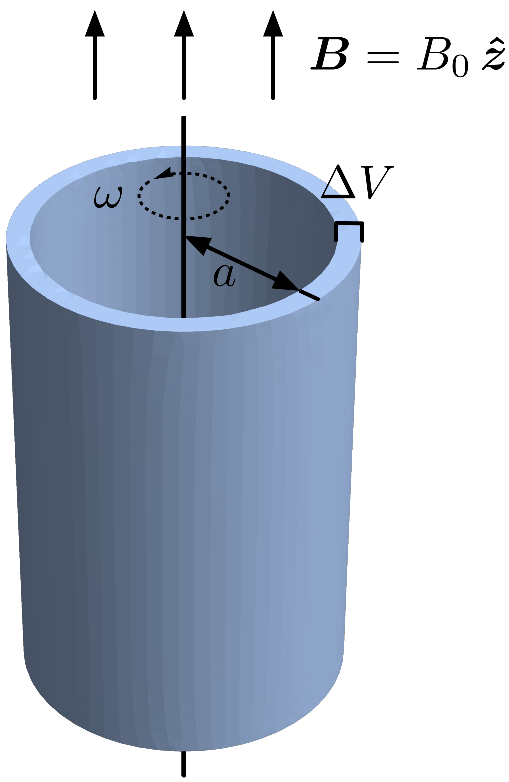
d. The surface charge density follows from Gauss flux law near the surface of the sphere

$$4\pi\sigma = (E_r(a + 0) - E(a - 0)) = \frac{\varphi(0)}{a} + \frac{\omega B a}{3}(3 - 5P_2(\cos\theta)) \tag{11}$$

Electromagnetism 3

Potential difference across a rotating cylinder

A neutral thin dielectric cylindrical shell of inner radius a and thickness δ (with $\delta \ll a$) rotates non-relativistically with constant angular velocity ω with $\omega a/c \ll 1$ (see below). The cylindrical shell sits in a constant homogeneous magnetic field directed along the z axis, $\boldsymbol{B} = B_o\,\hat{\boldsymbol{z}}$ (see below). A small potential difference of $\Delta V = V_{\rm out} - V_{\rm in}$ is observed between the outside and inside of the cylindrical shell as shown below. The cylinder has dielectric constant $\epsilon = 1 + \chi$ with $\chi \ll 1$.



(a) (6 points) Recall that the vector potential of a constant magnetic field is $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$. By making a Lorentz transformation of the gauge potential A^μ (in the Lorentz gauge), determine $\underline{A}^\mu(\underline{x})$ in the co-rotating frame of the cylinder. Compute the electric field in this frame by differentiating the vector potential $\underline{A}^\mu$.

Hint: Pick a specific point on the moving wall of the cylinder, and boost to the frame co-moving with that point. Work to leading order in $\omega a/c$ throughout.

(b) (4 points) By making a Lorentz transformation of $\boldsymbol{B}$ determine the electromagnetic fields in the co-rotating frame. Check that your electric field consistent with that of part (a).

(c) (4 points) In the co-rotating frame determine the charge density per area on the surfaces of the cylinder. Now find the corresponding surface charge density σ in the lab frame. Illustrate your result in the lab frame with a sketch.

(d) (4 points) Determine the potential difference $\Delta V = V_{\text{out}} - V_{\text{in}}$ in the lab frame. Indicate the direction of the (weak) electric field in the lab frame by making a sketch.

Solution

1. The gauge potential in the lab frame in a covariant guage is

$$A^\mu = (\phi, A^x, A^y, A^z) = (0, -B_0 y/2, B_0 x/2, 0)\,. \tag{1}$$

We will pick a point $\boldsymbol{r} = a\hat{\boldsymbol{x}}$ where y is nearly zero. So, the wall of the cylinder is moving in the y direction with velocity $\boldsymbol{v} = \omega a\,\hat{\boldsymbol{y}}$. The component parallel to the boost is y, while the transverse directions are x and z, and are unaffected by the boost. The required boost to take us from the lab to the comoving frame is

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \simeq \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}\,. \tag{2}$$

where $\beta = (\omega a/c)$. We have

$$\begin{pmatrix} \underline{A}^0 \\ \underline{A}^y \end{pmatrix} = \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ (B_0 x/2) \end{pmatrix} \tag{3}$$

So in the transformed frame we have

$$\underline{A}^\mu = [-(\omega a/c)(B_0 x/2), (B_0 y/2), (B_0 x/2)] \tag{4}$$

To complete the transformation, we should express the fields in terms of the new coordinates

$$\underline{t} = t - (v/c) y\,, \tag{5}$$

$$\underline{y} = -(v/c) t + y\,, \tag{6}$$

while the z and x coordinates are unchanged. Thus

$$y = (v/c) t + \underline{y} \simeq (v/c)\underline{t} + \underline{y} \tag{7}$$

leading ultimately to

$$\underline{A}^\mu = \tfrac{1}{2} B_0 \left[-\beta \underline{x}, -\beta \underline{t} - \underline{y}, \underline{x}, 0 \right]\,. \tag{8}$$

In components we have

$$\underline{\phi} = -\tfrac{1}{2} B_0 \underline{x}\,, \qquad \underline{\boldsymbol{A}} = \tfrac{1}{2} B_0 (-\beta \underline{t} - \underline{y}, \underline{x}, 0)\,. \tag{9}$$

The electic field is computed directly

$$\underline{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial}{\partial \underline{t}} \underline{\boldsymbol{A}} - \underline{\nabla}(\underline{\phi})\,, \tag{10}$$

$$= \tfrac{1}{2}\beta B_0\, \underline{\hat{\boldsymbol{x}}} + \tfrac{1}{2}\beta B_0\, \underline{\hat{\boldsymbol{x}}}\,, \tag{11}$$

$$= \beta B_0\, \underline{\hat{\boldsymbol{x}}}\,. \tag{12}$$

The first factor of $\frac{1}{2} B_0\, \underline{\hat{\boldsymbol{x}}}$ comes from differentiation $\underline{\boldsymbol{A}}$ with respect to time in that frame. The magnetic field is unchanged by the boost, can be seen by computing $\underline{\nabla} \times \underline{\boldsymbol{A}}$.

2. Next we make a Lorentz transformation of the field itself. The parallel components of the field to the boost direction do not transform

$$\underline{E}_{\parallel} = E_{\parallel}\,, \tag{13}$$

$$\underline{B}_{\parallel} = B_{\parallel}\,, \tag{14}$$

while the transverse components transform in a way that is reminiscent of the transformation of coordinates

$$\underline{\boldsymbol{E}}_{\perp} = \gamma \boldsymbol{E}_{\perp} + \gamma \boldsymbol{\beta} \times \boldsymbol{B}_{\perp}\,, \tag{15}$$

$$\underline{\boldsymbol{B}}_{\perp} = -\gamma \boldsymbol{\beta} \times \boldsymbol{E}_{\perp} + \gamma \boldsymbol{B}_{\perp}\,. \tag{16}$$

For small boosts and no electric field we find

$$\underline{\boldsymbol{E}} = \boldsymbol{\beta} \times \boldsymbol{B}\,, \qquad \underline{\boldsymbol{B}} = \boldsymbol{B}\,. \tag{17}$$

With $\boldsymbol{\beta} = \beta \hat{\boldsymbol{y}}$ and $\boldsymbol{B} = B_0 \hat{\boldsymbol{z}}$ we have

$$\underline{\boldsymbol{E}} = B_0 \beta\, \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = B_0 \beta\, \underline{\hat{\boldsymbol{x}}}\,, \tag{18}$$

$$\underline{\boldsymbol{B}} = B_0 \underline{\hat{\boldsymbol{z}}}\,. \tag{19}$$

The electric field agrees with the previous item.

3. In the rotating frame the electric field polarizes the dielectric

$$\underline{\boldsymbol{P}} = \chi \underline{\boldsymbol{E}}\,. \tag{20}$$

The jump in the polarization determines the surface charge:

$$-\boldsymbol{n} \cdot (\underline{\boldsymbol{P}}_{\rm out} - \underline{\boldsymbol{P}}_{\rm in}) = \sigma\,. \tag{21}$$

So on the outer surface of the cylinder ($\underline{\boldsymbol{P}}_{\rm out} = 0$) we find

$$\underline{\sigma}_{\rm out} = \chi B_0 \beta\,, \tag{22}$$

while on the inner surface we have the opposite sign, since the outward directed normal is in the negative $\hat{\boldsymbol{x}}$ direction

$$\underline{\sigma}_{\rm in} = -\chi B_0 \beta\,. \tag{23}$$

From the current transformation rule for J^{μ}

$$\begin{pmatrix} c\rho \\ j_y \end{pmatrix} = \begin{pmatrix} 1 & (v/c) \\ (v/c) & 1 \end{pmatrix} \begin{pmatrix} c\underline{\rho} \\ \underline{j}_y \end{pmatrix}\,, \tag{24}$$

we see that to lowest order (since there is no current in the co-rotating frame) $\rho \simeq \underline{\rho}$. A sketch of the charge distribution is shown in the figure Fig. 1.

4. Given the charge distribution it is straightforward to find the potential change. The electric field inside the cylinder is just that of a parallel plate capacitor with surface charge density $\sigma = \chi B_0 \beta$, i.e. $E = \chi B_0 \beta$. The direction of the electric field in the lab frame directed towards the center. The voltage change is positive

$$\Delta V = V_{\rm out} - V_{\rm in} = -\int_{\rm in}^{\rm out} \boldsymbol{E} \cdot {\rm d}\boldsymbol{\ell} = \chi B_0 (\omega_0 a/c) \delta\,. \tag{25}$$

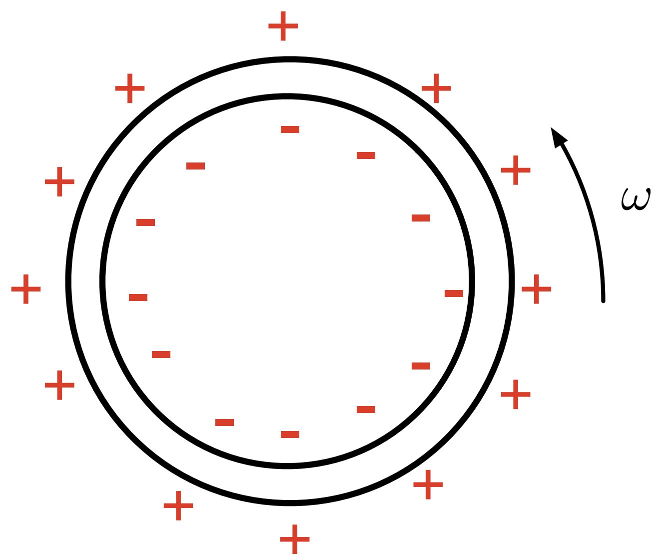


Figure 1: Distribution of charges in the rotating cylinder. The induced electric field in the dielectric is inward directed. The potential difference $\Delta V = V_{\text{out}} - V_{\text{in}}$ is positive.

Quantum Mechanics 1

Time-reversal symmetry and the Kramers degeneracy

Wigner proved that a symmetry in quantum mechanics is implemented by either a unitary or an anti-unitary transformation. In this problem we will discuss anti-unitary transformations and in particular the time-reversal symmetry.

The transformation

$$|\alpha\rangle \to |\tilde{\alpha}\rangle = \Theta|\alpha\rangle\,, \qquad |\beta\rangle \to |\tilde{\beta}\rangle = \Theta|\beta\rangle\,, \tag{1}$$

is said to be **anti-unitary** if

$$\begin{aligned} &\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*\,,\\ &\Theta\,(\,c_1|\alpha\rangle + c_2|\beta\rangle\,) = c_1^*\Theta|\alpha\rangle + c_2^*\Theta|\beta\rangle\,, \end{aligned} \tag{2}$$

where c^* denotes the complex conjugation of a complex number c. In such a case the operator Θ is an anti-unitary operator.

(a) (5pts) Show that $\langle\beta|\mathcal{O}|\alpha\rangle = \langle\tilde{\alpha}|\Theta\mathcal{O}^\dagger\Theta^{-1}|\tilde{\beta}\rangle$ for an arbitrary linear operator $\mathcal{O}$.

(b) (3pts) We can always choose a convention such that $\Theta|x\rangle = |x\rangle$, where $|x\rangle$ is the position basis. Let $\langle x|\alpha\rangle$ be the wavefunciton of the state $|\alpha\rangle$, i.e.,

$$|\alpha\rangle = \int dx|x\rangle\langle x|\alpha\rangle\,. \tag{3}$$

What is the wavefunction for the state $\Theta|\alpha\rangle$?

(c) (1pt) Let $|\ell, m\rangle$ be the orbital angular momentum eigenstate. More specifically, the eigenvalues of L^2 and L_z are $\ell(\ell+1)\hbar$ and $m\hbar$, respectively. As in the previous problem, we choose the convention that $\Theta|\theta,\phi\rangle = |\theta,\phi\rangle$, where $|\theta,\phi\rangle$ is the angular coordinate basis such that $Y_\ell^m(\theta,\phi) = \langle\theta,\phi|\ell,m\rangle$. What is $\Theta|\ell,m\rangle$? Hint: The complex conjugation of the spherical harmonics is $(Y_\ell^m(\theta,\phi))^* = (-1)^m Y_\ell^{-m}(\theta,\phi)$.

(d) (4pts) The defining property of a time-reversal operator T in quantum mechanics is that it changes the sign of the time coordinate t upon conjugation. More precisely, a Hamiltonian is said to have a time-reversal symmetry T if

$$\mathsf{T}\exp(iHt)\mathsf{T}^{-1}|\psi\rangle = \exp(-iHt)|\psi\rangle\,, \tag{4}$$

on any state $|\psi\rangle$. Here H is the Hamiltonian. Compute $\mathsf{T}H\mathsf{T}^{-1}$.

(e) (2pt) Suppose $|E\rangle$ is an eigenstate of the Hamiltonian with energy eigenvalue E. What is the energy eigenvalue of $\mathsf{T}|E\rangle$?

(f) (5pts) Suppose the time-reversal operator T obeys the following algebra in a subsector $\mathcal{H}'$ of the Hilbert space:

$$\mathsf{T}^2|\psi\rangle = -|\psi\rangle\,, \quad \forall|\psi\rangle \in \mathcal{H}'\,. \tag{5}$$

(For example, this would be the case on the subsector of states with half-integer ℓ in problem (c).) Prove that all energy eigenstates in $\mathcal{H}'$ are degenerate.

(a) (5pts) Show that $\langle\beta|\mathcal{O}|\alpha\rangle = \langle\tilde{\alpha}|\Theta\mathcal{O}^\dagger\Theta^{-1}|\tilde{\beta}\rangle$ for an arbitrary linear operator $\mathcal{O}$.

Solution: Define $|\gamma\rangle \equiv \mathcal{O}^\dagger|\beta\rangle$. Then the bra state is $\langle\beta|\mathcal{O} = \langle\gamma|$. Hence

$$\begin{aligned}\langle\beta\mathcal{O}\alpha\rangle &= \langle\gamma|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\gamma}\rangle \\ &= \langle\tilde{\alpha}|\Theta\mathcal{O}^\dagger|\beta\rangle = \langle\tilde{\alpha}|\Theta\mathcal{O}^\dagger\Theta^{-1}\Theta|\beta\rangle = \langle\tilde{\alpha}|\Theta\mathcal{O}^\dagger\Theta^{-1}|\tilde{\beta}\rangle\,.\end{aligned} \tag{6}$$

(b) (3pts) We can always choose a convention such that $\Theta|x\rangle = |x\rangle$, where $|x\rangle$ is the position basis. Let $\langle x|\alpha\rangle$ be the wavefunciton of the state $|\alpha\rangle$, i.e.,

$$|\alpha\rangle = \int dx|x\rangle\langle x|\alpha\rangle\,. \tag{7}$$

What is the wavefunction for the state $\Theta|\alpha\rangle$?

Solution:

$$\Theta|\alpha\rangle = \Theta\left(\int dx|x\rangle\langle x|\alpha\rangle\right) = \int dx\Theta|x\rangle\langle x|\alpha\rangle^* = \int dx|x\rangle\langle x|\alpha\rangle^* \tag{8}$$

where we have used (2) in the second equality. Therefore, the wavefunction for $\Theta|\alpha\rangle$ is the complex conjugation of that of $|\alpha\rangle$, in the convention that $\Theta|x\rangle = |x\rangle$.

(c) (1pt) Let $|\ell, m\rangle$ be the orbital angular momentum eigenstate. More specifically, the eigenvalues of L^2 and L_z are $\ell(\ell+1)\hbar$ and $m\hbar$, respectively. As in the previous problem, we choose the convention that $\Theta|\theta,\phi\rangle = |\theta,\phi\rangle$, where $|\theta,\phi\rangle$ is the angular coordinate basis such that $Y_\ell^m(\theta,\phi) = \langle\theta,\phi|\ell,m\rangle$. What is $\Theta|\ell,m\rangle$? Hint: The complex conjugation of the spherical harmonics is $(Y_\ell^m(\theta,\phi))^* = (-1)^m Y_\ell^{-m}(\theta,\phi)$.

Solution: Since Θ acts on the wavefunction by complex conjugation and $(Y_\ell^m(\theta,\phi))^* = (-1)^m Y_\ell^{-m}(\theta,\phi)$, we immediately have

$$\Theta|\ell,m\rangle = (-1)^m|\ell,-m\rangle\,. \tag{9}$$

(d) (4pts) The defining property of a time-reversal operator T in quantum mechanics is that it changes the sign of the time coordinate t upon conjugation. More precisely, a Hamiltonian is said to have a time-reversal symmetry T if

$$\mathsf{T}\exp(iHt)\mathsf{T}^{-1}|\psi\rangle = \exp(-iHt)|\psi\rangle\,, \tag{10}$$

on any state $|\psi\rangle$. Here H is the Hamiltonian. Compute $\mathsf{T}H\mathsf{T}^{-1}$.

Solution: Assuming t is infinitesimal, we have

$$\mathsf{T}iHt\mathsf{T}^{-1}|\psi\rangle = (-iHt)|\psi\rangle\,. \tag{11}$$

Since T is anti-unitary, we have $\mathsf{T}iHt\mathsf{T}^{-1} = -i\mathsf{T}Ht\mathsf{T}^{-1}$. Therefore,

$$\mathsf{T}H\mathsf{T}^{-1} = H\,. \tag{12}$$

(e) (2pt) Suppose $|E\rangle$ is an eigenstate of the Hamiltonian with energy eigenvalue E. What is the energy eigenvalue of $\mathsf{T}|E\rangle$?

Solution: From part (d), we learn that $\mathsf{T}H\mathsf{T}^{-1} = H$, and hence

$$H\mathsf{T}|E\rangle = \mathsf{T}H|E\rangle = E(\mathsf{T}|E\rangle)\,. \tag{13}$$

The energy eigenvalue of $\mathsf{T}|E\rangle$ is E again.

(f) (5pts) Suppose the time-reversal operator T obeys the following algebra in a subsector $\mathcal{H}'$ of the Hilbert space:

$$\mathsf{T}^2|\psi\rangle = -|\psi\rangle\,, \quad \forall|\psi\rangle \in \mathcal{H}'\,. \tag{14}$$

(For example, this would be the case on the subsector of states with half-integer ℓ in problem (c).) Prove that all energy eigenstates in $\mathcal{H}'$ are degenerate.

Solution: Consider an energy eigenstate $|E\rangle$ and its time-reversal counterpart $\mathsf{T}|E\rangle$. We will prove by contradiction by assuming that the energy level is non-degenerate. Then these two states must be equivalent, i.e.,

$$|E\rangle = e^{i\delta}\mathsf{T}|E\rangle \tag{15}$$

for some phase $e^{i\delta}$. Acting T on both sides again, we find

$$\mathsf{T}|E\rangle = \mathsf{T}e^{i\delta}\mathsf{T}|E\rangle = e^{-i\delta}\mathsf{T}^2|E\rangle = -e^{-i\delta}|E\rangle\,. \tag{16}$$

Comparing the two equations, we find a contradiction. Therefore the energy level has to be degenerate.

Quantum Mechanics 2

Time-independent perturbation theory

A quantum particle of mass m moves in a *weak* one-dimensional periodic potential $V(x)$ with period a as shown below. The Fourier expansion of the potential is:

$$V(x) = u \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \cos \frac{2\pi j x}{a} \,,$$

For simplicity, assume a very large but finite length $L = Na$ of the system, with integer N: $N \to \infty$, and periodic boundary conditions.

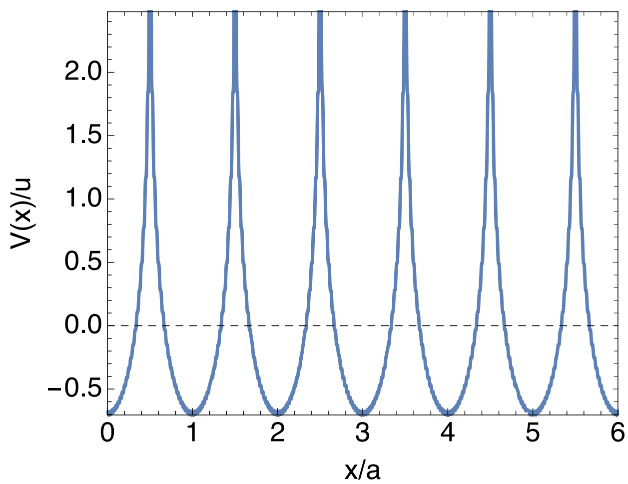


Figure 1: The weak periodic potenial under consideration. The problem determines how the weak potential perturbs a free particle.

(a) *(4 pts)* In the absence of $V(x)$, write down the zero-order energy eigenvalues and eigenfunctions $E_k^{(0)}$ and $\psi_k^{(0)}(x)$ that are also the eigenstates of the momentum p of the particle, recalling the periodic boundary conditions. What is the first-order correction $E_k^{(1)}$ to energies of these states due to $V(x)$?

(b) *(6 pts)* Calculate explicitly the second-order correction $E_0^{(2)}$ to energy $E_0^{(0)}$ of the ground state of the particle. Formulate an explicit condition on u that allows to treat the potential as weak. From the general expression for the second-order corrections $E_k^{(2)}$ to energies $E_k^{(0)}$, find the values k_n of the wavevector k for which the non-degenerate perturbation theory breaks down.

(c) *(6 pts)* Use the degenerate perturbation theory to find the energies E_n with accuracy up to the first order in u and the appropriate zero-order wavefunctions $\chi_n(x)$ for the special values of wavevector $k = k_n$ found in part (b). Based on these results, describe very briefly the qualitative structure of the particle energy spectrum in a periodic potential.

(d) *(4 pts)* For the lowest k_n (with $n = 1$), find the energies E_1 with accuracy up to the second order in u.

In this problem, you may leave the number series appearing in the results as they are, but you can also take into account some known formulas:

$$\sum_{l=1}^{\infty} \frac{1}{l^4} = \frac{\pi^4}{90}\,, \quad \sum_{l=1}^{\infty} \frac{1}{l^3(l+1)} = \zeta(3) - \frac{\pi^2}{6} + 1\,, \quad \sum_{l=1}^{\infty} \frac{1}{l^2(l+1)^2} = \frac{\pi^2 - 9}{3}\,.$$

Solution

(a) The eigenstates of the of the momentum p of the particle are the plane waves characterized by some wavevector k. Normalized on the finite interval of length L with the periodic boundary conditions, the corresponding eigenfunctions are

$$\psi_k^{(0)}(x) = \frac{1}{\sqrt{L}} e^{ikx}.$$

The energy of this state is just the kinetic energy of the particle:

$$E_k^{(0)} = \frac{\hbar^2 k^2}{2m}.$$

In perturbation theory, the first-order correction to energies is the average value of the perturbation potential in the unperturbed states. In our case, this correction vanishes:

$$E_k^{(1)} = \int dx [\psi_k^{(0)}(x)]^* V(x) \psi_k^{(0)}(x) = \frac{1}{L} \int dx V(x) = 0,$$

since there is no constant component in the Fourier expansion of the potential, and all oscillating components average to zero.

(b) The make use of the general expression for $E_k^{(2)}$:

$$E_k^{(2)} = \sum_{l \neq k} \frac{|\langle l|V|k\rangle|^2}{E_k^{(0)} - E_l^{(0)}},$$

one needs to find the general expression for the matrix elements of the perturbation potential:

$$\langle l|V|k\rangle = \int dx [\psi_l^{(0)}(x)]^* V(x) \psi_k^{(0)}(x) = \frac{1}{L} \int dx V(x) e^{i(k-l)x}.$$

Rewriting the Fourier expansion of the potential $V(x)$ in this expression in terms of the exponentials:

$$V(x) = \frac{u}{2} \sum_{j=1}^{\infty} \sum_{\pm} \frac{(-1)^j}{j} e^{\pm i 2\pi j x/a},$$

we see directly that

$$\langle l|V|k\rangle = 0, \text{ if } l \neq k \pm \frac{2\pi j}{a}, \qquad \langle l|V|k\rangle = \frac{u}{2} \frac{(-1)^j}{j}, \text{ for } l = k \pm \frac{2\pi j}{a}.$$

With these matrix elements, and the fact that $E_0^{(0)} = 0$, equation above for $E_k^{(2)}$ gives finally:

$$E_0^{(2)} = -\frac{u^2}{4} \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{\pm} \frac{1}{E_l^{(0)}} |_{l=\pm 2\pi j/a} = -\frac{u^2}{(2\pi\hbar/a)^2/m} \sum_{j=1}^{\infty} \frac{1}{j^4} = -\frac{\pi^2}{90} \frac{u^2}{\hbar^2/(ma^2)}.$$

Based on this expression, one can conclude that the potential is "weak", i.e., can be treated with the perturbation theory, if the typical magnitude of the potential u is much smaller than the typical kinetic energy of the particle confined to one period of the potential:

$$u \ll \frac{\hbar^2}{ma^2} .$$

The non-degenerate perturbation theory breaks down if the perturbation couples the states with the same zero-order energies. From the expression for the energies $E_k^{(0)}$ and the matrix elements $\langle l|V|k\rangle$ derived above we see that this happens if

$$k = k_n = \frac{\pi n}{a} , \quad n = 1, 2, 3, ...$$

(c) For each n, there are two degenerate states with wavevectors $k = \pm pin/a$ that are coupled by the corresponding matrix elements of the perturbation potential. This means that the perturbation operator reduced to the degenerate subspace at $k = k_n$ is

$$V = \frac{u}{2}\frac{(-1)^n}{n}\sigma_x ,$$

and its eigenenergies are

$$E_n^{(\pm)} = E_k^{(0)}|_{k=\pi n/a} \pm \frac{u}{2n} .$$

The corresponding eigenfunctions are

$$\chi_n^{(-)} = \sqrt{\frac{2}{L}} \cos \frac{\pi n x}{a} , \quad \chi_n^{(+)} = \sqrt{\frac{2}{L}} \sin \frac{\pi n x}{a} ,$$

for odd n, while for even n the "plus" and "minus" functions are interchanged.

These expressions imply that the spectrum of a quantum particle in the periodic potential is split into separate energy band, separated by the energy gaps. The width of the energy gaps is proportional to the magnitude of the potential. Each band is closed on itself.

(d) The higher-order corrections to the energies and states of the degenerate subspaces are generated by the coupling of these states to states outside of the degenerate subspace. In the present case, this coupling creates two types of corrections. First is "diagonal" second-order corrections to energies, which is the same for the state of the degenerate subspace

$$\sum_{j=1}^{\infty} \frac{|\langle\frac{\pi}{a}|V|\frac{\pi}{a}(1+2j)\rangle|^2}{E_{\pi/a}^{(0)} - E_{\pi(1+2j)/a}^{(0)}} + \sum_{j=2}^{\infty} \frac{|\langle\frac{\pi}{a}|V|\frac{\pi}{a}(1-2j)\rangle|^2}{E_{\pi/a}^{(0)} - E_{\pi(1-2j)/a}^{(0)}} = -\frac{u^2}{2(2\pi\hbar/a)^2/m}\Big[\sum_{j=1}^{\infty}\frac{1}{j^2}\frac{1}{j(j+1)} + \sum_{j=2}^{\infty}\frac{1}{j^2}\frac{1}{j(j-1)}\Big]$$

$$= -\frac{u^2}{2(2\pi\hbar/a)^2/m}\sum_{j=1}^{\infty}\Big[\frac{1}{j^3(j+1)} + \frac{1}{j(j+1)^3}\Big] = -\frac{u^2}{(2\pi\hbar/a)^2/m}[2 - \frac{\pi^2}{6}] .$$

where the last step uses the summation formulas given in the problem and a similar relation that can be derived along the same lines as the given ones:

$$\sum_{l=1}^{\infty} \frac{1}{l(l+1)^3} = 3 - \zeta(3) - \frac{\pi^2}{6} .$$

The second part of the second-order potential induced in the degenerate subspace is the "off-diagonal" coupling between the two degenerate states:

$$\sum_{j=1}^{\infty} \frac{\langle \frac{\pi}{a}|V|\frac{\pi}{a}(1+2j)\rangle\langle \frac{\pi}{a}(1+2j)|V|-\frac{\pi}{a}\rangle}{E^{(0)}_{\pi/a}-E^{(0)}_{\pi(1+2j)/a}} + \sum_{j=2}^{\infty} \frac{\langle \frac{\pi}{a}|V|\frac{\pi}{a}(1-2j)\rangle\langle \frac{\pi}{a}(1-2j)|V|-\frac{\pi}{a}\rangle}{E^{(0)}_{\pi/a}-E^{(0)}_{\pi(1-2j)/a}}$$

$$= \frac{u^2}{2(2\pi\hbar/a)^2/m}\Big[\sum_{j=1}^{\infty}\frac{1}{j^2(j+1)^2} + \sum_{j=2}^{\infty}\frac{1}{j^2(j-1)^2}\Big] = \frac{u^2}{(2\pi\hbar/a)^2/m}[\frac{\pi^2}{3}-3]\,.$$

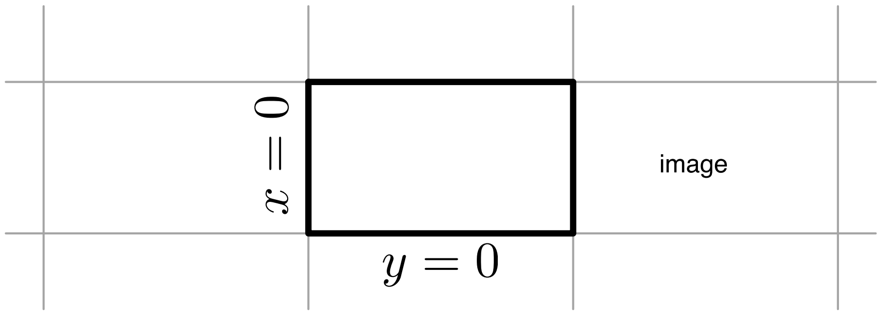
This means finally that the energies E_1 with accuracy up to u^2 terms are:

$$E_1^{(\pm)} = E^{(0)}_{\pi/a} - \frac{u^2}{(2\pi\hbar/a)^2/m}[2-\frac{\pi^2}{6}] \pm \Big(\frac{u}{2} - \frac{u^2}{(2\pi\hbar/a)^2/m}[\frac{\pi^2}{3}-3]\Big)\,.$$

Quantum Mechanics 3

Quantum particle in an equilateral triangle

Consider non-relativistic particle of mass M freely moving on a two dimensional xy plane, bouncing off hard walls. First, consider four hard walls which form a rectangle with sides a and b: $x \in [0, a]$, $y \in [0, b]$.



(a) *(3 pts)* Write down the energy spectrum $E_{m,n}$, and the corresponding normalized wavefunctions $\psi_{m,n}(x, y)$ of the particle inside the rectangle.

(b) *(7 pts)* Solve now the problem in part (a) explicitly by the "method of images", *i.e.* consider the particle wavefunction $\psi(x, y)$ on the whole xy plane. Since the particle is free, it should be possible to decompose $\psi(x, y)$ into the plane-wave components

$$\chi_{\mathbf{k}}(x, y) = e^{i(k_x x + k_y y)},$$

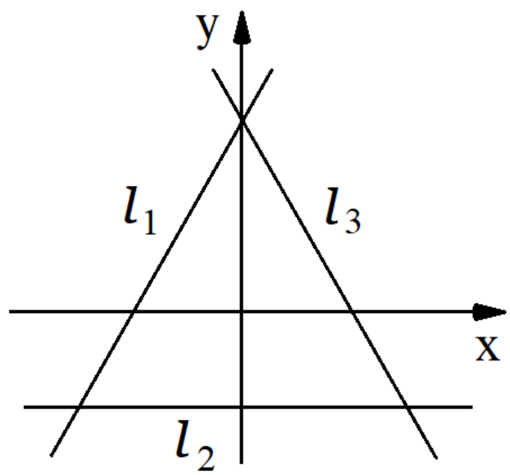
with, at this point, an arbitrary wavevector $\mathbf{k}$. Next, introduce the operators R_1, R_2, R_3, R_4 of reflections across the lines $x = 0$, $y = 0$, $x = a$, $y = b$, and impose the condition that the wavefunctions change sign upon individual reflections.

Show that the compositions of reflections $R_4 R_2$ and $R_3 R_1$ represent the coordinate shifts. From the parameters of these shifts, find the quantization conditions of the wavevector $\mathbf{k}$, and the corresponding energies $E_{m,n}$. Then, applying all the relevant reflections to $\chi_{\mathbf{k}}(x, y)$ find the wavefunctions $\psi_{m,n}(x, y)$.

Next, consider three hard walls forming an equilateral triangle of side a as shown below. The goal is to use the method of images as discussed in part (b) to find the energy spectrum $E_{m,n}$ of the particle inside the triangle. The triangle is placed on the plane so that its sides lie along the lines

$$l_1 \,:\, x\sqrt{3} - y = -\frac{a}{\sqrt{3}}\,, \quad l_2 \,:\, y = -\frac{a}{2\sqrt{3}}\,, \quad l_3 \,:\, x\sqrt{3} + y = \frac{a}{\sqrt{3}}\,,$$

i.e., the vertices of the triangle are: $v_{12} = l_1 \cap l_2 = (-a/2, -a/2\sqrt{3})$, $v_{13} = l_1 \cap l_3 = (0, a/\sqrt{3})$, $v_{23} = l_2 \cap l_3 = (a/2, -a/2\sqrt{3})$.



(c) *(4 pts)* The reflection R across an arbitrary line defined by the equation $y = kx + b$ transforms the coordinates x, y on a plane as

$$R\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+k^2}\begin{pmatrix} 2k(y-b) + (1-k^2)x \\ 2(kx+b) + (k^2-1)y \end{pmatrix}. \tag{1}$$

Use this result to derive the transformations of a wavefunction $\psi(x, y)$ under the reflections R_1, R_2, R_3 across the lines l_1, l_2, l_3 in the method of images.

(d) *(2 pts)* Show also (either geometrically or using Eq. (1) and the simplest choice of the coordinate system) that the two successive reflections across two arbitrary lines intersecting at angle ϕ is equivalent to a rotation. What is the angle of the rotation?

(e) *(4 pts)* Find the energy spectrum $E_{m,n}$ of the particle inside the triangle. *Hint:* Since two reflections produce a rotation, the simplest combinations of the reflections R_1, R_2, R_3 that result in the coordinate shifts consist of four reflections, e.g., $R_2R_1R_3R_1$, $R_2R_3R_2R_1$, and $R_2R_1R_2R_3$.

Solution.

(a) The wavefunction $\psi(x,y)$ of the stationary state solves the Schrödinger equation:

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi = E\psi \tag{2}$$

here the energy E is to be found from the requirement that ψ is non-singular inside the domain bounded by the walls, and vanishes at the walls:

$$\psi|_{\text{walls}} = 0 \tag{3}$$

For a rectangle $0 \le x \le a$, $0 \le y \le b$, two coordinates are independent and the wavefunctions are given by the product of the standard normalized Fourier harmonics that give the wavefunctions of a particle in an infinite square well potential:

$$\psi_{m,n}(x,y) = \frac{2}{\sqrt{ab}}\sin(k_x x)\sin(k_y y) \tag{4}$$

with

$$k_x a = \pi n\,, \qquad k_y b = \pi m\,, \qquad m,n \in \mathbb{Z} \tag{5}$$

Normally, Fourier expansion of a function on an interval involves both sin and cos functions, but the vanishing conditions gets rid of the cosines. The integers m,n are defined up to a sign change (which produces an irrelevant change of sign of the overall wavefunction), hence we can assume $m>0$, $n>0$. The corresponding energy eigenvalues are:

$$E_{m,n} = \frac{\hbar^2\pi^2}{2M}\left(\frac{m^2}{b^2}+\frac{n^2}{a^2}\right)\,. \tag{6}$$

(b) To obtain these results by the method of images we note that the wavefunction of a particle in the rectangle can be extended to the whole $x-y$ plane, if one ensures that it vanishes at the lines $x=0$, $x=a$, $y=0$, $y=b$. This can be achieved by requiring that the wavefunctions change sign under the reflections across these lines, i.e.:

$$R_1\psi(x,y) = -\psi(-x,y), \quad R_3\psi(x,y) = -\psi(2a-x,y),$$

$$R_2\psi(x,y) = -\psi(x,-y), \quad R_4\psi(x,y) = -\psi(x,2b-y),$$

respectively. These formulas show that the composition R_4R_2 is a shift by $(0,2b)$, while the composition R_3R_1 is a shift by $(2a,0)$, and the wavefunctions should be preserved under these combinations of reflections:

$$R_3R_1\psi(x,y) = \psi(x+2a,y), \quad R_4R_2\psi(x,y) = \psi(x,y+2b).$$

This means that the wavefunctions are double-periodic on the plane, and the wavevectors $\mathbf{k}$ of the plane-wave components $\chi_{\mathbf{k}}(x,y) = e^{i(k_x x + k_y y)}$ are quantized by the conditions:

$$2ak_x = 2\pi n\,, \qquad 2bk_y = 2\pi m\,.$$

These conditions coincide with (5), thus reproducing the energies $E_{m,n}$ (6).

Finally, since the translations by $(2a, 0)$ and $(0, 2b)$ are the identity transformations, there are three independent reflection operations: R_1, R_2, and R_1R_2, which produce the total wavefunction out of the plane-wave component $\chi_{\mathbf{k}}(x, y)$:

$$\psi_{m,n}(x, y) \propto (\chi_{\mathbf{k}}(x, y) - \chi_{\mathbf{k}}(-x, y) - \chi_{\mathbf{k}}(x, -y) + \chi_{\mathbf{k}}(-x, -y)) \ . \tag{7}$$

Again, this wavefunction coincides with the wavefunction (4) up to normalization.

(c) Now we can pass to the case of the triangle. As in part (b), we extend the wavefunction to the whole plane and require that it changes sign when reflected across the lines l_1, l_2, l_3. The equation for the transformation of coordinates by reflection given in the problem implies then the following properties of the wavefunction:

$$\begin{aligned}
&R_1 : \psi\left(-x/2 + \sqrt{3}y/2 - a/2, \sqrt{3}x/2 + y/2 + a/2\sqrt{3}\right) = -\psi(x, y)\,, \\
&R_2 : \psi\left(x, -y - a/\sqrt{3}\right) = -\psi(x, y)\,, \\
&R_3 : \psi\left(-x/2 - \sqrt{3}y/2 + a/2, -\sqrt{3}x/2 + y/2 + a/2\sqrt{3}\right) = -\psi(x, y)\,.
\end{aligned} \tag{8}$$

For two arbitrary lines, a convenient choice of the coordinate system is for the x axis to coincide with one of the lines, and chose the intersection point with an arbitrary second line to be the origin of the coordinate system. In this case, the reflection across the first line acts on (x, y) simply as $(x, y) \mapsto (x, -y)$, and the equation of the second line is $y = kx$, where k is given by the angle ϕ between the lines: $k = \tan\phi$. As follows from the equation given in the problem, the total transformation S of the coordinates due to the two successive reflections then is:

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+k^2}\begin{pmatrix} (1-k^2)x - 2ky \\ 2kx + (1-k^2)y \end{pmatrix} = \begin{pmatrix} \cos 2\phi\,, & -\sin 2\phi \\ \sin 2\phi\,, & \cos 2\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\,, \tag{9}$$

where in the last equality, we took into account that for $k = \tan\phi$: $(1-k^2)/(1+k^2) = \cos 2\phi$, and $2k/(1+k^2) = \sin 2\phi$. This transformation is indeed rotation by the angle that is twice the angle between the lines. Under the employed assumptions that the reflection lines are fixed, the rotation is in the direction from the first to the second line.

(d) Each quartic combination of reflections such as $R_2R_1R_3R_1$, $R_2R_3R_2R_1$, or $R_2R_1R_2R_3$, can be viewed as two successive rotations in the opposite direction which is equivalent to a translation. The simplest way to find this translation in each case is to notice that the translation vector is given by the displacement of the first center of rotation under the second rotation. Using parameters of the rotations: angle $120^{\deg}$ from the firsts to the second line in each case, one can see directly that the combinations of reflections given above represent the following shifts:

$$(x, y) \mapsto (x, y - a\sqrt{3})\,,\ (x, y) \mapsto (x + 3a/2, y - \sqrt{3}a/2)\,,\ (x, y) \mapsto (x - 3a/2, y - \sqrt{3}a/2)\,. \tag{10}$$

Since the wavefucntion should be preserved by four reflections, these coordinate shifts require the following quantization conditions:

$$-k_y a\sqrt{3} = 2\pi n\,,\ \pm 3ak_x/2 - \sqrt{3}ak_y/2 = 2\pi m_\pm\,, \quad n, m_\pm \in \mathbb{Z}\,, \tag{11}$$

for the wavevectors of the plane-wave components $e^{i(k_x x+k_y y)}$ of the wavefunction. These equations imply that $m_+ + m_- = n$, and denoting m_+ as $-m$, we have $m_- = n + m$. With this, the energy spectrum of the triangular cavity finally is

$$E_{m,n} = \frac{8\hbar^2\pi^2}{9Ma^2}\left(n^2 + m^2 + nm\right) . \tag{12}$$

Since only the quartic combinations of reflections represent the identity transformations, the proper wavefunction of the energy eigensatte is a linear combination of the exponent $e^{i(k_x x+k_y y)}$ and its single, double, and triple reflections. Since the single reflection R_2 changes the sign of k_y, the state that corresponds to the pair of integers (m, n) is identical to the state with the indices $(-m, -n)$.

One can further investigate the decomposition of the spectrum with respect to the action of the symmetry of the triangle (rotations by $2\pi/3$ and reflections).

Statistical Mechanics 1

Maxwell-Boltzmann Gas from Microcanonical Ensemble

The Boltzmann (B) statistics arises from counting the ways N distinguishable particles can be distributed among $\{j = 1, 2, \cdots, n\}$ states of energies ϵ_j with g_j-fold degeneracies. It becomes the Maxwell-Boltzmann (MB) distribution for indistinguishable particles on dividing by $N!$:

$$\Omega(N_1, N_2, \cdots, N_n) \equiv \Omega_{MB} = \frac{1}{N!}\Omega_B = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!} \tag{1}$$

(a) (4 points) Sketch a derivation of the rhs of Eq. 1. In this equation what are the conditions on g_j and N_j for the Maxwell-Boltzmann statistics (as opposed to quantum statistics) to be valid? Explain.

(b) (4 points) Maximize the entropy $S(E, N) = \ln(\Omega)$, with the constraints that both total particle number N and the total energy E are constant, to derive an expression for $f_j(\epsilon_j) = N_j/g_j$ in terms of the two corresponding Lagrange multipliers.

(c) (5 points) The temperature and chemical potential are defined from the derivatives of the entropy:

$$\left(\frac{\partial S}{\partial E}\right)_N \equiv \frac{1}{T}, \qquad \left(\frac{\partial S}{\partial N}\right)_E \equiv -\frac{\mu}{T}. \tag{2}$$

Use these derivatives to relate the Langrange multipliers of part (b) to T and μ as defined by eq. 2.

(d) (4 points) Derive an expression for the free energy of a monoatomic ideal gas in volume V as a function of temperature and N.

(e) (3 points) Use the expression for the free energy you obtained in part (d) to derive the following: (i) the ideal gas law and (ii) total energy E.

Solution

(a) First consider one energy level ϵ_1 consisting of g_1 states. Then if N_1 particles are placed at this level there are $g_1^{N_1}$ possibile states. Consider a partitioning of N particles into the n levels: e.g. particle 3,4 in level 1; particle 1,2,5 in bin level 2; etc. So $N_1 = 2$ and $N_2 = 3$ in this exxample. With this partition there are $g_1^{N_1} g_2^{N_2} \ldots$ states. Each permuation of the particles which exchanges particles amongst the levels (e.g. exchanging particle 1 and 3 in the example above) leads to a new set of $g_1^{N_1} g_2^{N_2} \ldots$ states. The number of such level changing permutations is

$$\frac{N!}{N_1! N_2! \ldots N_n!} \tag{3}$$

Multiplying this factor by $g_1^{N_1} g_2^{N_2} \ldots$ we get the total number of states. Diving by $N!$ gives Eq. 1 after dividing by $N!$.

For MB statistics to be valid, when I distrbute the N_1 particles amongst the g_1 states there should only very rarely be two particles in one state leading to the requirement that $g_1 \gg N_1$ or $f_j = N_j/g_j \ll 1$.

(b) Write $\ln(\Omega)$ using Stirling's formula, and define the constraint functions ϕ and ψ:

$$\ln(\Omega) \simeq \sum_i \left[N_i \ln g_i - N_i \ln N_i + N_i \right], \tag{4}$$

$$\phi(N_1, N_2, \cdots, N_n) = \sum_i N_i, \tag{5}$$

$$\psi(N_1, N_2, \cdots, N_n) = \sum_i N_i \epsilon_i \,. \tag{6}$$

Maximizing $S = k \ln(\Omega)$, with constraints $\phi = N$ and $\psi = E$ with Lagrange multipliers α and β (the sign of β has been chosen for later convenience) yields

$$\begin{aligned} & \frac{\partial}{\partial N_j} \ln(w) + \alpha \frac{\partial \phi}{\partial N_j} - \beta \frac{\partial \psi}{\partial N_j} = 0 \\ \Rightarrow \quad & \ln g_j - \ln N_j + \alpha - \beta \epsilon_j = 0 \\ \Rightarrow \quad & \ln \left(\frac{N_j}{g_j} \right) = \alpha - \beta \epsilon_j \end{aligned} \tag{7}$$

Thus we have finally

$$f_j \equiv \frac{N_j}{g_j} = e^{\alpha - \beta \epsilon_j} \,. \tag{8}$$

Clearly we want to interpret the multiplies as $\alpha = \beta\mu$ and $\beta = 1/T$. This is indeed the case as we will see in the next item.

(c) Substituting $N_j = g_j e^{\alpha - \beta\epsilon_j}$ we have

$$
\begin{aligned}
S(E,N) =& \sum_j \big(- N_j \ln f_j + N_j \big) && (9)\\
=& \sum_j g_j e^{-\beta\epsilon_j + \alpha} (\beta\epsilon_j - \alpha) + N && (10)\\
=& \beta E + (1-\alpha) N \,, && (11)
\end{aligned}
$$

where $\beta(E,N)$ and $\alpha(E,N)$ are defined implicitly via

$$
\begin{aligned}
E =& \sum_j g_j \epsilon_j e^{-\beta\epsilon_j + \alpha} \,, && (12)\\
N =& \sum_J g_j e^{-\beta\epsilon_j + \alpha} \,. && (13)
\end{aligned}
$$

When computing derivatives of the entropy with respect to E and N, one has to take into account that α and β also depend on (E,N). One can either use the Jacobian technique to convert from pair (α,β) to (E,N), or use the following shortcut. Differentiation of the entropy yields

$$
\begin{aligned}
\left(\frac{\partial S}{\partial E}\right)_N =& \left(\frac{\partial \beta}{\partial E}\right)_N E + \beta - \left(\frac{\partial \alpha}{\partial E}\right)_N N \,, && (14)\\
\left(\frac{\partial S}{\partial N}\right)_E =& \left(\frac{\partial \beta}{\partial N}\right)_E E + 1 - \alpha - \left(\frac{\partial \alpha}{\partial N}\right)_E N \,. && (15)
\end{aligned}
$$

To eliminate derivatives of α and β, one can use the following identities

$$
0 = \left(\frac{\partial N}{\partial E}\right)_N = \sum_j g_j e^{-\beta\epsilon_j + \alpha} \left[-\epsilon_j \left(\frac{\partial \beta}{\partial E}\right)_N + \left(\frac{\partial \alpha}{\partial E}\right)_N \right] = -\left(\frac{\partial \beta}{\partial E}\right)_N E + \left(\frac{\partial \alpha}{\partial E}\right)_N N \,, \tag{16}
$$

$$
1 = \left(\frac{\partial N}{\partial N}\right)_E = \sum_j g_j e^{-\beta\epsilon_j + \alpha} \left[-\epsilon_j \left(\frac{\partial \beta}{\partial N}\right)_E + \left(\frac{\partial \alpha}{\partial N}\right)_E \right] = -\left(\frac{\partial \beta}{\partial N}\right)_E E + \left(\frac{\partial \alpha}{\partial N}\right)_E N \,, \tag{17}
$$

$$
\tag{18}
$$

which upon substitution in the above equations result in

$$
\left(\frac{\partial S}{\partial E}\right)_N = \beta \,, \quad \left(\frac{\partial S}{\partial N}\right)_E = \alpha \,. \tag{19}
$$

Comparison to the definitions gives

$$
\beta = \frac{1}{T} \qquad \alpha = \frac{\mu}{T} \tag{20}
$$

as could have been anticipated.

(c) Use the density of states in the continuum limit to integrate

$$Z_1 = \int_0^\infty d\epsilon g(\epsilon) e^{-\epsilon/T} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\sqrt{\epsilon}\; e^{-\epsilon/T} = V\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2}$$

The free energy is

$$F = -T\ln Z = -T\ln\left(\frac{Z_1^N}{N!}\right) \tag{21}$$

Or

$$F = -NT\left[\ln\left(\frac{Z_1}{N}\right) + 1\right] \tag{22}$$

Leading to

$$F = -NT\left[\ln(V/N) + \frac{3}{2}\ln(T) + \ln\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} + 1\right]$$

(d) The pressure and the total internal energy can be computed from the free energy F as a function of Z, and the former gives the ideal gas law:

$$F = -NT\left[\ln(V/N) + \frac{3}{2}\ln(T) + \ln\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} + 1\right]$$

$$P = -\frac{\partial F}{\partial V} = \frac{NT}{V}$$

$$E = F + TS = F - T\frac{\partial F}{\partial T} = NT^2\frac{\partial}{\partial T}\ln(Z) = NT^2\frac{3}{2T} = \frac{3}{2}NT$$

Statistical Mechanics 2

Theory of Magnetization

Consider an equilibrium magnetic system in a fixed external magnetic field H. Its free energy $G(m,T)$ as a function of magnetization m and temperature T can be written as

$$G(T,m) = a(T) + \frac{1}{2}b(T)m^2 + \frac{1}{4}cm^4 - Hm\,. \tag{1}$$

In the relevant range of the temperatures T, the constant c is positive, while $b(T) = (T^2 - b_0)$ changes its sign.

In the initial part of the problem (parts **(a), (b), (c)**), the magnetic field is zero, $H = 0$.

(a) [2pt] Describe the change in the system as the temperature increases from low to high.

(b) [4pt] Calculate the equilibrium magnetization $m(T)$ as a function of the temperature T.

(c) [4pt] How does the specific heat of the system depend on the temperature T?

Now assume that magnetic field H is nonzero but very small.

(d) [3pt] Calculate the magnetic susceptibility $(\partial m/\partial H)_T$ *above* the phase transition.

(e) [3pt] Now calculate the magnetization $m(T,H)$ to the first order in the magnetic field H below the phase transition. For what range of temperatures is the small-H approximation valid?

Hint: use approximation $m(T,H) = m(T,0) + \delta m$ *and find* δm *to first order in* H.

(f) [4pt] Finally, do *not* assume that the magnetic field is small. Find the magnetization as a function of field H at the critical temperature.

Solution

(a) [2pt]
The free energy function has two minima for low temperatures where $b(T) < 0$), and only one minimum at high temperatures where $b(T) > 0$. The system undergoes a second-order phase transition at the $T = T_c = \sqrt{b_0}$ point where the coefficient $b(T)$ changes its sign. Below that point, the minimum of the free energy is achieved at some nonzero value of the order parameter $|m^*(T)| \neq 0$.
(b) [4pt]
For $T > T_c$, there is only one minimum of the free energy $G(T, m)$ at $m^*_> = 0$. Below the critical temperature $T < T_c$, the minimum of the free energy is determined by the condition

$$\frac{\partial G(m,T)}{\partial m}\bigg|_{m=m^*} = (bm + cm^3)_{m=m^*} = 0\,. \tag{2}$$

Since $b(T) < 0$ for $T < T_c$, the local minima of the free energy are achieved at the nonzero equilibrium values of magnetization $m^*_<(T) = \pm\sqrt{\frac{|b(T)|}{c}}$.
(c) [4pt]
In order to find the heat capacity, one has to find the equilibrium value of the free energy $G^*(T) = G(T, m^*(T))$. For the temperature above the transition $T > T_c$ and zero magnetic field, $G^*_>(T) = a(T)$ and the heat capacity is

$$c_> = -T\frac{\partial^2 G^*_>}{\partial T^2} = -Ta''\,. \tag{3}$$

Below the phase transition ($T < T_c$), the free energy is

$$G^*_<(T) = G(T, m^*_<(T)) = a - \frac{[b(T)]^2}{4c}\,. \tag{4}$$

Note that the temperature derivative of $G(T)$ is continuous at $T = T_c$,

$$\frac{dG^*_>}{dT}\bigg|_{T=T_c} = \frac{dG^*_<}{dT}\bigg|_{T=T_c} = a'\,, \tag{5}$$

therefore the transition is at least of the second order. The heat capacity below the transition acquires a contribution from the $b^2(T)$ term:

$$c_< = -T\left(a'' - \frac{(b')^2 + bb''}{2c}\right) = -T\left(a'' - \frac{3T^2 - b_0}{c}\right) \tag{6}$$

The specific heat is discontinuous at the phase transition temperature,

$$\Delta c = c_>(T_c) - c_<(T_c) = \frac{b_0 - 3T^2}{c} = -\frac{2b_0}{c}\,, \tag{7}$$

so the transition is the second order.

(d) [3pt]
For small nonzero magnetic field, the magnetization is also small at $T > T_c$ and is readily determined from the equilibrium equation

$$0 = \left(\frac{\partial G}{\partial m}\right)_T = -H + b(T)m + \cancel{cm^3} \approx -H + b(T)m \quad \Longrightarrow \quad m_>(T,H) = \frac{H}{b(T)} = \frac{H}{T^2 - b_0} \tag{8}$$

and the susceptibility $\chi = \left(\frac{\partial m}{\partial H}\right)_T = (T^2 - b_0)^{-1}$ appears to become singular as $T \to T_c$. However, this is only because the cubic term (cm^3) is neglected in this approximation.

(e) [3pt]
Using the suggested approximantion below the transition temperature in equation (8), one obtains magnetization

$$0 = -H + b(T)(m^* + \delta m) + c(m^* + \delta m)^3 = -H + b(T)\delta m + 3cm^{*2}\delta m + O(\delta m^2) \tag{9}$$

$$\Longrightarrow \quad \delta m(T,H) \approx \frac{H}{b(T) + 3cm^{*2}} = \frac{H}{2|b(T)|} = \frac{H}{2(b_0 - T^2)}\,,$$

which becomes "infinite" if $T \to T_c = \sqrt{b_0}$. This approximation is valid only if the correction is small,

$$\delta m \ll |m^*| \quad \Leftrightarrow \quad \frac{H}{2|b(T)|} \ll \sqrt{\frac{|b(T)|}{c}} \quad \Leftrightarrow \quad H^2 \ll \frac{4|b(T)|^3}{c} \tag{10}$$

This condition can be also rewritten as the condition on the temperature,

$$\frac{|T - T_c|}{T_c} \gg \frac{\left(cH^2/4\right)^{1/3}}{2b_0}\,, \tag{11}$$

i.e., the system should not be too close to the phase transition.

(f) [4pt]
If the condition (11) is violated, the cubic term (cm^3) cannot be neglected. Instead, the linear term (bm) can be neglected and

$$0 = \left(\frac{\partial G}{\partial m}\right)_T \approx -H + cm^3 \quad \Longrightarrow \quad m(H) \approx \left(\frac{H}{c}\right)^{1/3}\,, \tag{12}$$

at or in the immediate vicinity of the phase transition point $T \approx T_c$.

Statistical Mechanics 3

The Landsberg Limit

Solar cell performance is usually limited by semiconductor physics, and much current research goes into clever semiconductor design to improve photovoltaic efficiency. In contrast to semiconductor details, the second law of thermodynamics is inflexible. In this problem you will examine thermodynamic limits on solar cell efficiency.

(a) [4pt] Consider a high-temperature source of blackbody radiation (e.g. the sun) at temperature T_H. Write the partition function of its radiation assuming the sun is an ideal black body (black-body radiation, or BBR).

(b) [4pt] Find the relation between the internal energy density $\langle E \rangle / V$ of the BBR and its pressure p on a black, *ideally absorbing* surface.

Hint: There are several ways to work this out; since all that is needed is the ratio $pV/\langle E \rangle$, it is not necessary to calculate factors common to both p and $\langle E \rangle / V$.

(c) [3pt] Calculate the ratio $\langle E \rangle / S$ of energy $\langle E \rangle$ to entropy S (densities) for this source.

Hint: It may be helpful to recall that the chemical potential for photons is zero, and use the result from part (b).

(d) [4pt] Now consider a generic device (i.e. a solar cell) held at T_C that absorbs the blackbody radiation and performs useful work at a rate $\dot{W}$ (e.g. driving charges through a potential difference in an external circuit). The device is held at temperature T_C via thermal contact (heat exchange $\dot{Q}$) with a large heat bath, receives radiant energy at rate $\dot{E}_p$ from the high-temperature source, and can also emit blackbody radiation with characteristic temperature T_C to a sink at rate $\dot{E}_s$, as illustrated in figure 1. Write down constraints on these quantities from the first and second laws of thermodynamics.

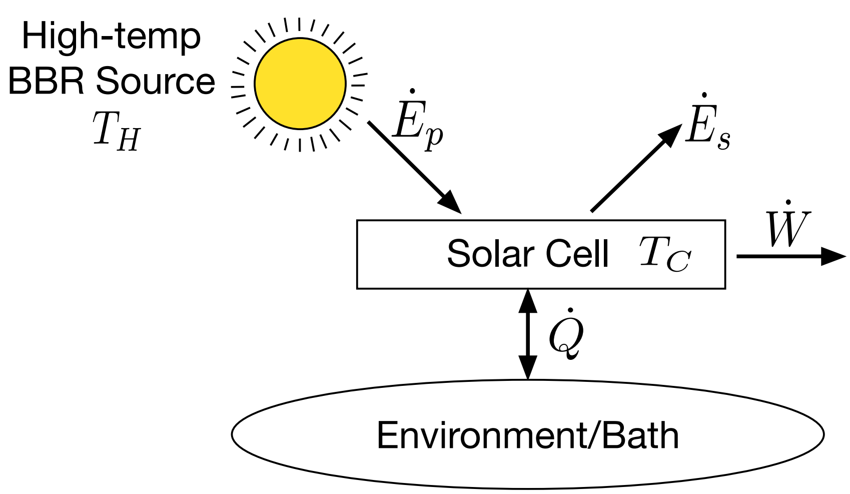


Figure 1: Energy balance for a solar cell in problem.

(e) [5pt] Use your results from (c) and (d) to calculate an upper limit to the efficiency of this device, defined as $\dot{W}/\dot{E}_p$. Consider which characteristics of the device have to be optimal to achieve that.

Hint: In the ideal case, no additional entropy is generated in the device, but the device still must receive and dispose of the incident entropy of the blackbody radiation.

Solution

(a) [4pt]

With the chemical potential $\mu = 0$, the grand partition function and the canonical partition function are the same. The Helmholtz free energy A, calculated from the partition function Q, is given by

$$\begin{aligned} A &= -k_B T \ln Q \\ &= -k_B T \ln \left(\prod_{\substack{\text{modes} \\ \omega_j}} \left[\sum_{n=0}^{\infty} e^{-\beta n \hbar \omega_j} \right] \right) = -k_B T \ln \left(\prod_{\substack{\text{modes} \\ \omega_j}} \left[\frac{1}{1 - e^{-\beta \hbar \omega_j}} \right] \right) \\ &= k_B T \sum_{\substack{\text{modes} \\ \omega_j}} \ln \left(1 - e^{-\beta \hbar \omega_j} \right) . \end{aligned} \tag{1}$$

For a volume much larger than the relevant radiation wavelengths, the sum can be taken over to an integral using the continuous density of states in $\mathbf{k}$-space $g(k) d^3\mathbf{k} = 2\frac{V}{(2\pi)^3} d^3\mathbf{k} = 2\frac{V}{(2\pi)^3} 4\pi k^2 dk$, where the leading factor of two accounts for two distinct transverse polarization states for the electromagnetic field. With the dispersion relation $\omega = ck$

$$A = k_B T \int_0^\infty 4\pi k^2 dk \, \frac{2V}{(2\pi)^3} \ln \left(1 - e^{-\beta \hbar c k} \right) . \tag{2}$$

(b) [4pt]

To calculate $pV/\langle E \rangle$, the precise numerical value of the partition function integral is not needed if one recognizes that it can be re-arranged via integration by parts to reveal a simple relationship between A and $\langle E \rangle$

$$\begin{aligned} A &= -k_B T \int_0^\infty 4\pi \frac{k^3}{3} dk \, \frac{2V}{(2\pi)^3} \frac{1}{1 - e^{-\beta \hbar c k}} e^{-\beta \hbar c k} (-\beta \hbar c) \\ &= -\frac{1}{3} \int_0^\infty 4\pi k^2 dk \, \frac{2V}{(2\pi)^3} \frac{1}{e^{\beta \hbar c k} - 1} \hbar c k \\ &= -\frac{1}{3} \sum_{\substack{\text{modes} \\ \omega_j}} \langle n(\omega_j) \rangle \, \hbar \omega_j \\ \Rightarrow A &= -\frac{1}{3} \langle E \rangle , \end{aligned} \tag{3}$$

where we've identified the number of photons per mode as $\langle n(\omega_j) \rangle = (e^{\beta \hbar \omega} - 1)^{-1}$ for photons following Bose-Einstein statistics. Since the chemical potential $\mu = 0$, the expression (2) is also the grand potential of the multicanonical photon ensemble $\Omega = A - \mu N = -pV$, one obtains the ratio

$$pV/E = \frac{1}{3} . \tag{4}$$

An alternative approach is to integrate momentum distribution of photons; integrating the angular dependence in case of pressure will produce the factor (1/3).

(c) [3pt]
Since $A = \langle E \rangle - TS = -pV = -\frac{1}{3}\langle E \rangle$, we have

$$S = \frac{\langle E \rangle - A}{T} = \frac{4}{3}\frac{\langle E \rangle}{T} \quad \Longrightarrow \quad \boxed{\frac{\langle E \rangle}{S} = \frac{3}{4}T} \tag{5}$$

with obviously the same ratio for the energy and entropy densities.

(d) [4pt]
In steady state, the rate of change of the energy $\dot{E}$ and the entropy $\dot{S}$ of the solar cell must be zero. For the energy, we have from figure 1

$$\dot{E} = \dot{E}_p - \dot{Q} - \dot{E}_s - \dot{W} = 0\,. \tag{6}$$

For entropy, we must also consider that entropy can be generated due to irreversible processes in the solar cell. Treating heat exchange with the bath as reversible since the bath and cell remain in thermal equilibrium, we have

$$\dot{S} = \dot{S}_p - \frac{\dot{Q}}{T_C} - \dot{S}_s + \dot{S}_g = 0\,, \tag{7}$$

where $\dot{S}_p$ is the entropy flux into the cell due to the incident blackbody radiation, $\dot{S}_s$ is entropy flux lost from the cell via blackbody radiation, and $\dot{S}_g$ is the rate of entropy generation in the cell.

(e) [5pt]
Using the result from a), we can write $\dot{S}_p = (4/3)\dot{E}_p/T_H$, $\dot{S}_s = (4/3)\dot{E}_s/T_C$. Inserting these relations in equation (7), multiplying by T_C, and using the Stefan-Boltzmann result $\dot{E}_s/\dot{E}_p = (T_C/T_H)^4$, we can combine equations (6) and (7) to find the efficiency. After some algebra, the result is

$$\eta = \frac{\dot{W}}{\dot{E}_p} = 1 - \frac{4}{3}\frac{T_C}{T_H} + \frac{1}{3}\left(\frac{T_C}{T_H}\right)^4 - \frac{T_C\dot{S}_g}{\dot{E}_p}$$
$$\Rightarrow \boxed{\eta \leq 1 - \frac{4}{3}\frac{T_C}{T_H} + \frac{1}{3}\left(\frac{T_C}{T_H}\right)^4}\,, \tag{8}$$

where the equality holds in the ideal case when $\dot{S}_g = 0$. The last term in (8) is completely negligible when $T_H \gg T_C$ but is critical to ensure the correct behavior of the expression for T_H only slightly larger than T_C. Without the last term, the equation would predict that converting blackbody radiation to work is impossible unless $T_H > 4/3T_C$, which is false.