Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

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General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

The adiabatic oscillator

Many physical systems involve motion at different time scales. We can treat such a system using a form of the WKB approximation. Consider an undamped harmonic oscillator with a slowly time dependent frequency $\omega(t)$, i.e. $|\dot{\omega}| \ll \omega^2$.

(a) (4 points) Write down the time-dependent Lagrangian for a particle of mass m moving along the x-axis in a harmonic potential with oscillation frequency $\omega(t)$.

Substitute the ansatz

$$x(t) = e^{a(t)+i\theta(t)} ,$$

into the Euler-Lagrange equations for $x(t)$, and without approximation determine the equations of motion for $a(t)$ and $\theta(t)$ by examining the real and imaginary parts.

(b) (4 points) Assuming that $\dot{a}^2 , |\ddot{a}| \ll \omega^2$, determine the leading-order solution for the real $x(t)$ given the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. Use this result to show that $|\dot{a}| \ll \omega$ is equivalent to $|\dot{\omega}| \ll \omega^2$.

Hint: $a(t)$ can be expressed in terms of $\dot{\theta}(t)$ without approximation.

(c) (4 points) Show that the energy E is *not* constant in time to first order in $\dot{\omega}$. Comment on the result in light of Noether's theorem.

(d) (4 points) Show, however, that the local average $\overline{E(t)/\omega(t)}$ *is* constant in time to first order in $\dot{\omega}$. The time average is taken over a local oscillation period $T = 2\pi/\omega(t)$ over which $\omega(t)$ does not change by much. Comment on the result in light of adiabatic invariants.

An example of such a system might be a pendulum consisting of a heavy mass on a thin cable with a time-dependent length $\ell(t)$. Imagine that the length of the cable varies due to the variation in the temperature throughout the day as

$$\ell(t) = \ell_0(1 + \beta \cos(\Omega t)) ,$$

where $t = 0$ at noon and $\Omega = 2\pi/1\,\text{day}$. Assume that $\beta \ll 1$ and that the oscillation amplitude is small.

(e) (4 points) If the pendulum is at its maximum displacement ϕ_0 at noon, what are the amplitude and phase of the oscillations at 6:00 p.m.?

Solution

(a) The Euler-Lagrange equation is simply $\ddot{x}+\omega(t)^2 x = 0$. Substituting the ansatz gives:

$$\left(\ddot{a} + i\ddot{\theta} + \dot{a}^2 - \dot{\theta}^2 + 2i\dot{a}\dot{\theta} + \omega(t)^2\right) x(t) = 0\,.$$

The real and imaginary parts give

$$\ddot{a} + \dot{a}^2 - \dot{\theta}^2 + \omega(t)^2 = 0 \quad , \qquad \ddot{\theta} + 2\dot{a}\dot{\theta} = 0\ .$$

(b) The equation from the imaginary part can be integrated:

$$\dot{a} = -\frac{\ddot{\theta}}{2\dot{\theta}} \ \Rightarrow\ e^{a(t)} = e^{a_0}\sqrt{\frac{\dot{\theta}_0}{\dot{\theta}(t)}}\,.$$

Assuming $\dot{a}^2\,,\ \ddot{a} \ll \omega^2$, we get $\dot{\theta}^2 = \omega(t)^2$, and hence

$$\theta(t) = \theta_0 \pm \int_0^t \omega(t')dt'\ .$$

This gives

$$x(t) = x_0\sqrt{\frac{\omega(0)}{\omega(t)}}\left(\cos\left(\int_0^t \omega(t')dt'\right) + q\sin\left(\int_0^t \omega(t')dt'\right)\right)\ , \tag{1}$$

where

$$q = \frac{1}{\omega(0)}\left(\frac{\dot{\omega}(0)}{2\omega(0)} + \frac{v_0}{x_0}\right)\ .$$

(c) The energy function is $h(q,\dot{q},t) = p\dot{q} - \mathcal{L}$ is where $p = \partial\mathcal{L}/\partial\dot{q}$ is a function of $\dot{q}$. Using the Euler Lagrange equations it is easy to show that

$$\frac{\partial h}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}\,, \tag{2a}$$

$$= m\dot{\omega}\omega x^2\,. \tag{2b}$$

The first line is the general result, while the second line is specific to the time dependent harmonic oscillator. There is no time translational symmetry in this problem so energy is not conserved. The Noether theorem establishes the link between time translational invariance and energy conservation.

(d) Now we modify the analysis, writing:

$$\overline{\frac{\partial E/\omega}{\partial t}} = \overline{m\dot{\omega}x^2 - \frac{E}{\omega^2}\dot{\omega}} \tag{3}$$

$$\simeq \frac{\dot{\omega}}{\omega^2}(m\omega^2\overline{x^2} - E)\,, \tag{4}$$

$$\simeq 0 + \mathcal{O}((\dot{\omega}/\omega^2)^2)\,. \tag{5}$$

In the first line we used Eq. (2) to evaluate $\dot{E}$. In passing to the second line we used that $\omega, \dot{\omega}, E$ are approximately constant over the timescale of the average. To reach the third line we recognized that since the whole result is already proportional to $\dot{\omega}/\omega^2$, we may use the lowest order time-independent simple harmonic oscillator result for the remaining average

$$\overline{\frac{1}{2}m\omega^2 x^2} = \frac{E}{2}\,.$$

Discussion: The theory of adiabatic invariance (in 1d) is concerned with a Hamiltonian system $H(p, q, \lambda)$ depending on a parameter λ, which is initially considered to be constant in time. For a given orbit in phase space, characterized by the energy E and parameter λ, we define integral

$$I(E, \lambda) = \frac{1}{2\pi}\oint p\, dq\,, \tag{6}$$

over the trajectory of the orbit. The line integral in Eq. 6 is (using the Stoke's theorem) the area in phase space enclosed by the orbit

$$I(E, \lambda) = \frac{1}{2\pi}\int_{\rm enclosed-orbital-phase-space-area} dp\, dq\,.$$

If now the parameter λ is changed on a time scale which is long compared to the orbital period, the quantity $I(E, \lambda)$ is unchanged in time to first order. The energy E changes (as required by the Noether theorem) and λ changes, but I remains approximately constant.

For the harmonic oscillator the phase space orbits are ellipses characterized by the energy E and frequency ω. The maxima and minima along the q and p axes of the ellipse are

$$q_\pm = \pm\sqrt{\frac{2E}{m\omega^2}}\,, \qquad p_\pm = \pm\sqrt{2mE}\,. \tag{7}$$

The area of the ellipse is $\pi q_+ p_+$. Thus, the area in phase space enclosed by the elliptic orbit is

$$I(E, \lambda) = \frac{\pi q_+ p_+}{2\pi} = \frac{E}{\omega}\,. \tag{8}$$

Now, as the frequency is changed in time, adiabatic invariance implies that $\overline{E(t)/\omega(t)}$ will remain approximately constant.

In a quantum mechanical WKB approximation for the eigen-energies, the integral in Eq. (6) is discrete, $I = E/\omega = (n+\frac{1}{2})\hbar$. In the quantum mechanical adiabatic approximation the energy level n remains fixed as $\omega(t)$ and $E(t)$ slowly change in time. Thus, there is a consistency between classical and quantum mechanical adiabatic approximations when n is large and the WKB approximation applies.

(e) The frequency is

$$\omega(t) = \sqrt{\frac{g}{\ell(t)}} \equiv \omega_0\sqrt{\frac{1}{1+\beta\cos(\Omega t)}} \approx \omega_0[1 - \frac{1}{2}\beta\cos(\Omega t)]\,, \quad \omega_0 \equiv \sqrt{\frac{g}{\ell_0}}\,.$$

Then we have

$$\theta(t) \approx \theta_0 + \omega_0[t - \frac{\beta}{2\Omega}\sin(\Omega t)] \ , \quad \omega(0) = \omega_0\sqrt{\frac{1}{1+\beta}} \ , \quad \dot{\omega} \approx \frac{\omega_0\beta\Omega}{2}\sin(\Omega t) \ .$$

The WKB solution must be mapped onto the angle from the axis $\phi(t)$. The adiabatic theorem says that E/ω is constant. Denoting the amplitude of ϕ oscillations $\phi_{\max}(t)$, then since $E \propto mg\ell\phi_{\max}^2$, and $\omega \propto 1/\sqrt{\ell}$, we have

$$\phi_{\max}(t) \propto \frac{1}{(\ell(t))^{3/4}} \, . \tag{9}$$

Thus

$$\phi(t) = \phi_0 \left(\frac{1+\beta}{1+\beta\cos(\Omega t)}\right)^{\frac{3}{4}} \cos(\omega_0[t - \frac{\beta}{2\Omega}\sin(\Omega t)]) \ . \tag{10}$$

At 6:00 p.m. we have $\Omega t = \frac{1}{2}\pi$, and hence the amplitude ϕ_{sunset} and phase shift φ are

$$\phi_{\text{sunset}} = \phi_0 \, (1+\beta)^{\frac{3}{4}} \quad , \qquad \varphi = -\frac{\omega_0\beta}{2\Omega} \ .$$

Eq. (10) can also be derived directly from the equation of motion. The Lagrangian of the pendulum

$$L = \frac{1}{2}m(\ell(t)\dot{\phi})^2 - \frac{1}{2}mg\ell(t)\phi^2 + \underbrace{\mathcal{O}(m\dot{\ell}^2)}_{\sim(\text{small})^2} \tag{11}$$

where here and below we neglect terms of second order $\sim \dot{\ell}^2$. The equation of motion is

$$\ddot{\phi} + 2\left(\frac{\dot{\ell}}{\ell}\right)\dot{\phi} = -\omega^2(t)\phi \, . \tag{12}$$

with $\omega^2(t) = g/\ell(t)$. Because of the $\dot{\ell}\dot{\phi}$ term (which is first order in $\dot{\ell}$), Eq. (1) is not directly applicable. However, substituting $x(t) \equiv \ell(t)\phi(t)$ into Eq. (12) yields

$$\ddot{x} = -\omega^2(t)x \tag{13}$$

after dropping terms of order $\dot{\ell}^2$ and $\ddot{\ell}$ which are second order in the adiabatic approximation. Thus Eq. (1) is applicable to $\ell(t)\phi(t)$ and this reasoning ultimately yields Eq. (10) for $\phi(t)$.

Classical Mechanics 2

Stellar Orbits

The gravitational potential of the Milky Way galaxy can be reasonably approximated by the axisymmetric form

$$\Phi(r,z) = \frac{1}{2}v_0^2 \ln\left(r^2 + \frac{z^2}{q^2}\right), \tag{1}$$

where $r = \sqrt{x^2+y^2}$ is the radial distance from the z-axis (i.e. we are using cylindrical coordinates r, ϕ, z), and v_0 and q are constants. In this problem we will study the orbits of stars in this potential.

(a) (2 points) For a star test particle of mass m, determine two constants of motion.

(b) (6 points) Find the location and period of stable circular orbits.

(c) (6 points) Now, consider orbits which are nearly circular $r(t) = r_{circ} + \delta r(t)$, $z = z_{circ} + \delta z(t)$. Determine the frequency of small oscillations in δr and δz. What do we mean by "small"? (That is, what are the criteria on δr and δz for this analysis to be valid?)

(d) (6 points) Our sun is at a distance of about 8 kpc (2.5×10^{17} km) from the center of the Milky Way galaxy, on an orbit that is approximately circular with a period of 225 million years. The sun undergoes vertical oscillations with an period of 87 million years, and radial oscillations with a period of 160 million years. Determine matter density profile. If you are uncertain of your results from (b) and (c) leave the result in terms of v_0 and q for partial credit. Newton's constant is $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}.

Solution

(a) (2 points) For a star test particle of mass m, identify and define two constants of motion.

The potential is time-independent so total energy is conserved and since it is azimuthally symmetric the z-component of the angular momentum is conserved. These are

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + m\Phi(r,z)\,, \tag{2}$$

$$L_z = mr^2\dot{\phi}\,. \tag{3}$$

(b) (6 points) To find the location and period of stable circular orbits, we note that

$$E = \frac{1}{2}m(\dot{r}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + m\Phi(r,z)\,, \tag{4}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{z}^2 + \frac{L_z^2}{2mr^2} + m\Phi(r,z)\,, \tag{5}$$

so that

$$V_{eff}(r,z) = \frac{L_z^2}{2mr^2} + m\Phi(r,z)\,. \tag{6}$$

Taking the first and second derivatives of the effective potential w.r.t. r and z, we see there is a minimum at

$$r_{circ} = \frac{L_z}{mv_0}\,, \quad z_{circ} = 0 \tag{7}$$

This corresponds to a circular orbit in the $z = 0$ plane. This means that for each value of L_z there is a unique stable circular orbit in the $z = 0$ plane of the galaxy, just like for Coulomb or Keplerian potentials $\sim 1/r$ (the location of the orbit for a given L_z will, of course, differ for those potentials). It's also worth noting that for the $z = 0$ orbit we've found here, the result is identical to what we'd have found if we started with spherically a symmetric potential ($q = 1$ in Eq. (1)) so from this single orbit, we can't learn about the asphericity of the potential.

The period of orbits is

$$\tau = \int \frac{d\phi}{\dot{\phi}} = \frac{2\pi m r_{circ}^2}{L_z} = \frac{2\pi L_z}{mv_0^2} = \frac{2\pi r_{circ}}{v_0}\,. \tag{8}$$

The coefficient v_0 in the potential is therefore the characteristic velocity of a particle on circular orbit (i.e. $v_{circ} \sim 2\pi r_{circ}/\tau = v_0$). Interestingly, in this potential the typical velocities of stars are roughly constant with radius. This is in contrast to the potential for a central mass distribution for which the circular velocity decreasing with increasing r (for $\Phi(r) = -GM/r$, $v_{circ} = \sqrt{GM/r}$).

(c) (6 points) Now, consider orbits which are nearly circular $r(t) = r_{circ} + \delta r(t)$, $z = z_{circ} + \delta z(t)$. Determine the frequency of small oscillations in δr and δz. What do we mean by "small"? (That is, what are the criteria on δr and δz for this analysis to be valid?)

The Euler-Lagrange equations give equations of motion

$$\ddot{r} - r\dot{\phi}^2 + \frac{\partial \Phi}{\partial r} = 0\,, \tag{9}$$

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0\,, \tag{10}$$

$$\ddot{z} + \frac{\partial \Phi}{\partial z} = 0\,. \tag{11}$$

Thus for near circular orbits we have,

$$\ddot{\delta r} + \left(3\frac{L_z}{m^2 r^4} + \frac{\partial^2 \Phi}{\partial r^2} + \frac{\partial^2 \Phi}{\partial r \partial z}\right)\Big|_{r=r_{circ},z=0} \delta r = 0\,, \tag{12}$$

$$\ddot{\delta z} + \left(\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial r \partial z}\right)\Big|_{r=r_{circ},z=0} \delta z = 0\,, \tag{13}$$

which reduces to

$$\ddot{\delta r} + \omega_r^2 \delta r = 0\,, \tag{14}$$

$$\ddot{\delta z} + \omega_z^2 \delta z = 0\,, \tag{15}$$

where

$$\omega_r = \sqrt{2}\frac{v_0}{r_{circ}}\,, \tag{16}$$

$$\omega_z = \frac{1}{q}\frac{v_0}{r_{circ}}. \tag{17}$$

We have dropped higher order terms in δr and δz. Rewriting the potential in terms of r_{circ}, δr, and δz, we see

$$\Phi(r,z) = \frac{1}{2}v_0^2 \ln\left(r_{circ}^2\left(1 + \frac{\delta r}{r_{circ}}\right)^2 + r_{circ}^2\frac{\delta z^2}{q^2 r_{circ}^2}\right)\,, \tag{18}$$

$$= \frac{1}{2}v_0^2\left(\ln\left(r_{circ}^2\right) + \ln\left(1 + 2\frac{\delta r}{r_{circ}} + \frac{\delta r^2}{r_{circ}^2} + \frac{\delta z^2}{q^2 r_{circ}^2}\right)\right)\,. \tag{19}$$

So, “small" oscillations in this case refers to oscillations with amplitudes $\delta r \ll r_{circ}$ and $\delta z \ll q r_{circ}$. The more squashed the disk is (i.e. the smaller q is, resulting in more rapid gradients of the potential with z), the stricter the requirement on δz in comparison to δr.

(d) (6 points) Our sun is at a distance of about 8 kpc (2.5×10^{17} km) from the center of the Milky Way galaxy, on an orbit that is approximately circular with a period of 225 million years. The sun undergoes vertical oscillations with a period $\approx$ 87 million years and radial oscillations with a $\approx$ 160 million year period. Determine matter density profile. If you are uncertain of your results from (b) and (c) leave the result in terms of v_0 and q for partial credit. Newton's constant is $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}.

Poisson's equation tells us

$$4\pi G\rho(r,z) = \nabla^2\Phi(r,z)\,, \tag{20}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{\partial^2\Phi}{\partial z^2}\,. \tag{21}$$

So that

$$\rho(r,z) = \frac{v_0^2}{4\pi G q^2}\left(\frac{r^2 + (2-1/q^2)z^2}{(r^2+z^2/q^2)^2}\right)\,. \tag{22}$$

From (b) and (c) we can determine the parameters of the gravitational potential.

$$v_0 = \frac{2\pi r_{circ}}{\tau} \approx 220\,\mathrm{km/s}\,, \tag{23}$$

which gives the normalization for the density profile of

$$\frac{v_0^2}{4\pi G r_{circ}^2} \approx 9.5\times10^{-22}\,\mathrm{kg/m^3} \approx 0.5\,\mathrm{GeV/cm^3}\,, \tag{24}$$

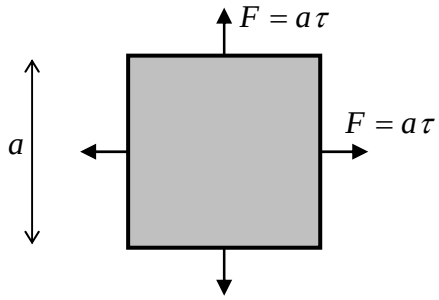
and

$$q = \frac{\omega_r}{\sqrt{2}\omega_z} \approx 0.4\,. \tag{25}$$

Classical Mechanics 3

Waves vs. oscillations

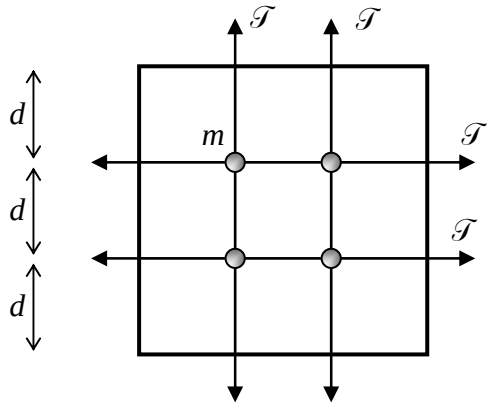
A uniform flexible membrane of area $a \times a$, with mass μ per unit area, is stretched on a thin plane frame, with an isotropic tension τ per unit length – see figure below.



A (6 points). Use any approach you like to derive the 2D wave equation describing small transverse displacements $z(x, y, t)$ of the membrane. (Here x and y are the coordinates in the plane's frame.)

B (3 points). Use the derived equation to calculate the frequency spectrum of standing waves in the system; sketch a few lowest wave modes.

Now consider a discrete-point analog of the system, with 4 particles of equal masses m, connected with light, flexible strings that are stretched with equal tensions $\mathcal{T}$ – see figure below. (Just as in the case of the membrane, the thin frame does not allow the string ends to deviate from its plane.)



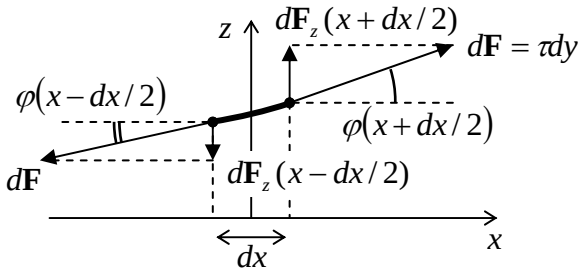
C (4 points). Use any approach you like to derive the system of equations describing small transverse oscillations $z_{i,j}(t)$ of the particles.

D (5 points). Use the derived system to calculate the modes and frequencies of the oscillations. (*Hint*: You may take clues from the system's symmetry and from the solution of Part B.)

E (2 points). Compare the calculated frequencies with those of the membrane quantitatively, and comment.

Solution

A (6 points). Let us consider the membrane that has deviated, in dynamics, from the frame's plane $z = 0$, by a small distance $z(x, y, t)$, and calculate the z-components of the forces $d\mathbf{F}$ exerted on a small rectangular fragment of the membrane, of area $dxdy << a^2$, by its adjacent parts. First let us consider the forces acting on two opposite edges of the fragment normal to the x-axis (both of length dy) within the plane $[x, z]$ – see Fig. below.



If z is not only small ($|z| << a$), but also changes significantly only on distances of the order of a, for the part of dF_z that is due to any of these forces we may write

$$\frac{d(F_z)_{[x,z]}}{dy} \approx \tau\varphi_x,$$

where τ is the membrane's tension (stretching force per unit width), and

$$\varphi_x \equiv \frac{\partial z}{\partial x}$$

is the (small) tilt of the membrane, within the $[x, z]$ plane – see Fig. above. This force, acting on one edge of our fragment of length dx, is nearly compensated by that acting on it opposite edge, so that the net force exerted on the fragment by these two forces is only due to the tilt gradient:

$$\frac{(dF_z)_{[x,z]}}{dy} \approx \tau\varphi_x\left(x+\frac{dx}{2}\right) - \tau\varphi_x\left(x-\frac{dx}{2}\right) = \tau\left[\frac{\partial z}{\partial x}\left(x+\frac{dx}{2}\right) - \frac{\partial z}{\partial x}\left(x-\frac{dx}{2}\right)\right].$$

If the deviation z is as sufficiently small and smooth as assumed, the expression in the square brackets may be well approximated as $(\partial^2 z/\partial x^2)dx$. Now adding a similar contribution from the forces acting, in the $[y, z]$ plane, on other two edges of our fragment, we get

$$dF_z = (dF_z)_{[x,z]} + (dF_z)_{[y,z]} = \tau\left(\frac{\partial^2 z}{\partial^2 x} + \frac{\partial^2 z}{\partial^2 y}\right)dxdy \equiv \tau\nabla^2_{x,y} z\, dxdy\,.$$

With this expression, the 2nd Newton law for the transverse motion of the membrane fragment, of mass $dm = \mu dxdy$, takes the form

$$\mu\frac{\partial^2 z}{\partial t^2} = \tau\nabla^2_{x,y} z. \qquad (1)$$

This is the standard 2D wave equation. Alternatively, it may be derived from the Lagrangian function of the system, calculated in the quadratic approximation in the function $z(x, y, t)$ and its partial derivatives.

B (2 points). In our case, equation (1) has to be solved with boundary condition $z|_{\text{on frame}} = 0$. Directing the x- and y-axes along the edges of the frame, with the origin in one of its corners, the boundary condition becomes

$$z = 0 \quad \text{at} \begin{cases} 0 \le x \le a, & \text{for } y = 0,\ a, \\ 0 \le y \le a, & \text{for } x = 0,\ a. \end{cases} \tag{2}$$

The boundary problem (1)-(2) may be readily solved by the variable separation method, i.e. by looking for the solution in the standard form

$$z(x, y, t) = \sum_{n,m} c_{n,m} z_{n,m}, \qquad \text{with } z_{n,m} = X_n(x) Y_m(y) T_{n,m}(t),$$

where $c_{n,m}$ are constant coefficients determined by initial conditions. Plugging the partial solution $z_{n,m}$ into Eq. (1), and dividing both sides of the equation by $\tau X_n Y_m T_{n,m}$, we get

$$\frac{\mu}{\tau} \frac{1}{T_{n,m}} \frac{d^2 T_{n,m}}{dt^2} = \frac{1}{X_n} \frac{d^2 X_n}{dx^2} + \frac{1}{Y_m} \frac{d^2 Y_m}{dy^2}. \tag{3}$$

This equality may hold for all x, y, and t only if each of its 3 terms is a constant. Calling these constants, respectively, k^2, k_x^2, and k_y^2, so that Eq. (3) becomes

$$k^2 = k_x^2 + k_y^2, \tag{4}$$

we get similar linear ordinary differential equations for the functions $T_{n,m}$, X_n, and Y_m, whose solutions are sinusoidal functions of, respectively, ωt, $k_x x$, and $k_y y$, where

$$\frac{\mu}{\tau} \omega^2 = k^2, \qquad \text{i.e. } \omega = kv, \qquad \text{where } v \equiv \left(\frac{\tau}{\mu} \right)^{1/2}. \tag{5}$$

The constant v has the physical sense of the velocity of (dispersion-free, isotropic) propagation of transverse waves on the membrane.

Now requiring all functions X_n and Y_m to satisfy the boundary conditions following from Eq. (2):

$$X_n(0) = X_n(a) = 0, \qquad Y_m(0) = Y_m(a) = 0,$$

we get the following spectra of the corresponding wave numbers:

$$k_x = \frac{\pi}{a} n, \quad k_y = \frac{\pi}{a} m, \qquad \text{with } n, m = 1, 2, ..., \tag{6}$$

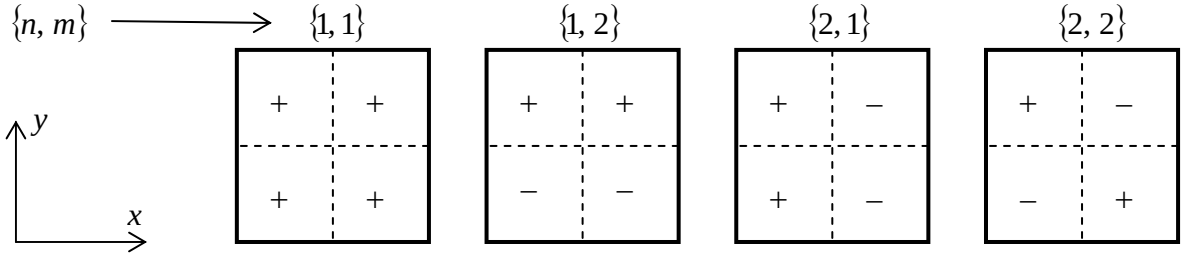
so that the partial solutions of the problem (each describing a specific “mode”, in our case a specific standing wave on the membrane) are

$$z_{n,m} = \sin \frac{\pi x n}{a} \sin \frac{\pi y m}{a} \cos(\omega_{n,m} t + \text{const}), \tag{7}$$

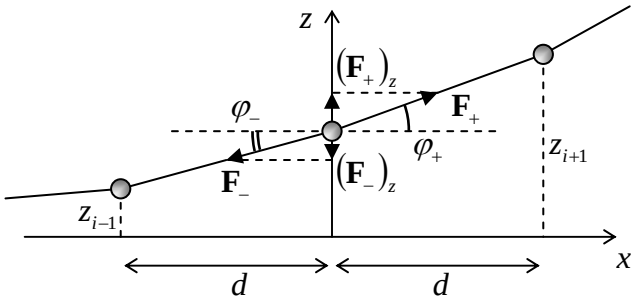
where $\omega_{n,m}$ is the frequency of the mode. The spectrum of these frequencies follows from Eqs. (4)-(6):

$$\omega_{n,m} = kv = \left(k_x^2 + k_y^2 \right)^{1/2} v = \omega_0 \left(n^2 + m^2 \right)^{1/2}, \qquad \text{where } \omega_0 \equiv \frac{\pi}{a} v = \frac{\pi}{a} \left(\frac{\tau}{\mu} \right)^{1/2}. \tag{8}$$

Figure below shows a sketch of the first four modes (7), in the order of their frequencies (8). (The signs show those of the functions $z_{n,m}$ in each of 4 quadrants, at a certain time instant.) Note that the modes {1, 2} and {2, 1} have the same frequency $\omega_{1,2} = \omega_{2,1} = \sqrt{5}\omega_0$, but differ by their orientation on the membrane.



C (4 points). The equation describing the motion of particles in the discrete version of the system may be derived similarly to that of the membrane. This analogy is more complete if we start not with the small, 4-particle system specified in the assignment, but with a large system of similar particles and strings, with the period d in each of the directions x, y. Figure below shows the diagram of forces exerted on an arbitrary internal particle (with number i in the x-direction, and number j in the y-direction) by the two x-oriented strings.



Similarly to what was done for the membrane, for small oscillations (with $|z| << d$) we may write

$$F_z = (F_+)_z - (F_-)_z \approx \mathcal{T}\varphi_+ - \mathcal{T}\varphi_- \approx \mathcal{T}\frac{z_{i+1,j} - z_{i,j}}{d} - \mathcal{T}\frac{z_{i,j} - z_{i-1,j}}{d} = \mathcal{T}\frac{z_{i+1,j} + z_{i-1,j} - 2z_{i,j}}{d}.$$

Now adding a similar contribution from the y-oriented string, we get the full transverse force exerted on the particle:

$$(F_z)_{i,j} = \mathcal{T}\frac{z_{i+1,j} + z_{i-1,j} + z_{i,j+1} + z_{i,j+1} - 4z_{i,j}}{d},$$

so the 2nd Newton law for it yields

$$m\frac{d^2 z_{i,j}}{dt^2} = \frac{\mathcal{T}}{d}\left(z_{i-1,j} + z_{i+1,j} + z_{i,j-1} + z_{i,j+1} - 4z_{i,j}\right). \qquad (9)$$

This equation has to be solved with the same boundary conditions $z(t)|_{\text{on frame}} = 0$ as for the membrane.

D (5 points). For a large-size system, solving the system of many mutually-coupled ordinary partial equations (9) is harder than solving the single partial differential equation (1). However, the small size and high symmetry of our system (of 4 particles) makes such a solution easy. Indeed, we may

consider the points where the strings contact the frame as additional particles, but with zero transverse displacements, so that they give zero contributions to the right-hand side of Eqs. (9) written for the adjacent internal particles. For example, for the top-right particle (assigning it the numbers $i = 1$, $j = 1$, with the natural sequential numbering of other particles) we may write

$$m\frac{d^2z_{1.1}}{dt^2} = \frac{\mathcal{T}}{d}\left(0 + z_{2,1} + 0 + z_{1,2} - 4z_{1,1}\right), \tag{10}$$

with the similar simplifications for other 3 particles.

Generally, plugging into the resulting system of 4 linear ordinary differential equations the oscillatory solution $z_{i,j} = c_{i,j}\exp\{i\omega t\}$, we get a system of 4 linear algebraic homogeneous equations for 4 distribution coefficients $c_{i,j}$, whose condition of consistency is

$$\begin{vmatrix} -4+\lambda & 1 & 1 & 0 \\ 1 & -4+\lambda & 0 & 1 \\ 1 & 0 & -4+\lambda & 1 \\ 0 & 1 & 1 & -4+\lambda \end{vmatrix} = 0, \qquad \text{where } \lambda \equiv \frac{m\omega^2}{\mathcal{T}/d}.$$

However, instead of trying to solve this equation, in our particular (highly symmetric) case, it is easier (as was suggested in the *Hint*) to assume that the 4 different oscillation modes of the discrete-particle system[1] have the same symmetry as the lowest modes of the membrane oscillations, sketched above. This means the following relations between the particle oscillations:

$$\text{mode } \{1, 1\}: \quad z_{11} = z_{12} = z_{21} = z_{22}, \tag{11}$$

$$\text{mode } \{1, 2\}: \quad z_{11} = -z_{12} = z_{21} = -z_{22}, \tag{12}$$

$$\text{mode } \{2, 1\}: \quad z_{11} = z_{12} = -z_{21} = -z_{22},$$

$$\text{mode } \{2, 2\}: \quad z_{11} = -z_{12} = -z_{21} = z_{22}.$$

Due to this symmetry, the corresponding oscillation frequencies may be calculated using just one equation, for example, Eq. (10) for $z_{1,1}$. For example, with the substitution of z_{12} and z_{21} from Eq. (11) for the {1, 1} mode, we get simply

$$m\frac{d^2z_{1.1}}{dt^2} = \frac{\mathcal{T}}{d}\left(0 + z_{1,1} + 0 + z_{1,1} - 4z_{1,1}\right), \qquad \text{i.e. } m\frac{d^2z_{1.1}}{dt^2} + 2\frac{\mathcal{T}}{d}z_{1,1} = 0.$$

This is just the usual equation of a harmonic oscillator with frequency

$$\omega_{11} = \sqrt{2}\omega_0, \qquad \text{where } \omega_0 \equiv \left(\frac{\mathcal{T}}{md}\right)^{1/2}. \tag{13}$$

Very similarly, using Eq. (12) for the {1, 2} mode, we may reduce the same Eq. (10) to

$$m\frac{d^2z_{1.1}}{dt^2} = \frac{\mathcal{T}}{d}\left(0 + z_{1,1} + 0 - z_{1,1} - 4z_{1,1}\right), \qquad \text{i.e. } m\frac{d^2z_{1.1}}{dt^2} + 4\frac{\mathcal{T}}{d}z_{1,1} = 0.$$

[1] As a reminder, the general theory of oscillations in a coupled linear system of N particles tells us that it may have only N normal modes.

The similar calculation for the mode {2, 1} yields the same final equation for z_{11}, so that the frequencies of this two modes are equal (just as they are for the membrane):

$$\omega_{12} = \omega_{21} = 2\omega_0 . \tag{14}$$

Finally, for the {2, 2} mode we get

$$m\frac{d^2 z_{1.1}}{dt^2} = \frac{\mathcal{T}}{d}\left(0 - z_{1,1} + 0 - z_{1,1} - 4z_{1,1}\right), \qquad \text{i.e. } m\frac{d^2 z_{1.1}}{dt^2} + 6\frac{\mathcal{T}}{d} z_{1,1} = 0 ,$$

giving the highest frequency of the spectrum:

$$\omega_{22} = \sqrt{6}\omega_0 . \tag{15}$$

E (2 points). The first way to compare the results for the continuous system (the membrane) and the system of the discrete particles is to look at the ratio of their frequencies. From Eqs. (8) and (13)-(15), we have:

$$\omega_{11} / \omega_{12} / \omega_{21} / \omega_{22} = \left(\sqrt{2} / \sqrt{5} / \sqrt{5} / \sqrt{8}\right)_{\text{continuous}} \quad \text{vs} \quad \left(\sqrt{2} / \sqrt{4} / \sqrt{4} / \sqrt{6}\right)_{\text{discrete}} .$$

So the ratios are quite close, with a somewhat larger deviation for the highest mode {2, 2}.

Moreover, looking at the systems, we may assume that they should have closest properties at the following relations of their parameters:

$$a = 3d, \quad \tau = \frac{\mathcal{T}}{d}, \quad \mu = \frac{m}{d^2}.$$

Plugging these relations into Eq. (8), we get

$$\omega_0 = \frac{\pi}{3}\left(\frac{\mathcal{T}}{md}\right)^{1/2} \approx 1.047\left(\frac{\mathcal{T}}{md}\right)^{1/2}_{\text{continuous}} \quad \text{vs } 1.000\left(\frac{\mathcal{T}}{md}\right)^{1/2}_{\text{discrete}} .$$

So, the difference of the values of this frequency scale (and hence of the fundamental frequencies $\omega_{1,1} = \sqrt{2}\omega_0$) in the two systems is below 5%.

Electromagnetism 1

A cylinder and a line charge

A line charge with linear charge density λ is placed at a distance R and parallel to a conducting cylinder of radius b ($R > b$), which is grounded.

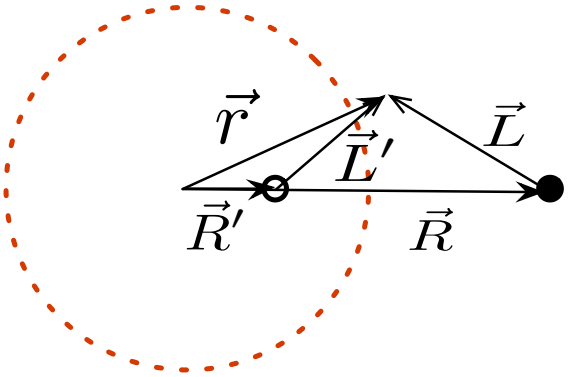
a. 8 points Determine the electrostatic potential at any point and discuss its limits.

b. 8 points Determine the charge per area $\sigma(\theta)$ induced on the surface of the cylinder. For $R/b = 2, 4$, evaluate and sketch $\sigma(\theta)$ as function of the azimuthal angle θ.

c. 4 points Determine the force per unit length on the line charge. Explain its behavior as the line charge approaches the cylinder.

Solution

a. We show below a planar configuration of the set up with the line charge at R and its harmonic image at R', i.e. $RR' = b^2$. The geometry is shown below.



The total potential at point $\vec{r}$ is then reduced to two line charges

$$V(\vec{r}) = \varphi + \varphi' + V_0 = -2(\lambda \,\mathrm{Log}\, L + \lambda' \mathrm{Log}\, L') + V_0\,, \tag{1}$$

with $\lambda' = -\lambda$ and $\vec{r} = \vec{L}' + \vec{R}' = \vec{R} + \vec{L}$. We can fix V_0 by noting that when L, L' are on the cylinder, then $L/L' = R/b$, and the potential vanishes if $V_0 = 2\lambda \,\mathrm{Log}\,(R/b)$. As a result, (1) reads

$$V(\vec{r}) = 2\lambda \,\mathrm{Log}\left(\frac{L'R}{Lb}\right)\,. \tag{2}$$

For $\vec{r} = r(\cos\theta, \sin\theta)$, $\vec{R} = (R, 0)$, and $\vec{R}' = (b^2/R, 0)$, we can find the potential more explicitly

$$V(\vec{r}) = \lambda \,\mathrm{Log}\left(\frac{r^2 + b^2(b/R)^2 - 2rR(b/R)^2\cos\theta}{r^2 + R^2 - 2rR\cos\theta}\right) + V_0\,. \tag{3}$$

For $L' \approx L$, the potential is about constant with $V(\vec{r}) \approx V_0$. For $L, L' \gg R \gg b$, we have $L' \simeq r$ and $\vec{L} \simeq \vec{r} - \vec{R}$, and then the potential is dipole-like in the plane $V(\vec{r}) = 2\lambda R\cos(\theta)/r + V_0$ with $\vec{L}' \approx \vec{r}$.

b. The surface density induced on the cylinder by the line charge is ($\cos\theta = \hat{r}\cdot\hat{R}$) given by the boundary conditions $4\pi\sigma = \boldsymbol{n}\cdot(\boldsymbol{E}_{\rm out} - \boldsymbol{E}_{\rm in})$:

$$\sigma(\theta) = -\frac{1}{4\pi}\frac{\partial}{\partial r}\left(2\lambda \mathrm{Log}\left(\frac{L'R}{Lb}\right)\right)\Big|_{r=b} = \frac{\lambda}{2\pi b}\left(\frac{1 - R^2/b^2}{1 + R^2/b^2 - 2R/b\cos(\theta)}\right)\,. \tag{4}$$

More explicitly, we have

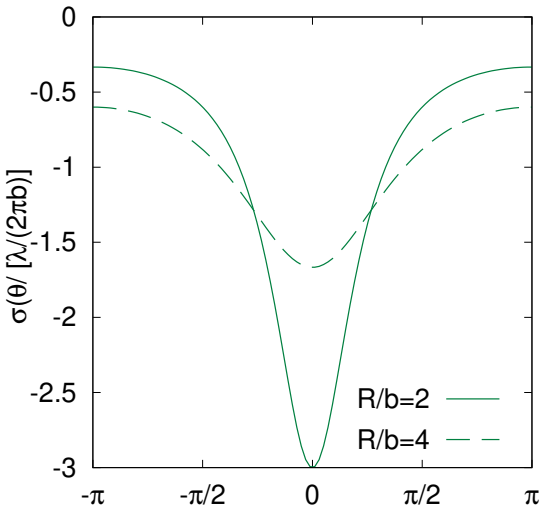


Figure 1: The charge per unit area $\sigma(\theta)$ given by Eq. (5).

$$\sigma(\theta) = -\frac{\lambda}{2\pi b}\left(\frac{3}{5-4\cos(\theta)}\right), \qquad \frac{R}{b} = 2\,,$$
$$\sigma(\theta) = -\frac{\lambda}{2\pi b}\left(\frac{15}{17-8\cos(\theta)}\right), \qquad \frac{R}{b} = 4\,, \tag{5}$$

which are sketched below in Fig. 1.

c. The electric field has two pieces $\vec{E} = \vec{E}_0(\vec{r}) + \vec{E}_{\rm ind}(\vec{r})$, where $\vec{E}_0 = -2\lambda \mathrm{Log}\, L$ is the Coulomb field of the original line of charge, and $\vec{E}_{\rm ind}$ is the induced field (the field of the image charge). The induced force between the polarized cylinder and the line of charge is

$$\vec{f} = \lambda\, \vec{E}_{\rm ind}\Big|_{\vec{r}=\vec{R}} = \lambda\left(\frac{-2\lambda}{R-R'}\right)\hat{R}\,, \tag{6}$$
$$= -\frac{2\lambda^2 \vec{R}}{R^2-b^2}\,, \tag{7}$$

and is manifestly attractive. As the line charge approaches the cylinder, the polarization increases and so does the attraction, explaining the singular behavior of (7) as $R \to b$.

Electromagnetism 2

Radiation from a current sheet:

(a) (5 points) The retarded Green function of the $1+1$ dimensional wave equation dimensional wave equation is defined as the solution to

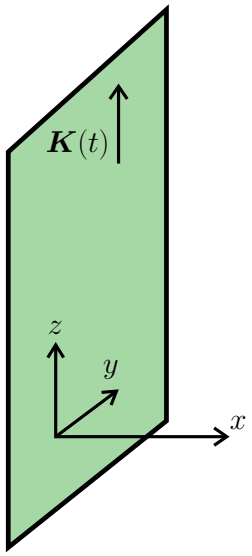
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) G(tx|t'x') = \delta(t-t')\delta(x-x')\,. \tag{1}$$

Show that the retarded Green function is

$$G(tx|t'x') = \frac{c}{2}\,\theta\left(t-t'+(x-x')/c\right)\,\theta(t-t'-(x-x')/c)\,. \tag{2}$$

Does $G(tx|t'x')$ satisfy the appropriate boundary conditions? Explain.

Now consider an infinite sheet spanning the yz plane, with uniform surface current, $\boldsymbol{K}(t) = K(t)\,\hat{\boldsymbol{z}}$.



(b) (5 points) Determine the gauge potentials $\varphi(t,x)$ and $\boldsymbol{A}(t,x)$ in the Lorenz gauge (without approximation) to the right and left of the sheet. Check that the Lorenz gauge condition is satisfied.

(c) (5 points) Determine the electric and magnetic fields, and the energy radiated per unit time and area by the current sheet.

(d) (5 points) The electric and magnetic fields close to the sheet can be calculated using the elementary methods of Ampere's and Faraday's laws. (i) For $x > 0$, use these methods to calculate the magnetic field $\boldsymbol{B}(t,x)$ and the electric field difference, $\Delta\boldsymbol{E} \equiv \boldsymbol{E}(t,x) - \boldsymbol{E}(t,0)$, close to the sheet. (ii) Show that the general results of (c) agree with these calculations in the appropriate limit. (iii) How close to the sheet does one need to be in order for the elementary methods to be applicable? Assume that the time-dependent currents are characterized by a time scale τ.

Solution:

(a) Writing

$$x^+ = \frac{x^0 + x^1}{\sqrt{2}}\,, \qquad x^- = \frac{x^0 - x^1}{\sqrt{2}}\,, \tag{3}$$

where $x^0 = ct$ and the inverse relations

$$x^0 = \frac{x^+ + x^-}{\sqrt{2}}, \tag{4}$$

$$x^1 = \frac{x^+ - x^-}{\sqrt{2}}, \tag{5}$$

we have

$$\frac{\partial}{\partial x^+} = \frac{\partial x^0}{\partial x^+}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x^+}\frac{\partial}{\partial x^1}\,, \tag{6}$$

$$= \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^0} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^1}\,, \tag{7}$$

and similarly

$$\frac{\partial}{\partial x^-} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^0} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^1}\,. \tag{8}$$

Thus

$$2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} = \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2\,, \tag{9}$$

is the wave operator. From here it is straightforward to verify that

$$\left(2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-}\right)\frac{c}{2}\theta(x^+)\theta(x^-) = c\delta(x^+)\delta(x^-) = \delta(t)\delta(x)\,. \tag{10}$$

Further $c/2\theta(x^+)\theta(x^-)$ has support only in the forward light cone, and therefore satisfies the required retarded boundary conditions. The full Green function is

$$G(tx|t'x') = \frac{c}{2}\,\theta\left(t - t' + (x - x')/c\right)\,\theta(t - t' - (x - x')/c)\,. \tag{11}$$

(b) The wave equation in three dimenisons reads

$$-\,\Box\boldsymbol{A} = \frac{\boldsymbol{j}}{c}\,, \tag{12}$$

and so for problem at hand (the current sheet) we have

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)A^z = \frac{K(t)}{c}\delta(x)\,. \tag{13}$$

So a formal for solution for $A^z(t,x)$ can be written

$$A^z(t,x) = \int dt'dx'\,G(tx|t'x')\,\frac{K(t')}{c}\delta(x')\,. \tag{14}$$

For $x > 0$ we have

$$A^z(t,x) = \int dt' \frac{c}{2}\, \theta(t - t' + x/c)\, \theta(t - t' - x/c) K(t')\,, \tag{15}$$

$$= \frac{c}{2} \int_{-\infty}^{t-x/c} dt' \frac{K(t')}{c}\,, \tag{16}$$

while for $x < 0$ we have

$$A^z(t,x) = \int dt' \frac{c}{2}\, \theta(t - t' + x/c)\, \theta(t - t' - x/c) K(t')\,, \tag{17}$$

$$= \frac{c}{2} \int_{-\infty}^{t+x/c} dt' \frac{K(t')}{c}\,. \tag{18}$$

The scalar potential φ is zero since the charge density is zero at all times. The gauge condition is then obviously satisfied

$$\frac{1}{c}\partial_t \varphi + \nabla \cdot \boldsymbol{A} = \partial_z A^z = 0\,. \tag{19}$$

(c) To the right of the sheet $x > 0$, we find

$$B^y = -\frac{\partial A^z}{\partial x} = +\frac{K(t - x/c)}{2c}\,, \tag{20}$$

$$E^z = -\frac{1}{c}\frac{\partial A^z}{\partial t} = -\frac{K(t - x/c)}{2c}\,. \tag{21}$$

We can compute the Poynting vector at $x = 0^+$:

$$\boldsymbol{S} = c\boldsymbol{E} \times \boldsymbol{B} = -\, cE^z B^y\, \hat{\boldsymbol{x}}\,, \tag{22}$$

$$= c \left(\frac{K(t)}{2c}\right)^2 \hat{\boldsymbol{x}}\,. \tag{23}$$

To the left of the sheet $x < 0$, the result is similar

$$B^y = -\frac{\partial A^z}{\partial x} = -\frac{K(t - |x|/c)}{2c}\,, \tag{24}$$

$$E^z = -\frac{1}{c}\frac{\partial A^z}{\partial t} = -\frac{K(t - |x|/c)}{2c}\,, \tag{25}$$

and the Poynting vector at $x = 0^-$ is

$$\boldsymbol{S} = c\boldsymbol{E} \times \boldsymbol{B} = -\, cE^z B^y\, \hat{\boldsymbol{x}}\,, \tag{26}$$

$$= c \left(\frac{K(t)}{2c}\right)^2 (-\hat{\boldsymbol{x}})\,. \tag{27}$$

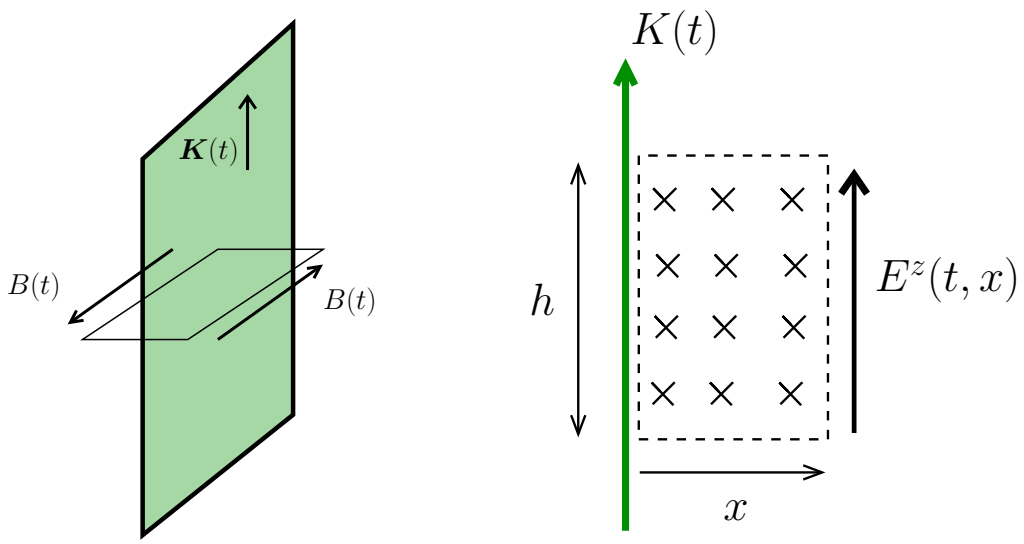


Figure 1: (a) Amperian loop used to determine the magnetic field. (b) Loop used to determine the electric field using Faraday's law.

The energy radiated per time and area $\mathcal{A}$ is found by integrating over both sides of the sheet yielding a factor of two, and the energy radiated per area per time is

$$\frac{1}{\mathcal{A}} \oint \boldsymbol{S} \cdot d\boldsymbol{a} = \frac{1}{2c} K^2(t) \,. \tag{28}$$

(d) (i) Finally we can use the elementary results to determine the electric and magnetic fields. Drawing an Amperian loop of length ℓ as shown in Fig. 1(a) we find

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \frac{I}{c} \,, \tag{29}$$

$$2B(t)\ell = \frac{K(t)}{c} \ell \,. \tag{30}$$

Reinserting the direction of the magnetic field we find

$$\boldsymbol{B}(t) = \begin{cases} \frac{K(t)}{2c} \hat{\boldsymbol{y}} & x > 0 \\ -\frac{K(t)}{2c} \hat{\boldsymbol{y}} & x < 0 \end{cases} . \tag{31}$$

One can also immediately check that the boundary conditions are satisfied:

$$\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}(t)}{c} \,, \qquad \text{or} \qquad B^y|_{x=0^+} - B^y|_{x=0^-} = \frac{K(t)}{c} \,. \tag{32}$$

To determine the induced electric field we use Faraday's Law in integral form

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{1}{c} \partial_t \int \boldsymbol{B} \cdot d\boldsymbol{a} \,. \tag{33}$$

Drawing a surface as shown in Fig. 1(b) for $x > 0$ we find

$$(E^z(t,x) - E^z(t,0))h = +\frac{1}{c}\partial_t B^y(t) h x \,, \tag{34}$$

where we have recognized that the normal to the surface is in the negative y direction for a loop as drawn in Fig. 1(b), and thus $\boldsymbol{B} \cdot d\boldsymbol{a} < 0$. Thus,

$$\Delta E^z(t,x) = \frac{1}{2c^2}\partial_t K(t) x \qquad x > 0 \tag{35}$$

A similar loop to the left of the wire yields (with careful attention to signs) the full result

$$\Delta E^z(t,x) = \frac{1}{2c^2}\partial_t K(t) |x| \,. \tag{36}$$

(ii) This agrees with the expansion of part (c) for $|x|/c$ smal compared to τ:

$$B^y \simeq \pm \frac{K(t)}{2c} \,, \tag{37a}$$

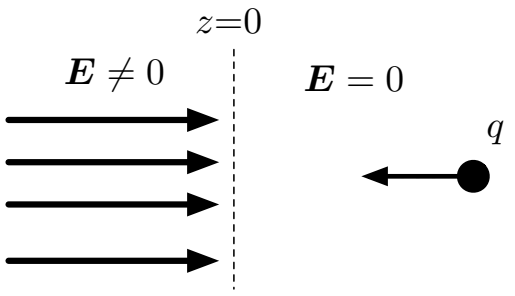
$$E^z(t,x) \simeq -\frac{K(t)}{2c} + \frac{|x|}{2c^2}\partial_t K(t) \,. \tag{37b}$$

(iii) For the expansion in Eq. (37) to make sense we must have $|x| < c\tau$.

Electromagnetism 3

Radiation during linear acceleration

(a) (6 points) An ultra-relativistic ($\gamma \gg 1$) positively charged particle of charge q and mass m is traveling with velocity $v_0 \equiv c \tanh y_0$ in the negative z direction from positive infinity as shown below. At $z = 0$ the particle enters a semi-infinite region ($z < 0$) of homogeneous electric field directed in the positive z direction, $\boldsymbol{E} = E\,\hat{\boldsymbol{z}}$.



(i) Determine the particle's position $z(\tau)$ as a function of proper time τ when the particle is in the electric field.

(ii) Determine how long (in the laboratory frame) the particle remains in the electric field.

(b) (6 points) Determine the total energy radiated by the particle as it accelerates in the electric field.

(c) (2 points) At what angle(s) relative to the z-axis is the radiation peaked? Explain.

(d) To determine the radiation of low frequency photons, the particle's acceleration may be treated with an impulsive approximation

$$\boldsymbol{a}(t) = 2v_0\,\hat{\boldsymbol{z}}\,\delta(t)\,. \tag{1}$$

(i) (4 points) For what range of frequencies is the impulsive approximation valid? Explain.

(ii) (2 points) Use dimensional and physical reasoning to deduce the dependence on ω of the distribution of radiated photons per unit frequency $\mathrm{d}N/\mathrm{d}\omega$ (at low frequency).

Solution:

(a.i) We will set $c = 1$ and record the relativistic Equation of Motion (EOM):

$$\frac{du^\mu}{d\tau} = \frac{q}{m} F^\mu{}_\nu u^\nu \,, \tag{2}$$

where $u^\mu = (\gamma, \gamma v)$ is the four velocity. Since the only non-vanishing component is $F^z{}_0 = E$ the EOM reads

$$\frac{du^z}{d\tau} = \frac{q}{m} E u^0 \,. \tag{3}$$

Parametrizing the velocity by the rapidity $v = \tanh(y)$ so that

$$u^z = \sinh(y) \,, \qquad u^0 = \cosh(y) \;, \tag{4}$$

the EOM now reads

$$\frac{dy}{d\tau} = \frac{q}{m} E \,. \tag{5}$$

Integrating we find

$$y = \alpha\tau \,, \qquad \alpha \equiv \frac{q}{m} E \,, \tag{6}$$

where we have chosen the integration constants so that the particle reaches zero velocity (rapidity) at $\tau = 0$. The particle has rapidity $y(\tau) = \pm y_0$ at times

$$\tau_\pm \equiv \pm \frac{y_0}{\alpha} \,, \tag{7}$$

and thus τ_+ and τ_- are the proper times that the particle exits and enters the electric field respectively.

To find the z coordinate we integrate the velocity

$$\frac{dz}{d\tau} = u^z \,, \tag{8}$$

using $u^z = \sinh(y)$ to find

$$z(\tau) = \int d\tau \sinh(\alpha\tau) \,, \tag{9}$$

$$= \frac{1}{\alpha} \cosh(\alpha\tau) + \text{const} \,. \tag{10}$$

Since at $\tau = \tau_\pm$ the particle is at $z = 0$, we adjust the integration constant to arrive at our final result.

$$z = \frac{1}{\alpha} \left(\cosh(\alpha\tau) - \cosh(\alpha\tau_+) \right) \;. \tag{11}$$

(a.ii) Similarly, we may determine the time as a function of proper time. Using

$$\frac{dt}{d\tau} = u^0 \,, \tag{12}$$

and $u^0 = \cosh(\alpha\tau)$, we find

$$t = \int d\tau \cosh(\alpha\tau)\,, \tag{13}$$

$$= \frac{1}{\alpha}\sinh(\alpha\tau) + \text{const}\,. \tag{14}$$

The integration constant is arbitrary and defines $t = 0$. The time that elapses between when the particle enters the electric field and when the particle exits the electric field is

$$t_{\text{in}} = t(\tau_+) - t(\tau_-)\,, \tag{15}$$

$$= \frac{2}{\alpha}\sinh(\alpha\tau_+)\,. \tag{16}$$

(b) We will determine the acceleration of the particle and then integrate the relativistic Larmour formula to determine the radiation. The total radiated per retarded time is a Lorentz invariant

$$\frac{dW}{dt} = \frac{q^2}{4\pi c^3}\frac{2}{3}\mathcal{A}_\mu\mathcal{A}^\mu\,, \tag{17}$$

where

$$\mathcal{A}^\mu = \frac{d^2x^\mu}{d\tau^2}\,, \tag{18}$$

is the proper acceleration.

Since

$$x^\mu = \frac{1}{\alpha}(\sinh(\alpha\tau) + \text{const}, \cosh(\alpha\tau) + \text{const})\,, \tag{19}$$

the proper acceleration is

$$\mathcal{A}^\mu = \alpha(\sinh(\alpha\tau), \cosh(\alpha\tau))\,, \tag{20}$$

and the proper acceleration squared is

$$\mathcal{A}_\mu\mathcal{A}^\mu = \alpha^2\,. \tag{21}$$

We used the familiar identity, $\cosh^2 - \sinh^2 = 1$. Integrating over time we find the total radiated power is proportional to t_{in}.

$$W = \int dt\frac{dW}{dt}\,, \tag{22}$$

$$= \int_{\tau_-}^{\tau^+} d\tau\frac{dt}{d\tau}\frac{e^2}{4\pi c^3}\frac{2}{3}\alpha^2\,, \tag{23}$$

$$= \frac{q^2}{4\pi c^3}\frac{2}{3}\alpha^2\left[t(\tau_+) - t(\tau_-)\right]\,, \tag{24}$$

$$= \frac{q^2}{4\pi c^3}\frac{2}{3}\alpha^2 t_{\text{in}}\,. \tag{25}$$

(c) The intent of this question is to understand the qualitative features of soft bremsstrahlung, which accompanies all collisions at high energies. For an impulsive collision with change in velocity from $\boldsymbol{v}_1 \to \boldsymbol{v}_2$ the energy radiated per frequency is

$$\omega \frac{dN}{d\omega d\Omega} \propto \left| \frac{\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{\beta}_2}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}_2} - \frac{\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{\beta}_1}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}_1} \right|^2 . \tag{26}$$

where for the problem at hand $\boldsymbol{\beta}_2 = v\hat{\boldsymbol{z}}$ while $\boldsymbol{\beta}_1 = -v\hat{\boldsymbol{z}}$. The important feature is the appearance of the "antenna" like factors

$$\frac{\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{\beta}}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}} . \tag{27}$$

Such factors produce radiation which is highly collimated in the direction of of motion. This means that the radiation will be peaked near $\boldsymbol{\beta}_2$ and $\boldsymbol{\beta}_1$, which corresponds to $\theta = 0$ (for $\boldsymbol{\beta}_2$) and $\theta = \pi$ (for $\boldsymbol{\beta}_1$) for the problem at hand.

(d.i) When the frequency ω is much smaller than t_{in}^{-1} the emitted radiation can not resolve the acceleration process, and perceives the acceleration as happening instantaneously.

(d.ii) Another important feature of Eq. (26) is that

$$\frac{dN}{d\omega} \propto \frac{1}{\omega} , \tag{28}$$

which says that the energy per frequency $\mathrm{d}W/\mathrm{d}\omega = \omega\, \mathrm{d}N/\mathrm{d}\omega$ is independent of frequency. This feature can be "derived" by recognizing that the $\mathrm{d}N/\mathrm{d}\omega$ has units of time, and $1/\omega$ is the only quantity with units of time in the impulsive approximation.

To summarize the results of (c) and (d) which highlight the generic features of soft bremsstrahlung, Fig. 1 shows a schematic plot of $\mathrm{d}N/\mathrm{d}\cos\theta$ and $\mathrm{d}N/\mathrm{d}\omega$

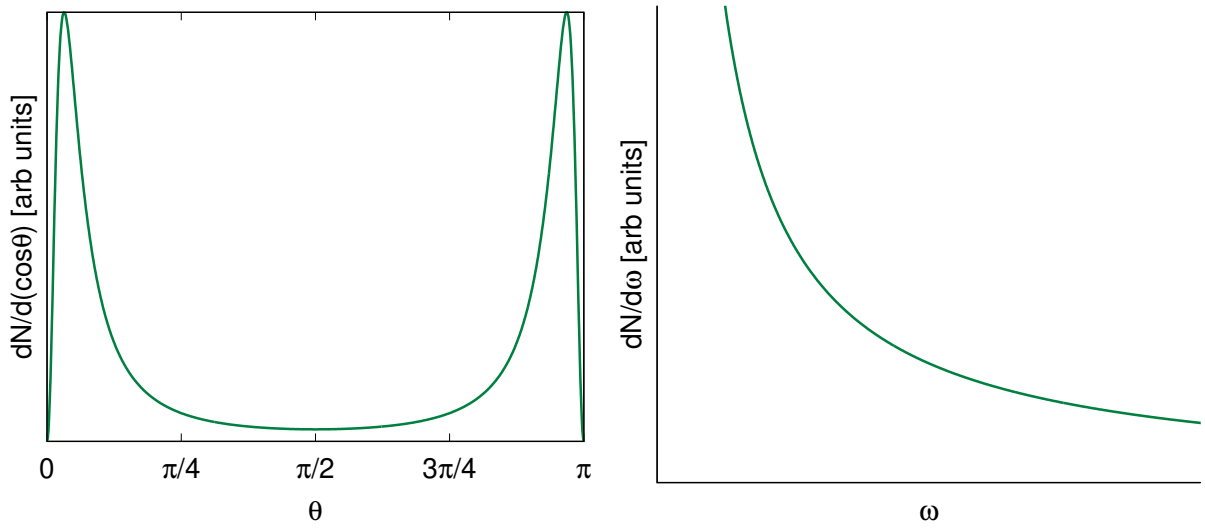


Figure 1: Two graphs showing the qualitative features of soft bremsstrahlung accompanying high energy collisions: (a) the angular dependence, and (b) the frequency dependence $dN/d\omega$,

Quantum Mechanics 1

An electron in E&M fields

An electron is confined to move within the $[x, y]$ plane. An external magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ is applied. Neglect spin.

(a) (2 points) What are the stationary values (i.e. eigenvalues) of the electron's energy?
Hint: work in the Landau gauge where $\mathbf{A} = Bx\hat{\mathbf{y}}$.

(b) (2 points) What is the ground state wavefunction (ignoring normalization)?

(c) (2 points) Show that the ground state degeneracy scales with the area of the system and the magnetic field.

Now an electric field is added, $\mathbf{E} = \mathcal{E}\hat{\mathbf{x}}$. This corresponds to a new term in the Hamiltonian, $H_{\text{elec}} = -e\mathcal{E}x$.

(d) (3 points) What are the stationary values of the electron's energy now?

(e) (3 points) How does the electric field change the ground-state wavefunctions?

(f) (3 points) Derive an expression for the probability current density, $\langle \mathbf{j}(x, y) \rangle$, evaluated in these states.

(g) (2 points) Evaluate the current density of part (f) at the point (x, y) where the wave function squared $|\psi(x, y)|^2$ is maximal.

(h) (1 point) Give a physical interpretation of the last result.

Solution

(a) In the Landau gauge $A_y = Bx$,

$$H = \frac{1}{2m}\left(p_x^2 + (p_y - eBx)^2\right) . \tag{1}$$

Since momentum along y is a good quantum number, we can adopt the ansatz for eigenstates $\psi_{k_y,n}(x) = e^{ik_y y}\phi_n(x)$, where $\phi_n(x)$ satisfies:

$$\frac{1}{2m}\left(p_x^2 + (\hbar k_y - eBx)^2\right)\phi_n(x) = E_{k_y,n}\phi_n(x) . \tag{2}$$

It is useful to define

$$x_0 \equiv \frac{\hbar k_y}{eB}, \tag{3}$$

so that Eq. (2) simplifies to:

$$\frac{1}{2m}\left(p_x^2 + (eB)^2(x - x_0)^2\right)\phi_n(x) = E_{k_y,n}\phi_n(x) . \tag{4}$$

This is exactly a a quantum harmonic oscillator. Its energies are given by:

$$E_{k_y,n} = \hbar\omega\left(n + \frac{1}{2}\right), \ \omega \equiv \frac{eB}{m} . \tag{5}$$

(b) Utilizing Eq. (4), the ground state wavefunction is a shifted harmonic oscillator wavefunction:

$$\psi_{k_y,n}(x) = e^{ik_y y}e^{-m\omega(x-x_0)^2/(2\hbar)}, \tag{6}$$

where ω is defined in Eq. (5) and x_0 is defined in Eq. (3).

(c) Assume periodic boundary conditions with dimensions L_x, L_y. The eigenvalues in Eq. (5) are independent of k_y. Thus, the ground state degeneracy is exactly equal to the number of possible k_y. The finite size in the y direction restricts k_y to the quantized values $2\pi m/L_y$ for integer m. Using the definition of x_0 in Eq. (3), the spacing between the states indexed by k_y and $k_y + 2\pi/L_y$ is $h/(eBL_y)$. Thus, in a finite sample, there will be eBL_xL_y/h states. This shows that the ground states degeneracy scales linearly with both the area of the system $(L_x \cdot L_y)$ and the magnetic field.

(d) Now we consider an electric field. Since momentum along y remains a good quantum number, we can maintain our ansatz for eigenstates, $\psi_{k_y,n}(x) = e^{ik_y y}\phi_n(x)$. With the addition of H_{elec}, $\phi_n(x)$ satisfies:

$$\frac{1}{2m}\left(p_x^2 + (eB)^2(x - x_0)^2 - 2me\mathcal{E}x\right)\phi_n(x) = \left(E_{k_y,n} + \Delta_{k_y,n}\right)\phi_n(x), \tag{7}$$

where $E_{k_y,n}$ is defined in Eq. (5). Completing the square yields:

$$\frac{1}{2m}\left(p_x^2 + (eB)^2\left((x - x_0 - x_1)^2 - 2x_0x_1 - x_1^2\right)\right)\phi_n(x) = \left(E_{k_y,n} + \Delta_{k_y,n}\right)\phi_n(x), \tag{8}$$

where we have defined

$$x_1 \equiv \frac{\mathcal{E}}{B\omega}\,. \tag{9}$$

Eq. (8) is a harmonic oscillator equation with an energy offset. It has the same characteristic frequency as before the electric field is turned on. Thus, it is evident that

$$\Delta_{k_y,n} = -e\mathcal{E}\left(x_0 - \frac{x_1}{2}\right), \tag{10}$$

which depends on k_y but not on n. The exact eigenspectrum is thus,

$$\hbar\omega\left(n + \frac{1}{2}\right) - e\mathcal{E}(x_0 - \frac{x_1}{2})\,, \tag{11}$$

where ω is the same as the case without an electric field (Eq. (5)) and x_0 and x_1 are defined in Eqs. (3) and (9).

(e) The eigenstates are again a shifted harmonic oscillator wavefunction:

$$\psi_{k_y,n}(x) = e^{ik_y y} e^{-m\omega(x-x_0-x_1)^2/(2\hbar)}\,. \tag{12}$$

(f) In the presence of an electromagnetic potential, the probability current, $\mathbf{j}$ is given by:

$$\mathbf{j} = \frac{1}{2m}\left(-i\hbar(\psi^*\nabla\psi - \psi\nabla\psi^*) - 2e\mathbf{A}|\psi|^2\right)\,. \tag{13}$$

Evaluating this equation for the eigenstate $\psi_{k_y,n}$ in Eq. (12) yields:

$$\mathbf{j}_{k_y,n} = \frac{1}{m}\left(\hbar k_y - eBx\right)\hat{\mathbf{y}}\,. \tag{14}$$

(g) When $\mathbf{E} = \mathcal{E}\hat{\mathbf{x}}$, the wavefunction $\psi_{k_y,n}$ is centered at $x_0 + x_1$ (from Eq. (12).) Plugging into Eq. (14) yields:

$$\mathbf{j}_{k_y,n} = -\frac{\mathcal{E}}{B}\,. \tag{15}$$

(h) Physically, we interpret this result as the classical drift velocity.

Quantum Mechanics 2

Orbital and spin dynamics in magnetic field

A. In this part of the problem, we consider a spinless particle with mass m and electric charge q, moving in an external magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$.

A.1 (4 points). Prove that the expression $\hat{\mathbf{v}} \equiv \left(\hat{\mathbf{p}} - q\mathbf{A}\right)/m$ is the legitimate operator of particle's velocity in the sense that $\hat{\mathbf{v}} = d\hat{\mathbf{r}}/dt$, where $\mathbf{r}$ is its radius-vector. Derive the commutation relations between the Cartesian components of the vector-operator $\hat{\mathbf{v}}$, in terms of the components of the vector $\mathbf{B}$.

A.2 (4 points). Prove that

$$m\frac{d\hat{\mathbf{v}}}{dt} = q\frac{\hat{\mathbf{v}}\times\mathbf{B} - \mathbf{B}\times\hat{\mathbf{v}}}{2}.$$

Discuss the relation between this equality and the classical expression for the Lorentz force.

B. In this part of the problem, we ignore the orbital motion of the particle, but take into account its spin-½, with a gyromagnetic ratio $\gamma \neq 0$.

One of the key components of the currently developed quantum information technology is the so-called *Hadamard gate* – a system that acts on a *qubit*, i.e. of a two-level quantum system, performing the following transformation of its basis states - in this field, traditionally called 0 and 1:

$$\hat{\mathcal{H}}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \qquad \hat{\mathcal{H}}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{1}$$

B.1 (8 points). Identifying the state 0 with the spin-½ fully polarized along the z-axis, and the state 1 with its opposite polarization, prove that the Hadamard transform may be implemented by applying to the particle a constant magnetic field $\mathbf{B}$ with the following Cartesian components:

$$B_x = \frac{B}{\sqrt{2}}, \qquad B_y = 0, \qquad B_z = \frac{B}{\sqrt{2}},$$

for a certain time interval t, and calculate this interval.

B.2 (4 points). Calculate $\langle\mathbf{S}(t)\rangle$ for the time evolution of the spin vector-operator $\hat{\mathbf{S}}$ in this field for the initial state 0. Give a geometric description of this evolution, and mark the trajectories describing the particular transformations (1).

Solution

A. 1). The velocity operator is the time-derivative of the position operator $\hat{\boldsymbol{r}}$, which can be calculated using Ehrenfest Theorem:

$$\frac{d}{dt}\hat{\boldsymbol{r}} = -\frac{i}{\hbar}\langle[\hat{\boldsymbol{r}}, \hat{H}]\rangle = -\frac{i}{2m\hbar}[\hat{\boldsymbol{r}}, (\hat{\boldsymbol{p}} - e\boldsymbol{A})^2]$$

$$= -\frac{i}{2m\hbar}\{[\hat{\boldsymbol{r}}, (\hat{\boldsymbol{p}} - e\boldsymbol{A})](\hat{\boldsymbol{p}} - e\boldsymbol{A}) + (\hat{\boldsymbol{p}} - e\boldsymbol{A})[\hat{r}, (\hat{\boldsymbol{p}} - e\boldsymbol{A})]\}$$

$$= \frac{1}{m}(\hat{\boldsymbol{p}} - e\boldsymbol{A})$$

The commutation relation of the components of the velocity operators is then:

$$[\hat{v}_i, \hat{v}_j] = \frac{1}{m^2}[(\hat{p}_i - qA_i), (\hat{p}_j - qA_j)]$$

$$= \frac{e}{m^2}([\hat{p}_j, A_i] - [\hat{p}_i, A_j]) = \frac{iq\hbar}{m^2}\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right) = \frac{iq\hbar}{m^2}\mathcal{E}_{i,j,k}B_k$$

2). We can calculate the time derivative of the velocity operator, again, using Ehrenfest Theorem. Here we express the Hamiltonian in terms of the velocity operator: $\hat{H} = \frac{(\hat{\boldsymbol{p}} - e\boldsymbol{A})^2}{2m} = \frac{m\hat{\boldsymbol{v}}^2}{2m}$. Then

$$m\frac{d}{dt}\hat{\boldsymbol{v}} = -\frac{im}{\hbar}[\hat{\boldsymbol{v}}, \hat{H}] = -\frac{im^2}{2\hbar}[\hat{\boldsymbol{v}}, \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{v}}]$$

$$= -\frac{im^2}{2\hbar}\left[\sum_i \hat{v}_i\hat{\boldsymbol{e}}_{\boldsymbol{i}}, \sum_i \hat{v}_i^2\right] = -\frac{im^2}{2\hbar}\sum_{i,j}[\hat{v}_i, \hat{v}_j^2]\,\hat{\boldsymbol{e}}_{\boldsymbol{i}}$$

$$= -\frac{im^2}{2\hbar}\sum_{i,j}\{\hat{v}_j[\hat{v}_i, \hat{v}_j] + [\hat{v}_i, \hat{v}_j]\hat{v}_j\}\,\hat{\boldsymbol{e}}_{\boldsymbol{i}}$$

Using result from 1), we get:

$$m\frac{d}{dt}\hat{\boldsymbol{v}} = \frac{q}{2}\sum_{i,j}\{\mathcal{E}_{i,j,k}\hat{v}_jB_k + \mathcal{E}_{i,j,k}B_k\hat{v}_j\}\,\hat{\boldsymbol{e}}_{\boldsymbol{i}} = \frac{q}{2}(\hat{\boldsymbol{v}} \times \boldsymbol{B} - \boldsymbol{B} \times \hat{\boldsymbol{v}})$$

B. 1). The Hamiltonian of the spin in B field is:

$$\hat{H} = -\gamma\boldsymbol{B} \cdot \boldsymbol{S} = -\gamma\frac{B}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

We can write down the Schrodinger equation:

$$i\hbar\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = -\gamma\frac{B}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Which yields eigenvalues of $\pm\gamma B/\hbar$, and normalized eigenvectors of $\left(1\pm\sqrt{2},1\right)/\sqrt{4\pm2\sqrt{2}}$. Thus

$$\psi_1 = \alpha\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}}e^{-i\frac{B\gamma}{\hbar}t} + \beta\frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}}e^{i\frac{B\gamma}{\hbar}t}$$

$$\psi_2 = \alpha\frac{1}{\sqrt{4+2\sqrt{2}}}e^{-i\frac{B\gamma}{\hbar}t} + \beta\frac{1}{\sqrt{4-2\sqrt{2}}}e^{i\frac{B\gamma}{\hbar}t}$$

Applying initial condition $\psi_1(0)=1$ and $\psi_2(0)=1$, we get

$$\psi_1 = \frac{1+\sqrt{2}}{2\sqrt{2}}e^{-i\frac{B\gamma}{\hbar}t} - \frac{1-\sqrt{2}}{2\sqrt{2}}e^{i\frac{B\gamma}{\hbar}t}$$

$$\psi_2 = \frac{1}{2\sqrt{2}}e^{-i\frac{B\gamma}{\hbar}t} - \frac{1}{2\sqrt{2}}e^{i\frac{B\gamma}{\hbar}t}$$

To realize Hadamard gating, we need $\psi_1=\psi_2$. If we rewrite the wave functions as:

$$\psi_1 = \cos\left(\frac{B\gamma}{\hbar}t\right) - \frac{i}{\sqrt{2}}\sin\left(\frac{B\gamma}{\hbar}t\right)$$

$$\psi_2 = -\frac{i}{\sqrt{2}}\sin\left(\frac{B\gamma}{\hbar}t\right)$$

we see that at $t=\frac{\pi\hbar}{2B\gamma}$, the cos term vanishes, and $\Psi\sim\frac{|0\rangle+|1\rangle}{\sqrt{2}}$. This achieves the functionality of a Hadamard gate.

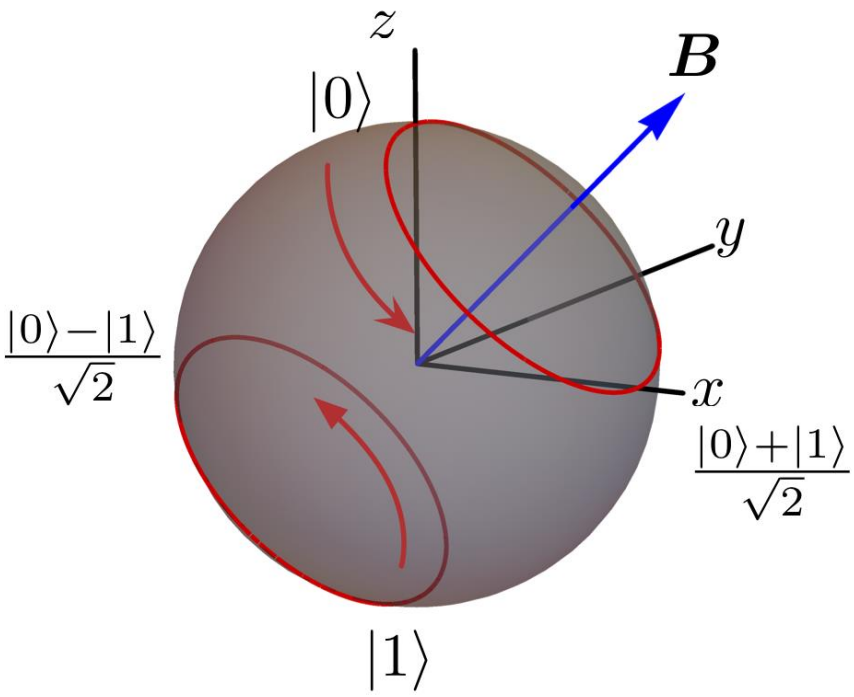
2). Let $\omega\equiv\frac{B\gamma}{\hbar}$. We can find out the expectation values of the x,y,z projections of the spin, as a function of time:

$$S_x(t) = \frac{\hbar}{2}(\psi_1^*,\psi_2^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{\hbar}{2}\sin^2(\omega t)$$

$$S_y(t) = \frac{\hbar}{2}(\psi_1^*,\psi_2^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{-\hbar}{2}\sqrt{2}\sin(\omega t)\cos(\omega t)$$

$$S_z(t) = \frac{\hbar}{2}(\psi_1^*,\psi_2^*)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{\hbar}{2}\cos^2(\omega t)$$

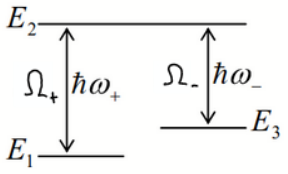
The rotation of the spin is depicted by the figure below.



The Hadamard operation rotates the spin from pointing to the z-axis to point to the x-axis. Similarly, the Hadamard operation also rotates the spin from pointing to the negative z-axis to point to the negative x-axis.

Quantum Mechanics 3

Dynamics of a driven 3-level atom



Consider a three-level atom with non-degenerate eigenstates $|\psi_j\rangle$ and energies E_j ($j = 1, 2, 3$) as shown in the figure (set $E_1 = 0$). The system is subject to weak resonant couplings $\hat{H}'_\pm(t) = \hbar\Omega_\pm(t)\cos\omega_\pm t$ between $|\psi_1\rangle$ and $|\psi_2\rangle$, and between $|\psi_3\rangle$ and $|\psi_2\rangle$, with $\omega_+ = (E_2 - E_1)/\hbar$ and $\omega_- = (E_2 - E_3)/\hbar$ respectively. The coupling amplitudes $\Omega_\pm(t)$ are adiabatically varying in time, and can be assumed to be real. Neglect any direct coupling between $|\psi_1\rangle$ and $|\psi_3\rangle$.

(a) **Equation of motion [5pts].** Use time-dependent perturbation theory to find the equations of motion for the amplitudes $a_j(t)$ in the evolution of $|\psi(t)\rangle = \sum_{j=1}^{3} a_j(t)|\psi_j\rangle$, and write these equations out in matrix form.

(b) **Rotating-wave Hamitonian [8pts].** (1) Re-write the equations of motion in terms of new amplitudes $b_1 = a_1$, $b_2 = a_2\exp(i\omega_+ t)$, $b_3 = a_3\exp(i[\omega_+ - \omega_-]t)$, neglecting any remaining oscillating terms. Why is this approximation justified? (2) Find the eigenenergies of the resulting effective Hamiltonian, and show that one of its three eigenstates, the so-called "dark state", has amplitudes $(\cos\theta, 0, -\sin\theta)$ in the new effective basis, where $\tan\theta = \Omega_+/\Omega_-$. Why does the term "dark state" make sense?

(c) **Sequential state transfer [3pts].** Suppose that you want to transfer the state of the system from $|\psi(0)\rangle = |\psi_1\rangle$ to $|\psi(\infty)\rangle = |\psi_3\rangle$. One way to do this is to first apply Ω_+ (with Ω_-=0) in order to perform a state rotation $|\psi_1\rangle \to |\psi_2\rangle$, and to then apply Ω_- (with Ω_+=0) for $|\psi_2\rangle \to |\psi_3\rangle$. Assuming that both $\Omega_\pm(t)$ are square pulses with amplitude Ω, solve the equation of motion for each state rotation and find the pulse duration τ necessary for making the transfer complete.

(d) **Adiabatic state transfer [4pts].** The existence of the dark state can be used to smoothly transfer the state of the system from $|\psi(0)\rangle = |\psi_1\rangle$ to $|\psi(\infty)\rangle = |\psi_3\rangle$ without going through $|\psi_2\rangle$. Explain why and how this can be done, and qualitatively describe suitable pulse profiles $\Omega_\pm(t)$. What can you say about the temporal ordering and overlap of the two pulses? Which of the two methods is more robust in view of possible fluctuations of the pulse parameters?

Solution

(a) **Equation of motion [5pts].** *Solution.* First write the total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}'(t)$ with $\hat{H}_0\psi_j = E_j\psi_j$. Solve Schrödinger's equation $i\hbar|\psi\rangle = \hat{H}|\psi\rangle$ for $|\psi(t)\rangle = \sum_{j=1}^{3} a_j(t)|\psi_j\rangle$, and use orthogonality $\langle\psi_j|\psi_k\rangle = \delta_{jk}$ to obtain

$$\hbar\frac{da_j}{dt} = E_j a_j(t) + \sum_k \langle\psi_j|H'(t)|\psi_k\rangle a_k(t)\,, \tag{1}$$

where the interaction matrix elements vanish except for the state pairs shown in the figure. In matrix form

$$i\hbar\,\frac{d}{dt}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & H_+(t) & 0 \\ H_+(t) & E_2 & H_-(t) \\ 0 & H_-(t) & E_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\,. \tag{2}$$

(b) **Rotating-wave Hamitonian [8pts].** *Solution.* Using the transformations ("rotating-wave transformation"), expressing $\cos\omega_\pm t$ in exponential form, and neglecting oscillatory time dependencies $\propto e^{i2\omega_\pm t}$ ("rotating-wave approximation") yields, after some algebra,

$$i\hbar\dot{b}_2 + \hbar\omega_+ b_2 = \frac{\hbar\Omega_+}{2}\left(e^{i2\omega_+ t} + 1\right) + E_2 b_2 + \frac{\hbar\Omega_-}{2}\left(e^{i2\omega_- t} + 1\right)\,, \tag{3}$$

and structurally similar equations for b_1 and b_3. Neglecting oscillating terms, this can be written as

$$i\hbar\,\frac{d}{dt}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 & \Omega_+ & 0 \\ \Omega_+ & 0 & \Omega_- \\ 0 & \Omega_- & 0 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\,. \tag{4}$$

The couplings $\Omega_\pm$, which determine the time evolution of the b_j, are much smaller than the splittings $\omega_\pm$, such that a cycle averaging is justified, and oscillatory terms can be neglected.

Eigenvalues and dark eigenstate follow from simple linear algebra. Looking for the partial solution of this system of three homogeneous linear differential equations in the form $\exp(iEt/\hbar)$ leads to eigenvalues

$$E_0 = 0, E_\pm = \pm\frac{\hbar}{2}\sqrt{A_-^2 + A_+^2}\,, \tag{5}$$

and eigenvectors

$$\left(-\frac{\Omega_-}{\Omega_+}, 0, 1\right), \quad \ldots, \quad \ldots\,, \tag{6}$$

the first of which has the suggested form after normalization. The term "dark state" is motivated by the fact that the state has no contribution from the upper excited level and is thus never subject to decay (and thus: the possibility of detection via scattered light).

(c) **Sequential state transfer [3pts].** *Solution.* The equation of motion for the first step can be reduced to

$$i\hbar \frac{d}{dt}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} . \tag{7}$$

Differentiation and substitution leads to $\ddot{b}_1 = -(\Omega/2)^2 b_1$ and $\ddot{b}_2 = -(\Omega/2)^2 b_2$, or $b_1 = \cos(\Omega t/2)$ and $b_2 = \sin(\Omega t/2)$ with the proper initial conditions. Therefore $\tau = \pi/\Omega$ ("Rabi π-pulse condition"). The same for the second step.

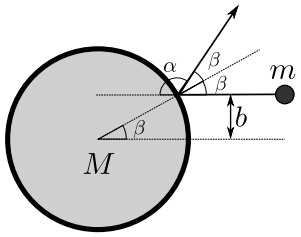
(d) **Adiabatic state transfer [4pts].** *Solution.* To transfer the population, note that $|\psi_1\rangle$ coincides with the dark state for $\theta = 0$. This means that $0 = \arctan\theta = \Omega_+/\Omega_-$, which can be fulfilled for $\Omega_+ \equiv 0$ while $\Omega_- \neq 0$. Conversely, $|\psi_3\rangle$ coincides with the dark state for $\theta = \pi/2$ or $\arctan\theta = \infty$, which means $\Omega_- \equiv 0$ while $\Omega_+ \neq 0$. This suggests the following pulse sequence ("STIRAP" or"stimulated rapid adiabatic passage"): Start with a pulse that couples the $2 \to 3$ transition (Ω_-), and then add a pulse that couples the $1 \to 2$ transition (Ω_+), i.e. a sequence that is is OPPOSITE ("counterintuituve") to the Rabi pulse sequence. Note that the pulses need to partially overlap, since once cannot have $\Omega_- = 0$ before Ω_+ is already at a finite value. There is no hard condition on the pulse area, duration, and exact timing; as long as one is able to completely extinguish the couplings at the beginning and end of the sequence, the transfer is complete.

Statistical Mechanics 1

Brownian motion

The goal of this problem is to explore several aspects of the Brownian motion of a classical particle.

In this model, the particle is a hard sphere of a radius R and mass M, which is hit by molecules of an ideal classical gas at temperature T. The size r and mass m of the gas molecules are negligible compared to those of the particle. Assume also that all the molecular collisions with the particle are elastic and specular (i.e. without any angular momentum transfer).



First, assume that the particle is initially at rest.

(A) [4pt] Calculate the *hit rate*, i.e., the number of collisions of the molecules with the particle per unit time.

(B) [4pt] Calculate the *average energy transferred* to the particle in one collision.
Now, assume that the particle has initial velocity some $\mathbf{V}$ with respect to the gas, which is much slower than the r.m.s. velocity of the molecules.

(C) [5pt] Calculate the *drag coefficient* η defined by the relation

$$\langle \mathbf{F} \rangle = -\eta \mathbf{V} \,, \tag{1}$$

where $\mathbf{F}$ is the effective force resulting from molecular hits.

Hint: For the calculation, you may want to consider how the Maxwell's distribution is modified in the reference frame of the particle.

(D) [4pt] Use the Langevin equation describing the molecular hits (but no other forces)

$$M\dot{\mathbf{V}} = \mathbf{F} = \langle \mathbf{F} \rangle + \widetilde{\mathbf{F}} = -\eta\, \mathbf{V} + \widetilde{\mathbf{F}} \,, \quad \text{where } \langle \widetilde{\mathbf{F}} \rangle = \langle (\mathbf{F} - \langle \mathbf{F} \rangle) \rangle = 0 \,, \tag{2}$$

and the results from (A,B,C) to calculate the *time dependence* of the average kinetic energy of the particle and its *equilibrium value*. Compare the latter to the equipartition theorem.

(E) [3pt] Use your results to estimate the molecular hit rate and the energy relaxation time for a 1-micron dust particle in air at room temperature.

Solution

(A) In central collisions, the deflection angle of a molecule is determined by the impact parameter

$$b = R\sin\beta = R\cos\frac{\alpha}{2} \tag{3}$$

The momentum transferred to the particle in a single collision is

$$\Delta\mathbf{p} = \Big(mv\big(1-\cos\alpha\big), mv\sin\alpha\Big) = 2mv\sin\frac{\alpha}{2}\,\Big(\sin\frac{\alpha}{2}, \cos\frac{\alpha}{2}\Big) \tag{4}$$

and the squared momentum transfer is

$$\big(\Delta\mathbf{p}\big)^2 = 4m^2v^2\sin^2\frac{\alpha}{2} = 4m^2v^2\Big(1-\frac{b^2}{R^2}\Big)\,, \tag{5}$$

In Part (B), we will also need the momentum transfer averaged over collisions with the same $|\mathbf{b}| = b$; it is parallel to the initial velocity $\mathbf{v}$:

$$\Delta\mathbf{p} = 2m\mathbf{v}\sin^2\frac{\alpha}{2} = 2m\mathbf{v}\Big(1-\frac{b^2}{R^2}\Big). \tag{6}$$

The Maxwell's distribution for the ideal gas at rest (normalized to one particle) is isotropic,

$$dn = nf_0(\mathbf{v})\,d^3\mathbf{v}\,, \quad f(\mathbf{v}) = C\exp\big[-\frac{mv^2}{2T}\big], \quad C^{-1} = \int d^3\mathbf{v}\,e^{-\frac{mv^2}{2T}} = \Big[\frac{2\pi T}{m}\Big]^{3/2}. \tag{7}$$

where $n = N/V$ is the average concentration (particle density), N is the total number of particles, and V is the total volume.

The total number of collisions per second is given by the integral over the velocity distribution

$$\frac{dN_{\rm coll}}{dt} = \int d^3\mathbf{v}\,f_0(\mathbf{v})\,n|\mathbf{v}|\int_0^R 2\pi b\,db\,, \tag{8}$$

where $n|\mathbf{v}|$ is the particle flux and $\int_0^R 2\pi b\,db$ is the integral over the 2D impact space. Although in this case the latter it is trivial and equal to πR^2, we have written explicitly because this representation will be useful later. Computing the integral with the Maxwell's distribution yields the rate of collisions

$$\frac{dN_{\rm coll}}{dt} = \pi R^2 nC\int_0^\infty dv\,4\pi\,v^3\,e^{-\frac{mv^2}{2T}} = 2nR^2\Big[\frac{2\pi T}{m}\Big]^{1/2}. \tag{9}$$

(B) Since all the collisions are independent, we can calculate the total squared momentum $(\Delta\mathbf{P})^2$ transferred to the particle in $N_{\rm coll}$ collisions as

$$\big\langle\big(\Delta\mathbf{P}_{\rm coll}\big)^2\big\rangle = \Big\langle\Big(\sum_{\rm coll}\Delta\mathbf{p}\Big)^2\Big\rangle = \Big\langle\sum_{\rm coll}\big(\Delta\mathbf{p}\big)^2\Big\rangle = N_{\rm coll}\big\langle\big(\Delta\mathbf{p}\big)^2\big\rangle\,. \tag{10}$$

In order to compute this quantity, we have to integrate the squared momentum transfer $(\Delta p)^2$ (5) over the impact parameter b (3). Otherwise similar to part A, the calculation of the mean squared momentum transfer per unit time can be expressed as

$$\frac{d}{dt}\langle(\Delta\mathbf{P}_{\rm coll})^2\rangle = \int d^3\mathbf{v}\, f_0(\mathbf{v})\, n|\mathbf{v}| \int_0^R 2\pi b\, db \,\cdot\, (\Delta\mathbf{p})^2\,, \tag{11}$$

and computing the integrals yields

$$\frac{d}{dt}\langle(\Delta\mathbf{P}_{\rm coll})^2\rangle = 4m^2nC\int 4\pi\, dv\, v^5\, e^{-\frac{mv^2}{2T}} \int_0^R 2\pi\, db\, b\left(1-\frac{b^2}{R^2}\right) = \frac{8}{\pi}m^2nR^2\Big[\frac{2\pi T}{m}\Big]^{3/2}. \tag{12}$$

Thus, the mean squared momentum transfer in a single collision is

$$\langle(\Delta\mathbf{p})^2\rangle = \frac{d\langle(\Delta\mathbf{P}_{\rm coll})^2\rangle/dt}{dN_{\rm coll}/dt} = 8mT\,. \tag{13}$$

and the corresponding energy transfer is

$$\langle\Delta\varepsilon_{\rm coll}\rangle = \frac{\langle(\Delta p)^2\rangle}{2M} = \frac{4m}{M}T\,. \tag{14}$$

(C) Computing the momentum transferred to the moving particle is similar to the calculations in Part (B). It is convenient to work in the reference frame of the particle, in which the gas velocity distribution is shifted by $(-\mathbf{V})$:

$$f(\mathbf{v}') = f_0(\mathbf{v}) = f_0(\mathbf{v}'+\mathbf{V}) \approx f_0(\mathbf{v}') + \frac{df_0}{d|\mathbf{v}|}\frac{d|\mathbf{v}|}{d\mathbf{V}}\cdot\mathbf{V} = f_0(\mathbf{v}')\Big[1-\frac{m\mathbf{v}'\cdot\mathbf{V}}{T}\Big]\,, \tag{15}$$

where $\mathbf{v}'$ is the molecule's velosity relative to the particle. With the expression for the transferred momentum (6), the total momentum transfer per second (aka force) can be computed as

$$\begin{aligned}
\frac{d\mathbf{P}_{\rm coll}}{dt} &= \int d^3\mathbf{v}'\, f(\mathbf{v}')\, n|\mathbf{v}'| \int_0^R 2\pi b\, db \,\cdot\, \Delta\mathbf{p} \\
&= \int d^3\mathbf{v}'\, f_0(\mathbf{v}')\Big[-\frac{m\mathbf{v}'\cdot\mathbf{V}}{T}\Big]\, n|\mathbf{v}'| \int_0^R 2\pi b\, db \,\cdot\, 2m\mathbf{v}'\Big(1-\frac{b^2}{R^2}\Big) \\
&= \Big(-\frac{2m^2nC}{T}\Big)\int d^3\mathbf{v}'\, |v'|e^{-\frac{mv'^2}{2T}}\, (\mathbf{v}'\cdot\mathbf{V})\mathbf{v}' \int_0^R 2\pi\, db\, b\left(1-\frac{b^2}{R^2}\right) \\
&= \Big(-\frac{2m^2nC}{3T}\,\mathbf{V}\Big)\int 4\pi\, dv'\, v'^5\, e^{-\frac{mv'^2}{2T}} \int_0^R 2\pi\, db\, b\left(1-\frac{b^2}{R^2}\right) \\
&= \frac{4}{3\pi}\,\frac{m^2nR^2}{T}\Big[\frac{2\pi T}{m}\Big]^{3/2}\,(-\mathbf{V})\,,
\end{aligned} \tag{16}$$

In the second line, the velocity-independent term was dropped and only the $\propto f_0\mathbf{V}$ term was kept due to rotational symmetry, which implies that the net force is zero for a particle at rest ($\mathbf{V}=0$). In the fourth line, the angular averaging $\langle(\mathbf{v}'\cdot\mathbf{V})\mathbf{v}'\rangle = \frac{1}{3}\langle v'^2\rangle\mathbf{V}$ was used.

It is instructive to observe the relation between the drag force coefficient

$$\eta = \frac{4}{3\pi}\frac{m^2 n R^2}{T}\Big[\frac{2\pi T}{m}\Big]^{3/2}. \tag{17}$$

and the the mean squared momentum transfer to the particle per unit time,

$$\frac{d\langle(\Delta\mathbf{P}_{\rm coll})^2\rangle/dt}{\eta} = 6T \tag{18}$$

which is a manifestation of the fluctuation-dissipation theorem (FDT).
(D) Rewriting the Langevin equation in terms of the particles' momentum, we get

$$\dot{\mathbf{P}} = -\frac{\eta}{M}\mathbf{P} + \widetilde{\mathbf{F}} \tag{19}$$

The the first term on the r.h.s. is the damping of the particle's momentum, while the second term induces fluctuation of its momentum and kinetic energy due to molecular collisions. The damping of the particle's kinetic energy is then

$$\Big[\frac{d}{dt}\langle E\rangle\Big]_{\rm damping} = \frac{1}{2M}\cdot 2\langle\mathbf{P}\cdot\dot{\mathbf{P}}\rangle = -\frac{2\eta}{M}\frac{1}{2M}\langle\mathbf{P}^2\rangle = -\frac{2\eta}{M}\langle E\rangle\,, \tag{20}$$

while the effect of the second term in Eq. (19) has been computed in Part (B). The full equation for $\langle E\rangle$ is

$$\frac{d}{dt}\langle E\rangle = -\frac{2\eta}{M}\langle E\rangle + \frac{1}{2M}\frac{d\langle(\Delta\mathbf{P}_{\rm coll})^2\rangle}{dt} \tag{21}$$

where the last term is the intermediated result (12) from Part (B). The stationary state is reached when $d\langle E\rangle/dt = 0$, and from the r.h.s. we find that the equilibrium value is

$$\langle E\rangle_{\rm eq} = \frac{1}{4}\frac{d\langle(\Delta\mathbf{P}_{\rm coll})^2/dt}{\eta} = \frac{3}{2}T\,, \tag{22}$$

which, of course, is consistent with the equipartition theorem. The relaxation time for the energy is $\tau = M/(2\eta)$, and the equilibrium value is reached by the exponential law

$$\langle E(t)\rangle = \langle E\rangle_{\rm eq} + \big(E_0 - \langle E\rangle_{\rm eq}\big)e^{-\frac{2\eta t}{M}}\,, \tag{23}$$

where E_0 is the initial kinetic energy of the particle.
(E) The molecular mass of the air is

$$m \approx \mu m_p = 29\cdot 1.67\cdot 10^{-24}\,\mathrm{g} \approx 4.8\cdot 10^{-23}\mathrm{g}\,, \tag{24}$$

where $\mu \approx 29$ is the molar mass of the air and $m_p \approx 1.67\cdot 10^{-24}$g is the proton mass. The concentration (number density) can be easily estimated as

$$n = \frac{p}{T} \approx \frac{10^5\mathrm{Pa}}{1.38\cdot 10^{-23}\mathrm{J/K}\cdot 300\mathrm{K}} \approx 2.4\cdot 10^{25}\mathrm{m}^{-3} = 2.4\cdot 10^{19}\mathrm{cm}^{-3} \tag{25}$$

and the "thermal velocity"

$$v_T = \sqrt{\frac{2\pi T}{m}} \approx \sqrt{\frac{2 \cdot 3.14 \cdot 1.38 \cdot e^{-16}\mathrm{erg/K} \cdot 300\mathrm{K}}{4.8 \cdot 10^{-23}\mathrm{g}}} \approx 7.4 \cdot 10^4 \mathrm{cm/s} \tag{26}$$

For a particle with radius $R = 0.5 \cdot 10^{-4}$cm, the hit rate is

$$\frac{dN_{\mathrm{coll}}}{dt} = 2nR^2 v_T \approx 2 \cdot 2.4 \cdot 10^{19}\mathrm{cm}^{-3} \cdot 0.25 \cdot 10^{-8}\mathrm{cm}^2 \cdot 7.4 \cdot 10^4 \mathrm{cm/s} \approx 0.9 \cdot 10^{16}\mathrm{s}^{-1}\,. \tag{27}$$

The drag coefficient is computed in a similar fashion

$$\eta = \frac{4}{3\pi}\frac{m^2 n R^2}{T}\left[\frac{2\pi T}{m}\right]^{3/2} = \frac{8}{3} m n R^2 v_T \approx 5.7 \cdot 10^{-7}\,\mathrm{g/s}\,, \tag{28}$$

and, assuming that the density of the particle is $\rho = 2\,\mathrm{g/cm}^3$, its mass is

$$M = \frac{4\pi}{3} R^3 \rho \approx \frac{4 \cdot 3.14}{3} \cdot 0.125 \cdot 10^{-12}\mathrm{cm}^3 \cdot 2\mathrm{g/cm}^3 \approx 1.05 \cdot 10^{-12}\,\mathrm{g}\,, \tag{29}$$

resulting in the energy relaxation rate

$$\tau = \frac{M}{2\eta} \approx \frac{1.05 \cdot 10^{-12}\mathrm{g}}{2 \cdot 5.7 \cdot 10^{-7}\,\mathrm{g/s}} \approx 0.92 \cdot 10^{-6}\mathrm{s}\,. \tag{30}$$

Statistical Mechanics 2

2D and 3D gases in equilibrium

A closed volume V, with inner wall surface of area A, contains $N >> 1$ similar, non-relativistic particles of mass m each. Any of the particles may be either moving freely inside the volume, as a component of a 3D classical gas, with degeneracy g_V of each orbital state, or condense on the inner walls, where it can also move freely as a component of a 2D gas – also classical, with a generally different degeneracy g_A of each orbital state. The condensation of a particle releases energy Δ.

A (3 points). Using any statistical approach you like, calculate the average number N_3 of particles in the 3D gas, as a function of its chemical potential μ and temperature T.

B (5 points). Perform a similar calculation of the average number N_2 of particles condensed on the surface, taking into account the 2D character of their motion.

C (2 points). Assuming that N_2, $N_3 >> 1$, use the conditions of thermal and chemical equilibrium of the 3D and 2D phases, and the results obtained in tasks A and B, to derive a system of algebraic equations relating these numbers.

D (3 points). Solve the obtained system of equations to calculate the chemical potential μ of the system, the number of particles in the 3D gas, and its pressure P, as explicit functions of N and T.

E (4 points). Analyze the results in detail. In particular, simplify them for very low and very high values of the ratio $k_B T/\Delta$, and sketch the function $P(T)$ at fixed N and Δ, paying special attention to the temperature region where the ratio is of the order of 1.

F (3 points). Discuss the physics of your results. Does this model describe a phase transition between the 2D and 3D gases? (Justify your answer.) If not, suggest an example how the model may be modified to describe such a transition. (No quantitative analysis is required.)

Hint / reminder: $\int_{-\infty}^{+\infty} \exp\{-\xi^2\} d\xi = \pi^{1/2}$.

Solution

A (3 points). Perhaps the simplest way to calculate N_V at given T and μ is to combine the well-known expression for the density of quantum states in the 3D momentum space,

$$dN_3 = \frac{g_V V}{(2\pi\hbar)^3} d^3 p\,, \tag{1}$$

with the elementary formula for the energy of a free particle,

$$\varepsilon = \frac{p^2}{2m}, \tag{2}$$

and the Boltzmann expression for the average occupancy of the state with energy ε, valid for both bosons and fermions in the classical limit $\langle n\rangle << 1$:

$$\langle n\rangle = \exp\left\{\frac{\mu-\varepsilon}{T}\right\}, \tag{3}$$

where temperature T is in energy units. Combining these formulas, we get

$$N_3 = \int \langle n\rangle dN_3 = \frac{g_V V}{(2\pi\hbar)^3} \exp\left\{\frac{\mu}{T}\right\} \int \exp\left\{-\frac{p^2}{2mT}\right\} d^3 p\,. \tag{4}$$

The last integral may be readily calculated in either spherical or Cartesian coordinates; in the latter approach, using the provided reminder, we get

$$\int \exp\left\{-\frac{p^2}{2mT}\right\} d^3 p = \left[\int_{-\infty}^{+\infty} \exp\left\{-\frac{p_x^2}{2mT}\right\} dp_x\right]^3 = \left[(2mT)^{1/2} \int_{-\infty}^{+\infty} \exp\{-\xi^2\} d\xi\right]^3 = (2\pi mT)^{3/2}, \tag{5}$$

so that for N_3 we get the result that may be represented in the following convenient form:[1]

$$N_3 = N_V(T) \exp\left\{\frac{\mu}{T}\right\}, \qquad \text{where } N_V(T) \equiv g_V V \left(\frac{mT}{2\pi\hbar^2}\right)^{3/2}. \tag{6}$$

B (5 points). For the classical, non-relativistic 2D gas on the wall surface, we still may use Eq. (3). However, if we want to keep for ε and μ the same reference as in Eq. (2) (equal to the energy of a particle at rest in the 3D gas), for the 2D gas we need to replace that expression with the formula

$$\varepsilon = \frac{p^2}{2m} - \Delta\,, \tag{7}$$

which takes into account the particle condensation energy Δ. Also Eq. (1) has to be replaced with the analogous 2D expression:

$$dN_2 = \frac{g_A A}{(2\pi\hbar)^2} d^2 p\,. \tag{8}$$

[1] Such notation, making calculations less bulky, is very popular in semiconductor physics, where effective charge carriers (electrons and holes) may exist in several different "valleys" of the energy dispersion law.

As the result, instead of Eq. (6) we get

$$N_2 = \int \langle n \rangle \, dN_2 = \frac{g_A A}{(2\pi\hbar)^2} \exp\left\{\frac{\mu + \Delta}{T}\right\} \int \exp\left\{-\frac{p^2}{2mT}\right\} d^2 p$$

$$= \frac{g_A A}{(2\pi\hbar)^2} \exp\left\{\frac{\mu + \Delta}{T}\right\} \left[\int_{-\infty}^{+\infty} \exp\left\{-\frac{p_x^2}{2mT}\right\} dp_x\right]^2 = \frac{g_A A}{(2\pi\hbar)^2} \exp\left\{\frac{\mu + \Delta}{T}\right\} \left[(2\pi mT)^{1/2}\right]^2 \qquad (9)$$

$$= N_A(T) \exp\left\{\frac{\mu + \Delta}{T}\right\}, \qquad \text{where } N_A(T) \equiv g_A A \frac{mT}{2\pi\hbar^2}.$$

C (2 points). Conceptually, Eqs. (6) and (9) give the correct average numbers of N_3 and N_2 if the chemical potential μ is externally fixed. However, if these numbers are very large, their fluctuations are (relatively) very small, so that we may use these formulas even if the (approximate, average) value of μ is determined by the given (exact) total number of particles N:

$$N_3 + N_2 = N \, . \qquad (10)$$

Moreover, for the gases in the chemical and temperature equilibrium, μ and T have to be equal. With this condition, Eqs. (6), (9) and (10) yield

$$N_V(T) \exp\left\{\frac{\mu}{T}\right\} + N_A(T) \exp\left\{\frac{\mu + \Delta}{T}\right\} = N \, . \qquad (11)$$

This equation is sufficient to calculate all characteristics of the system.

D (3 points). Since by their definition, $N_V(T)$ and $N_A(T)$ do not depend on μ, Eq. (11) may be readily solved to give

$$\exp\left\{\frac{\mu}{T}\right\} = \frac{N}{N_V(T) + N_A(T) \exp\{\Delta / T\}}, \qquad (12)$$

so that the equilibrium number of particles in the 3D gas is

$$N_3 = N_V(T) \exp\left\{\frac{\mu}{T}\right\} = N \frac{N_V(T)}{N_V(T) + N_A(T) \exp\{\Delta / T\}} = N \frac{\kappa (T/\Delta)^{1/2}}{\kappa (T/\Delta)^{1/2} + \exp\{\Delta / T\}}, \qquad (13)$$

where

$$\kappa \equiv \frac{g_V V}{g_A A} \left(\frac{m\Delta}{2\pi\hbar^2}\right)^{1/2} \qquad (14)$$

is a temperature-independent, dimensionless parameter of the system. Since the particles in the 3D phase obey the ideal-gas equation of state, their pressure is

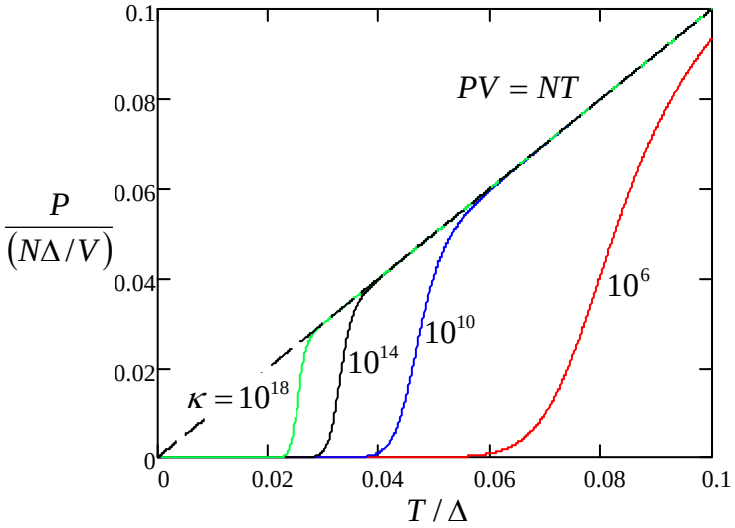
$$P = \frac{N_3 T}{V} = \frac{\kappa (T/\Delta)^{1/2}}{\kappa (T/\Delta)^{1/2} + \exp\{\Delta / T\}} \frac{NT}{V}. \qquad (15)$$

E (4 points). The last result is plotted in the Fig. below for several large values of the parameter κ. (Note that according to Eq. (14), since typically $g_V \sim g_A \sim 1$, this parameter is of the order of $V/Ar_c(\Delta)$ where $r_c(T) \equiv \hbar/(mT)^{1/2}$ is the well-known temperature-dependent quantum correlation length of a free particle. For typical molecules and temperatures, this length is of the order of 10^{-11}m, while the ratio V/A

is of the order of the linear size of the container, so that in human-scale experiments with molecular gases, κ is so large that even its logarithm is substantially larger than 1.) The plots show that the temperature dependence of P is very much different at temperatures below and above the value T_c that calculated from the transcendental equation

$$\kappa(T_c/\Delta)^{1/2} = \exp\{\Delta/T_c\}, \qquad \text{i.e. } T_c = \frac{\Delta}{\ln\left[\kappa(T_c/\Delta)^{1/2}\right]}. \tag{16}$$

(Because of the extreme slowness of the logarithm function at very large values of its argument, T_c is somewhat lower than, but still of the order of Δ.)



In particular, if temperature is well below T_c, virtually all particles are condensed at the surface: $N_3 << N_2 \approx N$, and the pressure provided by the few particles remaining in the 3D phase is exponentially low:

$$P \approx \frac{NT}{V}\kappa\left(\frac{T}{\Delta}\right)^{1/2}\exp\left\{-\frac{\Delta}{T}\right\} << \frac{NT}{V}, \qquad \text{at } T << T_c, \tag{17}$$

approximately following the Arrhenius law $\exp\{-\Delta/T\}$ typical for all thermally-activated effects.

On the other hand, at temperatures well above T_c, the calculated pressure in the 3D particles approaches the equation of state of the usual ideal gas,

$$P \approx \frac{NT}{V}. \tag{18}$$

Hence this simple model can describe the virtually full evaporation of the condensed phase ($N_2 << N_3 \approx N$) at sufficiently high temperatures.

F (3 points). Eqs. (13)-(15) and figure above show that in contrast to genuine phase transitions, at any finite κ the crossover between the 3D and 2D phases, taking place at $T \sim T_c$, is smooth even at $N \to \infty$. Such smooth crossovers are typical for models neglecting particle interaction. (The Bose-Einstein condensation, due to the implicit quantum interaction between the particles, is a very special exception.) Typically, any account of particle interaction, for example their mutual attraction, leads to a model describing a phase transition.

Statistical Mechanics 3

The 1D Potts model

In the so-called Potts model, a uniform 1D chain of N classical spins (in the absence of an external magnetic field) is described by the following interaction Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\eta_i, \eta_j} \ , \text{with } J > 0 \ , \tag{1}$$

where J is a positive coupling constant, η_i is a classical spin variable at the site i, describing the spin state, which may take integer values in the set $\{1, ..., q\}$, δ_{ab} is the Kronecker delta symbol, and the summation is over all pairs of adjacent spins. Consider the model with $q = 3$ in thermal equilibrium at temperature T.

Do parts parts (a), (b), and (c) for finite N, and then take the limit $N \to \infty$ for parts (d), (e), and (f). For parts (c), (d), (e), and (f), you should give an explicit closed-form expression, not an abstract expression involving a summation.

(a) (1 pts.) Write the general expression for the statistical sum (partition function) Z of the system.

(b) (2 pts.) Assuming periodic boundary conditions, express Z via the appropriate transfer matrix.

(c) (4 pts.) Use this expression to calculate Z. (Check your result carefully as all subsequent parts depend on this result.)

Take $N \to \infty$ for the remainder of the problem:

(d) (3 pts.) Calculate the free energy per site, F, and the average energy per site, E.

(e) (3 pts.) Calculate the specific heat capacity per site, C, and the entropy per site, S.

(f) (5 pts.) Calculate the values of E, C, and S in the limits $T \to 0$ and $T \to \infty$. Physically and quantitatively explain your results for E and S in both limits.

(g) (2 pts.) Does this system have a symmetry-breaking phase transition at finite temperature? Prove your answer.

Solution

(a) (1 pt.) Write the general expression for the statistical sum (= partition function) Z of the system. Ans. Let $\beta = 1/(k_B T)$, where k_B is the Boltzmann constant. In general, the partition function of a statistical system in thermal equilibrium at temperature T is

$$Z = \sum_{var.} e^{-\beta \mathcal{H}} \,, \tag{2}$$

where the sum is over the values of all of the dynamical variables in the system. Let $K = \beta J$. Then here

$$Z = \sum_{\{\eta_r\}} e^{K \sum_{\langle ij \rangle} \delta_{\eta_i, \eta_j}} \,, \tag{3}$$

where $\{\eta_r\}$ denotes the set of all variables η_r with r denoting a site on the lattice. Each of these variables η_r can take on values in the set $\{1,\ 2,\ 3\}$.

(b) (2 pts.) Assuming periodic boundary conditions, express Z via the appropriate transfer matrix. Ans. Denote the transfer matrix as $\mathcal{T}$, with matrix elements $\langle \eta_i | \mathcal{T} | \eta_j \rangle$. Given the periodic boundary conditions (BC),

$$Z = \mathrm{Tr}(\mathcal{T}^N) \,. \tag{4}$$

(c) (4 pts.) Use this expression to calculate Z. Ans. Let $y = e^K$. Then in the basis of states (1,2,3), the transfer matrix is

$$\begin{pmatrix} y & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & y \end{pmatrix} . \tag{5}$$

This is a real symmetric matrix, so it can be diagonalized by an orthogonal transformation R (with $R^T = R^{-1}$):

$$R \mathcal{T} R^{-1} = \mathcal{T}_d \equiv \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} , \tag{6}$$

where, as indicated, $\mathcal{T}_d$ is a diagonal matrix, and λ_p, $p = 1, 2, 3$ are the eigenvalues of $\mathcal{T}$. Thus, $\mathcal{T} = R^{-1} \mathcal{T}_d R$. Solving the indicial equation, we find these to be $\lambda_1 = \lambda_2 = y - 1$ and $\lambda_3 = y + 2$. Using the cyclic property of the trace, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, we have

$$\begin{aligned} \mathrm{Tr}(\mathcal{T}^N) &= \mathrm{Tr}[(R^{-1} \mathcal{T}_d R) \cdots (R^{-1} \mathcal{T}_d R)] \\ &= \mathrm{Tr}(\mathcal{T}_d^N) = (\lambda_1)^N + (\lambda_2)^N + (\lambda_3)^N \\ &= 2(y-1)^N + (y+2)^N \,, \end{aligned} \tag{7}$$

(where the $\cdots$ in the first line indicate an N-fold product). Hence,

$$Z = 2(y-1)^N + (y+2)^N \,. \tag{8}$$

(d) (2 pts.) Calculate the Gibbs free energy per site, G, and the internal energy per site, E. The Gibbs free energy per site is $G = -k_B T f$, where the dimensionless function f is

$$\begin{aligned} f &= \lim_{N\to\infty} \frac{1}{N} \ln Z \\ &= \ln(y+2)\,, \end{aligned} \tag{9}$$

so

$$G = -k_B T \ln(y+2)\,. \tag{10}$$

Note that only the dominant eigenvalue λ_3 contributes in this limit. The internal (configurational) energy per site E is given by

$$E = -\frac{\partial f}{\partial \beta} = -J\frac{\partial f}{\partial K}\,. \tag{11}$$

Now $\partial/\partial K = (\partial y/\partial K)\partial/\partial y$ and $(\partial y/\partial K) = y$, so

$$E = -Jy\frac{\partial f}{\partial y} = -\frac{Jy}{y+2}\,. \tag{12}$$

(e) (3 pts.) Calculate the specific heat capacity per site, C, and the entropy per site, S. Ans. The specific heat $C = dU/dT$ here. Now $dU/dT = -k_B(K^2/J)dU/dK$, so

$$C = k_B K^2 y \frac{\partial}{\partial y}\Big(\frac{y}{y+2}\Big) = \frac{2k_B K^2 y}{(y+2)^2}\,. \tag{13}$$

The entropy S can be calculated from the relation $G = E - TS$, i.e., $S = (E-G)/T = k_B\beta(E-G)$. Subsituting our results for G and E, we have

$$S = k_B\Big[-\frac{Ky}{y+2} + \ln(y+2)\Big]\,. \tag{14}$$

(f) (4 pts.) Calculate the values of E, C, and S in the low-temperature limit $T \to 0$ and the high-temperature limit $T \to \infty$. Ans. The limit $T \to \infty$ is $\beta \to 0$, i.e., $y \to 1$. In this limit,

$$E = -\frac{J}{3}, \quad C = 0, \quad S = k_B \ln 3 \quad \text{for } \beta \to 0\,. \tag{15}$$

The limit $T \to 0$ with $J > 0$ is $\beta \to \infty$ and $K \to \infty$. In this limit

$$E = -J, \quad C = 0, \quad S = 0 \quad \text{for } T \to 0\,. \tag{16}$$

(g) (4 pts.) Does this system have a symmetry-breaking phase transition at finite temperature? Prove your answer. Ans. No, this system does not have a symmetry-breaking phase transition at finite temperature. The proof, using what is known as the Peierls argument,

goes as follows. To simplify the proof, use the fact that the boundary conditions have no effect in the thermodynamic limit and hence use free boundary conditions. Assume that there is an incipient ordering, with a nonzero order parameter, i.e., magnetization, $\langle \eta_i \rangle = 1$ for all i. Clearly, this would break the symmetry of the theory, under which any of the three values of η_i is equally likely. We show that this incipient symmetry-breaking long-range order is not stable under a change that minimizes the Gibbs free energy $G = E - TS$. We can destabilize this incipient ordering by flipping the value of η_i to another value, say 2, for the interval $i \geq \ell$, where $1 \leq \ell \leq N$. The cost in energy is $\Delta E = J$ but since we can choose ℓ in any of N ways, the gain in entropy is $\Delta S = k_B \ln N$, so the total change in the Gibbs free energy is $\Delta G = J - k_B T \ln N$. Since $T > 0$, this is always negative as $N \to \infty$. So an incipient ordered state is not thermodynamically stable. Therefore, there is no symmetry-breaking phase transition of this system at finite temperature. (There is, in fact, a symmetry-breaking phase transition at zero temperature.)