Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

August 2017 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Some of the problems may cover multiple pages. Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

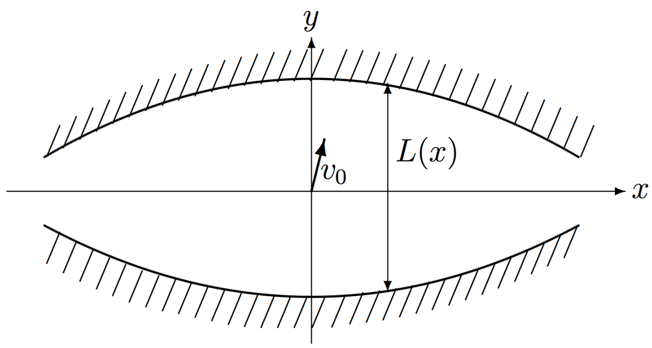
Write your ID number (not your name!) on the exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

A magnetic trap

Consider a non-relativistic particle of mass m moving freely between two fixed curved walls separated by a distance $L(x) = L_0(1 - x^2/a^2)$, where a is a constant with $a \gg L_0$. Assume that $v_x \ll v_y$ and that the particle crosses the midpoint with initial velocity $\vec{v}_0$, where $v_0^2 = v_{0x}^2 + v_{0y}^2$, as shown below. The initial angle is small, i.e. $\theta_0 \equiv \tan^{-1}(v_{0x}/v_{0y}) \ll 1$. There is no gravity.



(a) (6 points) Derive an approximate expression for the value of $x_{\max}$, the maximum distance the particle will reach in its motion in the x-direction. Describe what happens after that.

Hint: the adiabatic invariant, $J_y \equiv \oint p_y dy$, is approximately conserved during the motion.

Now consider a new problem without the walls of part (a). Consider a non-relativistic particle of mass m and charge q moving on a circular orbit in a plane perpendicular to a uniform static magnetic field $\vec{B} = B\hat{x}$ pointing in the $\hat{x}$-direction.

(b) (2 points) Write down an expression for the kinetic energy of the particle in terms of q, m, B, and the radius of the orbit.

(c) (6 points) Now imagine that the particle has a small velocity component in the x-direction, i.e. in the direction of the magnetic field. Moreover, imagine that the static magnetic field $\vec{B}$ acts in the x-direction, but has a tiny positive gradient in that direction, i.e. $\vec{B} = B(x)\,\hat{x}$. The Lagrangian of the system is

$$\mathcal{L} = \frac{m}{2}\left(v_\perp^2 + \dot{x}^2\right) + \frac{q}{2}Br^2\dot{\theta}\,,$$

where θ is the angle with respect to the y-axis in the yz-plane, r is the radius $r = \sqrt{y^2+z^2}$, and $v_\perp^2 \equiv \dot{r}^2 + r^2\dot{\theta}^2$ is the square of the velocity perpendicular to $\vec{B}$. Assume $\dot{r}$ is small, so that $v_\perp^2 \simeq r^2\dot{\theta}^2$. Show that Br^2 is approximately constant along the particle's motion.

(d) (6 points) Argue now that in analogy to part (a), there is again a maximum value for x. What does the configuration of the magnetic field need to be in order to confine the particle in this "magnetic trap".

Solution

(a) The energy is conserved. In this case, it is just the kinetic energy,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \,.$$

The adiabatic invariant, J_y, is approximately conserved, where

$$\begin{aligned} J_y &= \oint p_y dy \\ &= 2\ m\ \dot{y}\ L(x) \\ &\approx 2\ m\ L_0\ v_{0y} \\ &= 2\ m\ L_0\ v_0 \cos\theta_0 \end{aligned}$$

Combining the two invariants, we have

$$\begin{aligned} &\dot{x}^2 = v_0^2 - \dot{y}^2 = v_0^2\left(1 - \frac{L_0^2}{L(x)^2}\cos^2\theta_0\right) \\ &\dot{x} = 0 \implies \cos\theta_0 = \frac{L(x)}{L_0} = 1 - \frac{x^2}{a^2} \implies 1 - \frac{1}{2}\,\theta_0^2 = 1 - \frac{x^2}{a^2} \\ &\implies x_{\max} = \frac{a\theta_0}{\sqrt{2}} \end{aligned}$$

Once the particle reaches $x_{\max}$, it is reflected until it reaches $-x_{\max}$. It will oscillate between these two points indefinitely.

(b) The Lorentz force law applied to a particle on a circular orbit is

$$qv_\perp B = \frac{mv_\perp^2}{r}\,,$$

where $v_\perp$ is the velocity of the particle perpendicular to $\vec{B}$, so that

$$v_\perp = \frac{qBr}{m}\,.$$

The kinetic energy is then given by

$$E_\perp = \frac{1}{2}mv_\perp^2 = \frac{q^2B^2r^2}{2m}\,.$$

(c) The action variable corresponding to θ is

$$J_\theta = \oint p_\theta d\theta\,,$$

where the canonical momentum p_θ is given by

$$p_\theta = mr^2\dot{\theta} + \frac{qBr^2}{2}\,.$$

Since $\dot{\theta} = -\frac{qB}{m}$, we have

$$p_\theta = -\frac{1}{2}qBr^2\,,$$

so that

$$J_\theta = -\pi qBr^2\,.$$

J_θ is an adiabatic invariant, so that Br^2 is approximately a constant of motion.

(d) The total kinetic energy E, which is conserved, is given by

$$E = E_\perp + E_\parallel = \frac{1}{2}mv_\perp^2 = \frac{q^2B^2r^2}{2m} + \frac{1}{2}m\dot{x}^2\,.$$

Since Br^2 is some constant C, we have

$$\frac{1}{2}m\dot{x}^2 = E - \frac{q^2CB(x)}{2m}\,.$$

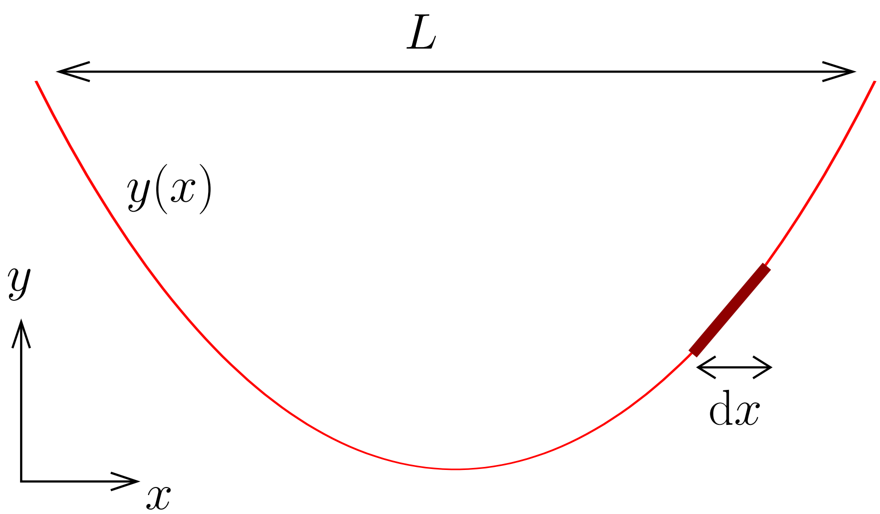
If $B(x)$ slowly increases as x increases, $\dot{x}$ will decrease, eventually going to 0. This means there is a maximum value for x for the particle. The particle will then turn around and move towards negative x. In order for the particle to be trapped, the magnetic field eventually also needs to increase in the negative x-direction.

Classical Mechanics 2

An engineered rope

Consider a flexible, but inextensible, rope with total mass m and an adjustable mass per length, $\lambda \equiv \mathrm{d}m/\mathrm{d}\ell$, where ℓ is the distance along the rope. The rope is to be hung between two points separated by a horizontal distance L.

The mass per length is adjusted so that λ is proportional to the tension in the rope when it is hung[1]. Thus for the rope shown below, $\lambda(x) = cT(x)$ with c a proportionality constant, and we have parametrized λ by the x coordinate, i.e. $\mathrm{d}m = \lambda\,\mathrm{d}\ell = \lambda(x)\sqrt{1+y'(x)^2}\,\mathrm{d}x$. This problem determines the shape of hanging rope $y(x)$ and the necessary mass density.



(a) (8 points) Take a small segment of the rope extending from $x - \mathrm{d}x/2$ to $x + \mathrm{d}x/2$. Draw a well labeled free body diagram for this segment including the tension and gravity. Use the force balance equations in the x and y directions to determine a differential equation for the shape of the rope $y(x)$ with the specified mass density, $\lambda(x) = cT(x)$.

(b) (5 points) Determine $y(x)$. What is the maximum distance, $L_{\max}$, such a rope could traverse?

(c) (3 points) Keeping the mass density $\lambda(\ell)$ fixed, if a small weight of mass $m_o \ll m$ is now hung from the center of the rope, does the height of the center of mass of the rope increase or decrease under the additional load? Explain physically without detailed calculations.

(d) (4 points) Now an additional small mass m_o is hung, but λ is re-adjusted to maintain the constraint, $\lambda(x) = cT(x)$. Determine $y(x)$ with the additional load.

[1]Assuming that the maximum tension is proportional to the mass density, this engineered rope has an equal probability of breaking anywhere along its length, and not just at the points of greatest tension.

Possibly useful integrals:

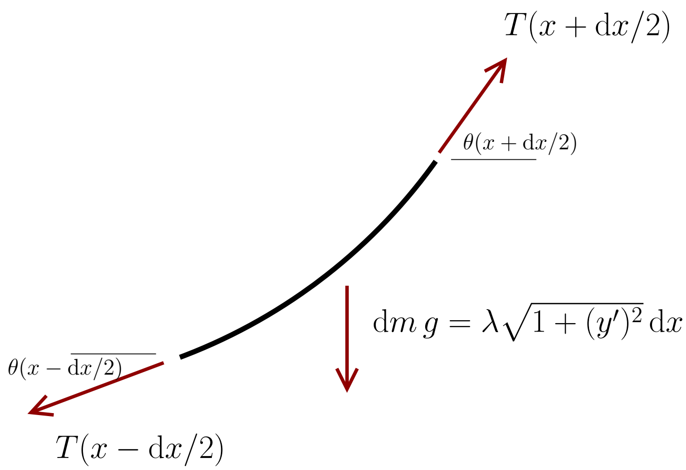
$$\int \frac{\mathrm{d}u}{1+u^2} = \operatorname{atan}(u) + C$$

$$\int \frac{\mathrm{d}u}{\sqrt{e^u - 1}} = 2\operatorname{atan}(\sqrt{e^u - 1}) + C$$

$$\int \mathrm{d}u \sec^2(u) = \tan(u) + C$$

Solution

(a) A free body diagram is shown below.



The horizontal force balance equation gives

$$\frac{d}{dx}(T(x)\cos\theta) = 0 \qquad \text{with} \qquad \cos\theta = \frac{1}{\sqrt{1 + (y'(x))^2}}\,, \tag{1}$$

which upon integration yields the condition

$$T(x) = T_0\sqrt{1 + (y')^2}\,. \tag{2}$$

The vertical components of Newton's law gives

$$\frac{d}{dx}(T(x)\sin\theta) = \lambda(x)g\sqrt{1 + (y')^2} \qquad \text{with} \qquad \sin\theta = \frac{y'}{\sqrt{1 + (y')^2}}\,. \tag{3}$$

Using the constraint

$$\lambda(x) = cT(x)\,, \tag{4}$$

we find the required differential equation

$$\frac{d}{dx}(y'(x)) = cg(1 + (y')^2)\,. \tag{5}$$

(b) Letting $u = y'(x)$ we integrate

$$\frac{du}{1 + u^2} = cgdx\,, \tag{6}$$

or

$$\tan^{-1}(u) = cgx + \text{const}\,. \tag{7}$$

By recognizing the $\tan^{-1}(y'(x)) = \theta(x)$ and taking $x = 0$ as the bottom of the rope, we demand that $\theta(x) = 0$ at $x = 0$. Thus,

$$\frac{dy}{dx} = \tan(cgx)\,. \tag{8}$$

Integrating again

$$\int dy = \int \frac{\sin(cgx)}{\cos(cgx)} dx\,, \tag{9}$$

we find

$$y(x) - y_{\rm min} = -\frac{1}{cg}\log(\cos(cgx))\,. \tag{10}$$

We may take $y_{\rm min} = 0$.

Clearly when

$$cgx \to \frac{\pi}{2}\,, \tag{11}$$

$y(x)$ becomes infinitely long. Thus $L/2$ is limited to

$$\frac{L_{\rm max}}{2} = \frac{\pi}{2cg}\,. \tag{12}$$

(c) The vertical height increases. Imagine pulling from downwards from the center. You clearly are doing work on the system. The work you do increases the gravitational potential energy of the rope. The height increases is related to the ΔPE

$$\Delta{\rm PE} = mg\Delta h_{\rm cm}\,. \tag{13}$$

(d) The differential equation of part (a) remains valid. However, when we solve for $y'(x)$ as in Eq. (7)

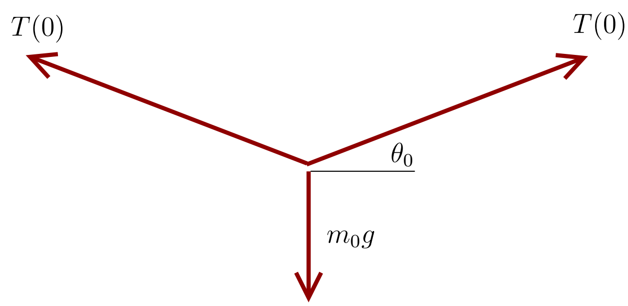
$$y'(x) = \tan(\theta(x)) = \tan(cgx + \theta_0)\,, \tag{14}$$

the constant θ_0 should no longer be set to zero. θ_0 is easily interpreted as the angle at the bottom of the rope.

At $x = 0$ the bottom of the rope we must satisfy

$$2T(0)\sin(\theta_0) = m_o g\,, \qquad T(0)\cos(\theta_0) = T_0\,, \tag{15}$$

which is determined by the free body diagram shown below:



Since the mass is small we have

$$\cos\theta_0 \simeq 1\,, \qquad \sin(\theta_0) \simeq \tan(\theta_0) \simeq \theta_0\,, \tag{16}$$

and thus

$$\theta_0 = \frac{m_0 g}{2T_0}\,. \tag{17}$$

We may determine the integration constant T_0 from the total mass. To first order in m_0 we may disregard the modification of $y(x)$ to determine this relation:

$$\begin{aligned}
m =& 2\int_0^{L/2} dx \lambda(x)\sqrt{1+(y'(x))^2}\,, && (18)\\
=& 2c\int_0^{L/2} dx T(x)\sqrt{1+(y'(x))^2}\,, && (19)\\
=& 2cT_0\int_0^{L/2} dx(1+y'(x)^2)\,, && (20)\\
=& 2cT_0\int_0^{L/2} \sec^2(cgx)\,, && (21)\\
=& 2cT_0\left[\frac{1}{cg}\tan(cgx)\right]_0^{\frac{L}{2}}\,, && (22)\\
m =& \frac{2T_0}{g}\tan(cgL/2)\,. && (23)
\end{aligned}$$

Putting together the ingredients we have

$$y(x) = \frac{-1}{cg}\log(\cos(cgx+\theta_0))\,, \tag{24}$$

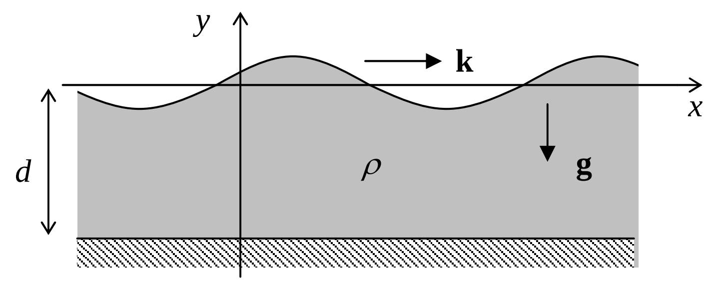
where

$$\theta_0 \simeq \frac{m_o}{m}\tan(cgL/2)\,. \tag{25}$$

Classical Mechanics 3

“Gravity” Waves

The usual waves on a water surface[1] may be reasonably well described modeling the water as an ideal (viscosity-free), incompressible fluid of density ρ, placed in a constant gravity field **g**. This problem addresses only sinusoidal waves, with all variables independent of one Cartesian coordinate - in the Fig. below, of z.



A (4 points). What general equations should be satisfied by an ideal fluid's density $\rho(\mathbf{r}, t)$ and velocity $\mathbf{v}(\mathbf{r}, t)$, where $\mathbf{r}$ is a fixed point in a lab reference frame (rather than the fluid particle's position)?

B (2 points). How may these equations be simplified:

- if the fluid is incompressible (ρ = const)?
- in the low-velocity limit?

C (4 points). Prove that the simplified equations are satisfied by the following expressions for the particle displacements:

$$q_x(\mathbf{r},t) = Ae^{ky}\cos(kx - \omega t), \qquad q_y(\mathbf{r},t) = Ae^{ky}\sin(kx - \omega t),$$

provided that the wave's amplitude A is small ($kA \ll 1$), and the fluid's depth d is large ($kd \gg 1$).

D (4 points). Calculate the waves' dispersion relation $\omega(k)$, and find their phase and group velocities.

E (4 points). Modify these results (including the formulas for q_x and q_y) for a fluid of a finite depth $d \sim 1/k$ (but still $d \gg A$), and analyze the resulting dispersion relation.

F (2 points). Qualitatively, how would these dispersion relations be affected by surface tension?

[1] Traditionally, they were called "gravity waves", but nowadays this term should probably be reserved for the recently observed "real" gravity waves in free space, described by the general relativity.

Solution

A. Generally, the function **v**(**r**, t) has to satisfy the (kinematic) continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \qquad \text{with } \mathbf{j} \equiv \rho \mathbf{v}, \tag{1}$$

and the (dynamic) Euler equation - essentially, the 2nd Newton law:[1]

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = -\nabla P + \mathbf{f}, \tag{2}$$

where $P = P(\mathbf{r}, t)$ is pressure, and **f** is the distributed external force by unit volume, in our current case equal to $\rho\mathbf{g}$.

B. For an incompressible fluid (ρ = const), the continuity equation (1) is reduced to

$$\nabla \cdot \mathbf{v} = 0, \tag{3}$$

while for small velocities ($\mathbf{v} \to 0$), the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in the Euler equation (2) is negligible in comparison with the partial derivative $\partial\mathbf{v}/\partial t$, so that the equation is reduced to

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \nabla(-P + \rho \mathbf{g} y). \tag{4}$$

C-D. Using the suggested relations for particle displacements to calculate their velocities,

$$v_x = \omega A e^{ky} \sin(kx - \omega t), \qquad v_y = -\omega A e^{ky} \cos(kx - \omega t),$$

we see that Eq. (3) is indeed identically satisfied:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \kappa\omega A e^{ky} \cos(kx - \omega t) - \kappa\omega A e^{ky} \cos(kx - \omega t) = 0.$$

The component of the vector Eq. (4) that is normal to the fluid's surface may also be always satisfied with a proper pressure profile (which has no back effect on the incompressible fluid's motion). However, the pressure at the fluid's surface has to be equal to the external (say, atmospheric) pressure, and hence cannot have a gradient's component directed along the surface. Generally, the surface position $y_s(x, t)$ has to be found from the transcendent equation[2]

$$y_s(x,t) = q_y(\mathbf{r},t) \equiv A e^{ky_s} \cos(kx - \omega t),$$

but due to the condition $A << 1/k$, we may use its linear approximation:

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[1] The expression in the square brackets of Eq. (2) is just the particle's acceleration, as observed in the lab frame, and is called the "convective", or "advective", or "material", or "Lagrangian" derivative of the velocity.

[2] Here, for convenience, the wave-free fluid level is taken for $y = 0$ - see Fig. above.

$$y_s(x,t) = A\cos(kx-\omega t), \qquad \text{so that } \frac{\partial y_s}{\partial x} = kA\sin(kx-\omega t) << 1.$$

As a result, the tangential component of Eq. (4) at the surface yields

$$\rho\omega^2 A\sin(kx-\omega t) = \rho g\frac{\partial y_s}{\partial x} \equiv \rho g k A\sin(kx-\omega t).$$

This relation is satisfied for all x and t, provided that

$$\omega^2 = gk, \qquad \text{i.e. } \omega = (gk)^{1/2}. \tag{5}$$

This is the requested dispersion relation. For the phase and group velocities it yields

$$u_{\text{phase}} \equiv \frac{\omega}{k} = \left(\frac{g}{k}\right)^{1/2}, \qquad u_{\text{group}} \equiv \frac{\partial\omega}{\partial k} = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2} = \frac{1}{2}u_{\text{phase}}.$$

Note that these waves do not have an acoustic branch (with $\omega \propto k$) for any range of frequencies.

E. At the bottom of a finite layer (in Fig. above, for $y = -d$), the vertical displacements of the fluid particles should vanish: $q_y = 0$. Evidently, this effect may be taken into account by generalizing the above solution as

$$q_x = A\frac{\sinh k(y+d)}{\sinh kd}\cos(kx-\omega t), \qquad q_y = A\frac{\sinh k(y+d)}{\sinh kd}\sin(kx-\omega t).$$

Repeating the above calculations, we see that this is indeed a solution of Eqs. (3)-(4), provided that Eq. (5) is generalized as

$$\omega^2 = gk\tanh kd .$$

Note that now for low frequencies ($kd << 1$, i.e. $\omega << (g/h)^{1/2}$) there is acoustic branch, with

$$\omega = (gd)^{1/2}k, \qquad \text{i.e. with } u_{\text{phase}} = u_{\text{group}} = (gd)^{1/2}.$$

For the Earth's oceans (with $d \sim 10$ km, $g \approx 10$ m/s^2), this velocity is very high: $u \sim 300$ m/s, of the order of the speed of sound in air.

F. The surface tension "tries" to keep any surface as flat as possible (to minimize its area A, and hence the surface energy $U_s = \gamma A$, positive for all mechanically stable fluids). Hence we may guess that for a given wave number k , it increases the wave frequency. Indeed, a straightforward analysis (which was not required in this exam) yields the following generalization of Eq. (5):

$$\omega^2 = gk + \frac{\gamma k^3}{\rho} \geq gk .$$

Electromagnetism 1

Radiation during a collision

A classical non-relativistic charged particle of charge q and mass m is incident upon a repulsive mechanical potential $U(r)$

$$U(r) = \frac{\mathcal{A}}{r^2},$$

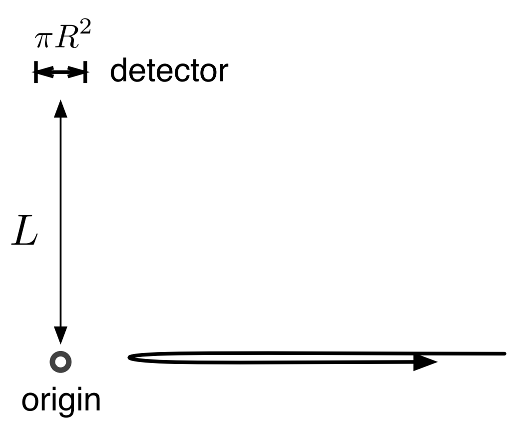
so that the force exerted on the particle is $\boldsymbol{F} = -\nabla U(r)$. The particle moves along the x-axis and strikes the central potential head on as shown below. The incident kinetic energy (i.e. the kinetic energy of the particle far from the origin) is K.

(a) (2 points) Determine the particle's classical trajectory $x(t)$. Adjust the integration constants so that the particle reaches its distance of closest approach at $t = 0$. Check that for late times $x(t)$ approaches $v_o\, t$ with the phyically correct value of v_o. Check that for small times $x(t)$ behaves as $x(t) \simeq x_o + \frac{1}{2}a_o t^2$ with the physically correct value of x_o.

(b) (4 points) Use dimensional reasoning and the Larmour formula to estimate the total energy lost to electromagnetic radiation during the collision. How does the energy lost scale with the incident energy?

(c) (2 points) Calculate quantitatively the energy lost to radiation during the collision processes. Some relevant integrals are given at the end of this problem.

Now consider a detector placed along the y-axis far from the origin as shown below. The front face of the detector has an area of πR^2, and the detector is placed at a distance L from the origin with $L \gg R$.



(d) (2 points) What is the direction of polarization of the observed light in the detector? Explain.

(e) (2 points) What is the typical frequency of the photons that are emitted at 90^o? Explain.

(f) (5 points) For the detector described above, determine the average number of photons received by the detector per unit frequency:

$$\frac{dN}{d\omega}\,. \tag{1}$$

Some relevant integrals are given at the end of the problem.

(g) (3 points) We have determined the photon radiation spectrum using classical electrodynamics. For what values of the parameters $\mathcal{A}$ and K is this approximation justified?

Useful integrals and formulas for EM1:

1. For positive integer n, we note the integrals

$$\int_{-\infty}^{\infty} du \frac{1}{(1+u^2)^n} = \pi\, c_n \tag{2}$$

where

$$c_1, c_2, c_3, c_4, \ldots = 1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \ldots \tag{3}$$

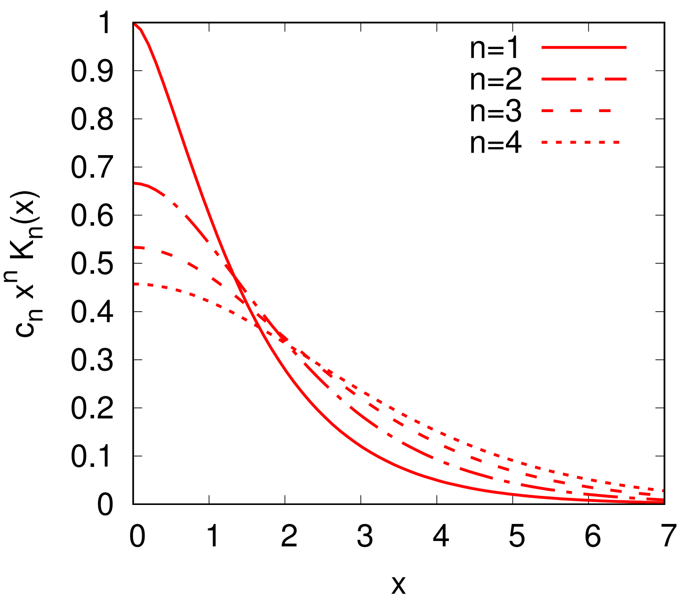
2. For positive integers n, we note the integrals

$$\int_{0}^{\infty} du \frac{\cos(xu)}{(u^2+1)^{n+\frac{1}{2}}} = c_n\, x^n K_n(x) \tag{4}$$

where

$$c_1, c_2, c_3, c_4, \ldots = 1, \frac{1}{3}, \frac{1}{15}, \frac{1}{105}, \ldots \tag{5}$$

and $K_n(x)$ are the modified Bessel functions. The RHS of Eq. (4) is illustrated below.



Solution

(a) It is convenient to use dimensionless variables. The dimensional constants for the classical problem are

$$K, \mathcal{A}, m\,, \tag{6}$$

from which we can select a unit for meters, seconds, and kilograms. The unit of velocity can be taken as

$$K = \frac{1}{2} m v_o^2 \Longrightarrow v_o \equiv \sqrt{\frac{2K}{m}}\,, \tag{7}$$

which phyiscally is the velocity as $r \to \infty$. The unit of meters is

$$K = \frac{\mathcal{A}}{x_o^2} \Longrightarrow x_o \equiv \sqrt{\frac{\mathcal{A}}{K}}\,, \tag{8}$$

which (by energy conservation) is the distance of closest approach[2]. The unit of seconds is therefore

$$t_o \equiv \frac{x_o}{v_o} \equiv \frac{\sqrt{(m\mathcal{A})/2}}{K}\,. \tag{9}$$

We need to solve for the trajectory $x(t)$. The velocity is given by the first integral (i.e. enegy conservation)

$$\frac{1}{2} m v^2(t) + \frac{\mathcal{A}}{x(t)^2} = K\,. \tag{10}$$

Switching to dimensionless variables,

$$\bar{v} = \frac{v}{v_o}\,, \qquad \bar{x} = \frac{x(t)}{x_o}\,, \tag{11}$$

the dimensionless form of energy conservation reads

$$\bar{v}^2 + \frac{1}{\bar{x}^2} = 1\,. \tag{12}$$

Solving Eq. (12) for $\bar{v}$ we have

$$\bar{v} = \sqrt{1 - \frac{1}{\bar{x}^2}}\,. \tag{13}$$

Finally, we write $\bar{v} = d\bar{x}/d\bar{t}$ and integrate Eq. (13) to find

$$\sqrt{\bar{x}^2 - 1} = \bar{t} + \text{constant}\,. \tag{14}$$

[2]Note that this is the distance of closest approach in the absence of energy loss due to radiation. In the limit of classical electrodynamics one first determines the trajectories of charged particles (ignoring the radiation), and then soves for the subsequent radiation. This is in effect ignoring radiations back reaction.

We choose the integration constant to be zero, so that at $t = 0$ the trajectory is at the turning point $\bar{x} = 1$ and find

$$\sqrt{\bar{x}^2 - 1} = \bar{t} \qquad \text{or} \qquad \bar{x}(\bar{t}) = \sqrt{1 + \bar{t}^2} \tag{15}$$

Restoring units, the trajectory is

$$x(t) = \sqrt{\frac{2Kt^2}{m} + \frac{\mathcal{A}}{K}}\,. \tag{16}$$

It is easy to check that this trajectory satisfies the appropriate limits.

(b) The energy lost to radiation is

$$E_{\rm loss} = \int_{-\infty}^{\infty} dt\, \frac{q^2}{4\pi} \frac{2a^2}{3c^3} \tag{17}$$

We need to use dimensional reasoning to estimate a and the time interval over which the acceleration is active.

Using the dimensional analysis of the previous section, the integral is of order

$$\int dt\, a^2 \sim \frac{v_o^2}{t_o}\,. \tag{18}$$

and thus

$$E_{\rm loss} \sim \frac{q^2}{4\pi t_o} \frac{v_o^2}{c^3} \sim \frac{q^2 K^2}{m\sqrt{Am}c^3}\,. \tag{19}$$

The energy lost scales as the velocity to the fourth power, $K^2 \propto v_o^4$.

(c) We next evaluate the integral in Eq. (17) precisely. For reference we record the acceleration:

$$a(t) = \frac{v_o}{t_o} \frac{d^2\bar{x}}{d\bar{t}^2} = \frac{v_o}{t_o} \frac{1}{(1+\bar{t}^2)^{3/2}}\,. \tag{20}$$

The relevant integral is

$$\int_{-\infty}^{\infty} dt\, a^2 = \frac{v_o^2}{t_o} \int_{-\infty}^{\infty} d\bar{t} \left(\frac{d^2\bar{x}}{d\bar{t}^2}\right)^2 = \frac{v_o^2}{t_o} \int_{-\infty}^{\infty} \frac{d\bar{t}}{(1+\bar{t}^2)^3} = \frac{v_o^2}{t_o} \left(\frac{3\pi}{8}\right)\,. \tag{21}$$

The energy lost is therefore

$$E_{\rm loss} = \frac{q^2}{4\pi} \frac{2}{3c^3} \int_{-\infty}^{\infty} dt\, a^2\,, \tag{22}$$

$$= \frac{q^2}{4\pi t_o} \frac{v_o^2}{c^3} \left(\frac{\pi}{4}\right)\,. \tag{23}$$

(d) The radiation electric field is

$$\boldsymbol{E}_{\rm rad}(t, \boldsymbol{r}) = \frac{q}{4\pi r c^2}\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{a}(t_e)\,, \tag{24}$$

where the emission time is

$$t_e = t - \frac{r}{c}\,. \tag{25}$$

For the problem at hand $\boldsymbol{a} = a(t)\hat{\boldsymbol{x}}$ and thus

$$\boldsymbol{n} \times \boldsymbol{n} \times \hat{\boldsymbol{x}} = -\hat{\boldsymbol{x}}\,. \tag{26}$$

So the radiation field is polarized in the $-\hat{\boldsymbol{x}}$ direction.

(e) The typical frequency is given by dimension reasoning

$$\omega_o \sim \frac{1}{t_o} \tag{27}$$

(f) To determine the yield of photons, we Fourier transform the radiation field and square this Fourier transform. The Fourier transform of the electric field (in the $-\hat{\boldsymbol{x}}$ direction) reaching the detector

$$E_{\rm rad}(\omega, r) = \frac{q}{4\pi r c^2}\int_{-\infty}^{\infty} dt e^{-i\omega t} a(t_e)\,. \tag{28}$$

After switching to variables to integrate over the emission time,

$$e^{i\omega t} = e^{i\omega(t_e + r/c)} = e^{ikr}e^{i\omega t_e}, \qquad k \equiv \frac{\omega}{c}, \tag{29}$$

the integral reads

$$E_{\rm rad}(\omega, r) = \frac{q\, e^{ikr}}{4\pi r c^2}\int_{-\infty}^{\infty} dt_e e^{-i\omega t_e} a(t_e)\,. \tag{30}$$

Thus, after switching to dimensionless variables

$$\bar{\omega} = \omega t_o \qquad \bar{t}_e = \frac{t_e}{t_o}\,, \tag{31}$$

Thus we find

$$E_{\rm rad}(\omega, r) = \frac{q\, e^{ikr}}{4\pi r c^2} v_o \int_{-\infty}^{\infty} d\bar{t}_e e^{-i\bar{\omega}\bar{t}_e}\, \frac{1}{(1+\bar{t}_e^2)^{3/2}}\,, \tag{32}$$

$$= \frac{q\, e^{ikr}}{4\pi r c^2} v_o \left[2\bar{\omega} K_1(\bar{\omega})\right]\,. \tag{33}$$

Squaring the radiation field, we find the yield of photons

$$\hbar\omega \frac{dN}{d\omega d\Omega} = \frac{c}{\pi}|r E_{\rm rad}(\omega, r)|^2\,. \tag{34}$$

Assembling the ingredients, and expressing the result in terms of the fine structure constant $\alpha = q^2/(4\pi\hbar c) = 1/137$, we find

$$\frac{dN}{d\omega d\Omega} = \frac{\alpha}{4\pi^2}\left(\frac{v_o}{c}\right)^2 \frac{1}{\omega}\left[2\bar{\omega}K_1(\bar{\omega})\right]^2 . \tag{35}$$

The solid angle is $\Delta\Omega = \pi R^2/L^2$, and thus we find

$$\frac{dN}{d\omega} = \frac{\pi R^2}{L^2}\frac{\alpha}{4\pi^2}\left(\frac{v_o}{c}\right)^2 \frac{1}{\omega}\left[2\bar{\omega}K_1(\bar{\omega})\right]^2 . \tag{36}$$

In the low frequency limit the term in brackets approaches

$$\left[2\bar{\omega}K_1(\bar{\omega})\right]^2 \to 2^2 , \tag{37}$$

and thus in low frequency limit we find

$$\frac{dN}{d\omega d\Omega} = \frac{\alpha}{4\pi^2}\left(\frac{2v_o}{c}\right)^2 \frac{1}{\omega} . \tag{38}$$

Notice that this expression is independent of $\mathcal{A}$, and is in fact indentical to the radiation for impulsive scattering where $v(t)$ changes instantaneously:

$$\boldsymbol{v}_{\rm impulse}(t) = \begin{cases} -v_o\,\hat{\boldsymbol{x}} & t < 0 \\ v_o\,\hat{\boldsymbol{x}} & t > 0 \end{cases} . \tag{39}$$

Indeed, in the low frequency limit the radiated waves do not have the temporal resolution to resolve events of order the collision time t_o. Thus, as far as the radiation of these low frequency waves is concerned, the collision happens instantaneously.

(g) To determine the validity of the classical approximation, we note that the typcal frequency is $1/t_o$. The energy of the emitted photon has to be small compared to the kinetic energy of the particle for the classical approximation to be valid

$$\hbar\omega \ll K . \tag{40}$$

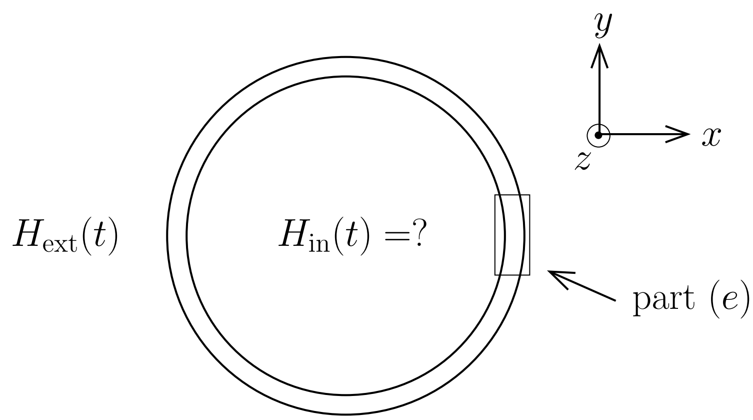
With $\frac{1}{t_o} = \frac{K}{\sqrt{(m\mathcal{A})/2}}$, we find

$$\frac{2\hbar^2}{m\mathcal{A}} \ll 1 . \tag{41}$$

Electromagnetism 2

A cylindrical shell in a magnetic field

Consider an infinitely long cylindrical ohmic shell of conductivity σ and radius a. The walls have thickness Δ, with $\Delta \ll a$. The shell is placed in a uniform, but time dependent, external magnetic field $H_{\rm ext}(t)$, which is directed along the z-axis as shown below. The goal of this problem is to determine the magnetic field inside the cylinder. The thickness Δ is sufficiently small that the induced current density may be considered (spatially) constant inside the shell wall in parts (a)–(c).



(a) (1 point) For a specified surface current $\boldsymbol{K} = K(t)\,\hat{\boldsymbol{\phi}}$, how is the magnetic field inside the shell related to the external magnetic field.

(b) (5 points) Determine a differential equation for the evolution of the magnetic field inside the cylinder. Check that your equation is dimensionfully correct.

(c) (5 points) For a sinusoidal external field, $H_{\rm ext}(t) = H_o e^{-i\omega t}$, determine the amplitude of the magnetic field's sinusoidal oscillations inside the cylinder. Make a graph of the ratio of the interior to exterior amplitudes as a function of frequency.

(d) (4 points) At higher frequency the induced current changes appreciably over the wall thickness Δ. Estimate the frequency where this (neglected) dynamics becomes important.

(e) (5 points) Determine the amplitude of magnetic field's sinusoidal oscillations inside the cylinder without assuming that the induced current is constant within the walls. Check that for small Δ you reproduce the results of part (c).

Hint: Magnify and analyze the highlighted region shown in the figure to relate the interior and exterior. Treat the walls of the cylinder as having infinite transverse (y and z) extent, so that all fields in the walls are functions x only.

Solution:

(a) First we note that for a specified current

$$\boldsymbol{n} \times (\boldsymbol{H}_{\text{ext}} - \boldsymbol{H}_{\text{in}}) = \frac{\boldsymbol{K}}{c} \tag{1}$$

Taking $\boldsymbol{n} = \hat{\boldsymbol{\rho}}$, $\boldsymbol{H} = H\,\hat{\boldsymbol{z}}$, $\boldsymbol{K} = K(t)\,\hat{\boldsymbol{\phi}}$, and noting that $\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{z}} = -\hat{\boldsymbol{\phi}}$ we have

$$H_{\text{ext}}(t) - H_{\text{in}}(t) = -\frac{K(t)}{c}\,. \tag{2}$$

One can (and should) also reason the signs in this equation using the right hand rule. Either way

$$H_{\text{in}}(t) = H_{\text{ext}}(t) + \frac{K(t)}{c}\,. \tag{3}$$

(b) The changing flux inside the cylinder induces a voltage. This voltage produces a current $K(t)$ given by Ohms Law. Given the current we can relate the internal and external magnetic fields through Eq. (3).

The voltage induced is

$$-\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{1}{c}\partial_t \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}\,. \tag{4}$$

For the geometry at hand

$$-E_\phi(2\pi a) = \frac{1}{c}\dot{H}_{\text{in}}(t)\pi a^2\,, \tag{5}$$

and thus

$$E = -\frac{a}{2c}\dot{H}_{\text{in}}\,. \tag{6}$$

From Ohm's Law, $\boldsymbol{J} = \sigma\boldsymbol{E}$, and the surface current $K = J\Delta$, we find

$$K = -\frac{a\Delta\sigma}{2c}\dot{H}_{\text{in}}\,. \tag{7}$$

Using the boundary conditions in Eq. (3) we have finally

$$\frac{a\Delta\sigma}{2c^2}\dot{H}_{\text{in}}(t) + H_{\text{in}}(t) = H_{\text{ext}}(t)\,. \tag{8}$$

We note that since $[\sigma] = s^{-1}$ it is easily seen that

$$\tau_m \equiv \frac{a\Delta\sigma}{2c^2}\,, \tag{9}$$

has units of time. To make sense of these numbers, note that magnetic diffusion coefficient for copper is of order

$$D \equiv \frac{c^2}{\sigma} \sim \frac{\text{cm}^2}{\text{millisec}}\,. \tag{10}$$

Thus the time constant of this equation is of order

$$\tau_m \sim \text{millsec}\left(\frac{\text{cm}^2}{a\Delta}\right)\,. \tag{11}$$

(c) Solving Eq. (8) for a sinusoidal steady state, $H_{\rm ext}(t) = H_{\rm o}e^{-i\omega t}$ and $H_{\rm in}(t) = H_{\rm in}e^{-i\omega t}$, we have

$$- i\omega\tau_m H_{\rm in} + H_{\rm in} = H_o\,. \tag{12}$$

Thus, $H_{\rm in} = H_o/(1 - i\omega\tau_m)$, and the oscillation amplitude is

$$|H_{\rm in}| = \frac{|H_o|}{\sqrt{1 + (\omega\tau_m)^2}}\,. \tag{13}$$

(d) At higher frequency the skin depth becomes important. The skin depth is of order

$$\delta(\omega) \sim \sqrt{\frac{D}{\omega}} \sim \sqrt{\frac{c^2}{\sigma\omega}}\,. \tag{14}$$

The dynamics changes when the skin depth is comparable to Δ

$$\delta(\omega) \sim \Delta\,. \tag{15}$$

Solving for ω, we find that the dynamics changes when

$$\omega \sim \frac{c^2}{\sigma\Delta^2}\,. \tag{16}$$

So, for a magnetic diffusion coefficient of order Eq. (10), we find

$$\omega \sim {\rm kHz}\left(\frac{{\rm cm}^2}{\Delta^2}\right)\,. \tag{17}$$

(e) Now we solve more precisely for the fields inside the walls. The magnetic fields obey the diffusion equation. This follows from Ampere's Law

$$\nabla \times \boldsymbol{B} = \frac{\sigma}{c}\boldsymbol{E}\,, \tag{18}$$

and Faraday's Law

$$- \nabla \times \boldsymbol{E} = \frac{1}{c}\partial_t \boldsymbol{B}\,. \tag{19}$$

Indeed, taking the curl of Ampere's Law, using $\nabla\times\nabla\times\boldsymbol{B} = \nabla(\nabla\cdot\boldsymbol{B}) - \nabla^2\boldsymbol{B}$ and $\nabla\cdot\boldsymbol{B} = 0$, we find the magnetic diffusion equation

$$D\nabla^2\boldsymbol{B} = \partial_t\boldsymbol{B}\,, \qquad D \equiv \frac{c^2}{\sigma}\,. \tag{20}$$

Since the wall thickness is very small compared to the radius, $\Delta \ll a$, we can approximate the geometry as one dimensional, up to correction of order Δ/a. the radial coordinate is in the x direction, and the ϕ direction (the direction of the electric field and current) is in the y direction. We choose $x = 0$ to be the inside wall of the cylindrical shell, so that

$x = \Delta$ is the outside wall of the cylindrical shell. The diffusion equation for sinusoidal field, $\boldsymbol{B}(t, \boldsymbol{x}) = B(x)e^{-i\omega t}\,\hat{\boldsymbol{z}}$, reads

$$\partial_x^2 B(x) = -i\frac{\omega}{D}B(x)\,. \tag{21}$$

The electric field is determined from Eq. (18)

$$-\frac{c}{\sigma}\frac{dB}{dx}(x) = E_y(x)\,. \tag{22}$$

Solving Eq. (21) we have

$$B(x) = C_0 e^{i\kappa x} + C_1 e^{-i\kappa x} \tag{23}$$

where

$$\kappa = \sqrt{\frac{i\omega}{D}} = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{D}}. \tag{24}$$

Equivalently, we will use

$$B(x) = C_0\cos(\kappa x) + C_1\sin(\kappa x)\,, \tag{25}$$

since is slightly simpler to analyze the boundary conditions in this form.

We have boundary conditions at $x = 0$

$$B(0) = H_{\text{in}} \tag{26}$$

and this sets $C_0 = H_{\text{in}}$. The electric field at the $x = 0$ boundary is given by Eq. (6)

$$E_y(0) = \frac{+i\omega a}{2c}H_{\text{in}}\,, \tag{27}$$

and this (through Eq. (22)) sets the derivative of $B(x)$ at $x = 0$

$$B'(0) = -\frac{\kappa^2 a}{2}H_{\text{in}}\,, \tag{28}$$

fixing the coefficient $C_1 = (\kappa a)/2$. To summarize

$$B(x) = H_{\text{in}}\left[\cos(\kappa x) - \frac{\kappa a}{2}\sin(\kappa x)\right]\,. \tag{29}$$

Finally since $B(\Delta) = H_{\text{ext}}$ we find

$$H_{\text{in}} = \frac{H_{\text{ext}}}{\cos(\kappa\Delta) - \frac{\kappa a}{2}\sin(\kappa\Delta)}\,, \tag{30}$$

and thus the amplitude is

$$H_{\text{in}} = \frac{H_{\text{ext}}}{|\cos(\kappa\Delta) - \frac{a}{2\Delta}\kappa\Delta\sin(\kappa\Delta)|}\,. \tag{31}$$

We can check that when $\kappa\Delta \ll 1$

$$|H_{\text{in}}| \simeq \frac{|H_{\text{ext}}|}{|1 - i\omega\tau_m|} = \frac{H_{\text{ext}}}{\sqrt{1 + (\omega\tau_m)^2}}\,, \tag{32}$$

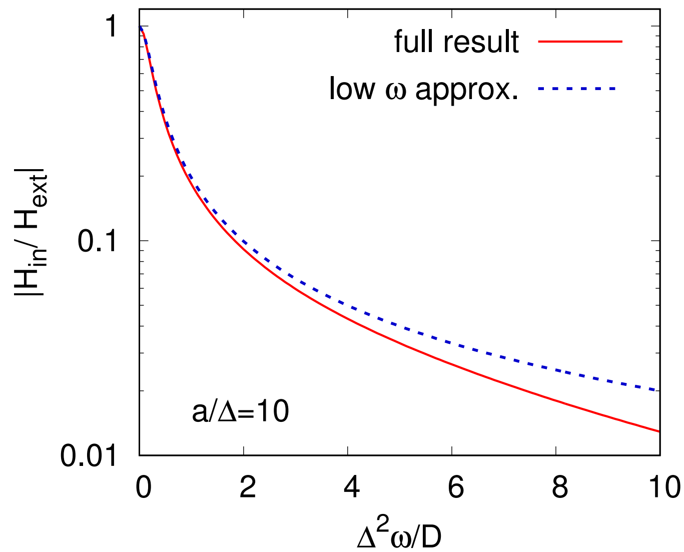


Figure 1: The field in the center divided by the external field. See text for further explanation

where we have recognized that

$$\frac{\kappa^2 a\Delta}{2} = i\omega\tau_m\,. \tag{33}$$

We have assumed that $a/\Delta \gg 1$. Thus for $(\kappa\Delta)^2 \gg \Delta/a$ we can neglect the $\cos(\kappa\Delta)$ term in comparison to the $\sin(\kappa\Delta)$ term in the denominator of Eq. (31). For $(\kappa\Delta)^2 \sim \Delta/a$ we may approximate $\cos(\kappa\Delta) \simeq 1$ up to correction of order a/Δ. Thus in a uniform approximation (i.e. an approximation which is valid for all $\kappa\Delta$) we have

$$|H_{\rm in}| = \frac{H_{\rm ext}}{|1 - i\omega\tau_m \frac{\sin(\kappa\Delta)}{\kappa\Delta}|}\,, \tag{34}$$

which is our final result.

Taking $a/\Delta = 10$ for instance, we plot the full result (Eq. (34)) and its low frequency approximation (Eq. (13)) in Fig. 1. At large frequency the skin-depth leads to exponential suppression, rather than the $1/\omega$ behaviour predicted by the low frequency approximation.

Electromagnetism 3

Angular momentum in a wave packet

Consider a wave packet with a transverse profile $E_o(x, y)$ propagating in the z direction (see eq. (3) for a complete specification of $\boldsymbol{E}$ and $\boldsymbol{B}$). Although the precise form of $E_o(x, y)$ is not needed below, for definiteness you may assume that the wave packet has a Gaussian profile for

$$E_o(x, y) = \mathcal{A}e^{-\frac{x^2+y^2}{4\sigma^2}} , \tag{1}$$

and is infinitely broad in the z direction. The following integrals may be useful:

$$\int_{-\infty}^{\infty} \mathrm{d}u \; e^{-\alpha u^2} = \sqrt{\frac{\pi}{\alpha}} , \tag{2a}$$

$$\int_{-\infty}^{\infty} \mathrm{d}u \; e^{-\alpha u^2} e^{iku} = \sqrt{\pi} e^{-\frac{k^2}{4\alpha}} . \tag{2b}$$

(a) (2 points) When all derivatives of $E_o(x, y)$ are neglected, show that[3]

$$\boldsymbol{E}^{(0)}(t, \boldsymbol{r}) = E_o(x, y) \, e^{i(kz-\omega t)} \, \frac{(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})}{\sqrt{2}} , \tag{3a}$$

$$\boldsymbol{B}^{(0)}(t, \boldsymbol{r}) = \hat{\boldsymbol{z}} \times \boldsymbol{E}^{(0)} , \tag{3b}$$

is a solution to the Maxwell equations for $\omega = ck$.

(b) (3 points) Calculate the time averaged energy per length in the wave packet, $\langle U \rangle$.

(c) (5 points) When the derivatives of $E_o(x, y)$ are not neglected, Eq. (3) is not a solution to the Maxwell equations. Determine the corrections to $\boldsymbol{E}^{(0)}$ and $\boldsymbol{B}^{(0)}$ to first order in gradients for $k\sigma \gg 1$.

Hint: try a solution for $\boldsymbol{E}$ (and analogously for $\boldsymbol{B}$) of the form

$$\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}^{(0)} + E^{(1)}(x, y)e^{i(kz-\omega t)}\hat{\boldsymbol{z}} , \tag{4}$$

and determine the correction $E^{(1)}(x, y)$ in terms of $E_o(x, y)$ and its derivatives.

(d) (4 points) Write the solution to part (c) as a linear superposition of the plane wave solutions to the Maxwell equations. First use the superposition to qualitatively explain the correction to the electric field (proportional to $\hat{\boldsymbol{z}}$), and then use the superposition to precisely reproduce this correction.

(e) (4 points) Calculate the z-component of the time averaged angular momentum per length in the wave packet, $\langle L^z \rangle$, to the lowest non-trivial order in $k\sigma$.

(f) (2 points) Determine the ratio $\langle L^z \rangle \, / \, \langle U \rangle$. Interpret the result using photons.

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[3] This is Gaussian or Heaviside-Lorentz units. In SI units the magnetic field reads, $\boldsymbol{B}^{(0)} = \frac{1}{Z_0}\hat{\boldsymbol{z}} \times \boldsymbol{E}^{(0)}$ where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 376\,\mathrm{Ohms}$ is the vacuum impedance.

Solution

(a) The Maxwell equations in free space read

$$\nabla \cdot \boldsymbol{E} = 0\,, \tag{5a}$$

$$-\frac{1}{c}\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = 0\,, \tag{5b}$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{5c}$$

$$-\frac{1}{c}\partial_t \boldsymbol{B} - \nabla \times \boldsymbol{E} = 0\,. \tag{5d}$$

Substituting

$$\boldsymbol{E} = E_o e^{i(kz-\omega t)} \boldsymbol{\epsilon}_+\,, \tag{6a}$$

$$\boldsymbol{B} = E_o e^{i(kz-\omega t)} \hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+\,, \tag{6b}$$

with E_o constant, gives the conditions

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{\epsilon}_+ = 0\,, \tag{7a}$$

$$\frac{i\omega}{c}\boldsymbol{\epsilon}_+ + ik\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+) = 0\,, \tag{7b}$$

$$\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+) = 0\,, \tag{7c}$$

$$\frac{i\omega}{c}(\hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+) - ik(\hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+) = 0\,. \tag{7d}$$

These equations are all clearly satisfied if $\omega = ck$ and $\boldsymbol{\epsilon}_+ = (\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})/\sqrt{2}$.

(b) The energy per length is

$$U = \int dx \int dy\, \frac{1}{2} \left\langle \boldsymbol{E}^2 + \boldsymbol{B}^2 \right\rangle\,, \tag{8}$$

$$= \int dx \int dy\, \frac{1}{4} \left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2 \right) \tag{9}$$

$$= \frac{1}{2} \int dx \int dy\, (E_o(x,y))^2\,, \tag{10}$$

$$= \mathcal{A}^2 (\pi\sigma^2)\,. \tag{11}$$

We used the fact that

$$|\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}|^2 = 2 \qquad |\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})|^2 = 2 \tag{12}$$

We also used the "time-averaging theorem"

$$\langle A(t)B(t) \rangle \equiv \left\langle \mathrm{Re}[Ae^{-i\omega t}]\ \mathrm{Re}[Be^{-i\omega t}] \right\rangle = \frac{1}{2}\mathrm{Re}[AB^*] \tag{13}$$

(c) We need to satisfy

$$\nabla \cdot \boldsymbol{E} = 0\,. \tag{14}$$

Substituting the suggested ansatz, this equation reads

$$\frac{1}{\sqrt{2}}\partial_x E_o(x,y) + \frac{i}{\sqrt{2}}\partial_y E_o(x,y) + ikE^{(1)}(x,y) = 0\,, \tag{15}$$

and thus

$$E^{(1)} = \frac{i}{\sqrt{2}k}\left(\frac{\partial E_o}{\partial x} + i\frac{\partial E_o}{\partial y}\right)\,. \tag{16}$$

For the magnetic field, we have

$$\boldsymbol{B}^{(0)} = E_o \frac{(-i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})}{\sqrt{2}} e^{i(kz-\omega t)}\,. \tag{17}$$

So since $\nabla \cdot \boldsymbol{B} = 0$,

$$\frac{-i}{\sqrt{2}}\partial_x E_o + \frac{1}{\sqrt{2}}\partial_y E_o + ikB^{(1)} = 0\,, \tag{18}$$

we find

$$B^{(1)} = \frac{i}{\sqrt{2}k}\left(-i\frac{\partial E_o}{\partial x} + \frac{\partial E_o}{\partial y}\right)\,. \tag{19}$$

(d) A general superposition (which is a pure plane in the z-direction) can be written

$$\boldsymbol{E}(t,\boldsymbol{r}) = \sum_{s=\pm}\int \frac{dk_x dk_y}{(2\pi)^2} E_o(\boldsymbol{k},s) e^{i(k_x x + k_y y + k_z z - \omega(\boldsymbol{k})t)} \boldsymbol{\epsilon}_s(\boldsymbol{k})\,, \tag{20}$$

where

$$\boldsymbol{k}\cdot\boldsymbol{\epsilon}_s(\boldsymbol{k}) = 0\,, \tag{21}$$

$$\boldsymbol{\epsilon}_s(\boldsymbol{k})\cdot\boldsymbol{\epsilon}_s^*(\boldsymbol{k}) = 1\,, \tag{22}$$

and of course

$$\omega(\boldsymbol{k}) = c\sqrt{k_x^2 + k_y^2 + k_z^2}\,. \tag{23}$$

The superposition we described above has

$$k_z \gg k_x, k_y \sim \frac{1}{\sigma}\,,$$

and is nearly circularly polarized. Qualitatively it is easy to see the need for a longitudinal correction to $\boldsymbol{E}^{(0)}$. The wave packet is a super-position of Fourier modes, one of which is shown in Fig. 1. The electric field points along the polarization vector, $\boldsymbol{\epsilon}(\boldsymbol{k})$. Sine the polarization vector is perpendicular to $\boldsymbol{k}$, it points partly in the z direction when k_x and k_y are non-zero. Thus, there must be a component of the electric field in the z-direction. We will now show how this reasoning quantitatively reproduces part (b).

First we note that to linear order in $k_\perp/k_z$

$$k = \sqrt{k_\perp^2 + k_z^2} \simeq k_z\,, \tag{24}$$

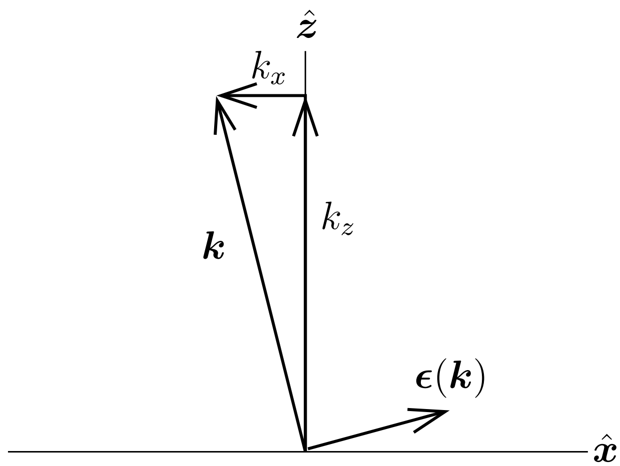


Figure 1: A typical Fourier mode in the wave packet and its polarization vector $\boldsymbol{\epsilon}(\boldsymbol{k})$.

implying that $\omega = ck \simeq ck_z$ are all approximately constant, and may be brought out of the integral in Eq. (20). We next decompose $\boldsymbol{k}$ and $\boldsymbol{\epsilon}$ into components transverse and parallel to $\hat{\boldsymbol{z}}$

$$\boldsymbol{k} \equiv \vec{k}_\perp + k_z \hat{\boldsymbol{z}}\,, \tag{25}$$

$$\boldsymbol{\epsilon} \equiv \vec{\epsilon}_\perp + \epsilon_z \hat{\boldsymbol{z}}\,. \tag{26}$$

Intuition from the plane wave solutions says that $|\vec{\epsilon}_\perp| \gg \epsilon_z$. Indeed, from the orthogonality condition

$$\vec{k}_\perp \cdot \vec{\epsilon}_\perp + k_z \epsilon_z = 0 \tag{27}$$

we find that

$$-\frac{\vec{k}_\perp \cdot \vec{\epsilon}_\perp}{k} = \epsilon_z\,. \tag{28}$$

The distribution is therefore

$$\boldsymbol{E}(t, \boldsymbol{r}) = e^{i(kz-\omega t)} \int \frac{dk_x dk_y}{(2\pi)^2} E_o(k_x, k_y) e^{i(k_x x + k_y y)} \left(\vec{\epsilon}_\perp - \frac{\vec{k}_\perp \cdot \vec{\epsilon}_\perp}{k} \hat{\boldsymbol{z}} \right)\,. \tag{29}$$

Taking $\vec{\epsilon}_\perp = (\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})/\sqrt{2}$, and using the properties of Fourier transforms, i.e.

$$\underbrace{ik_j}_{\text{Fourier space}} \quad \leftrightarrow \quad \underbrace{\partial_j}_{\text{coordinate space}}\,, \tag{30}$$

yields

$$\boldsymbol{E}(t, \boldsymbol{r}) = E_o(x, y)\,\frac{(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})}{\sqrt{2}} + \frac{i}{\sqrt{2}k}\left(\partial_x E_o(x, y) + i\partial_y E_o(x, y)\right)\hat{\boldsymbol{z}}\,. \tag{31}$$

Clearly we want

$$E_o(x, y) = \mathcal{A} e^{-(x^2+y^2)/(4\sigma^2)}\,, \tag{32}$$

and thus

$$E_o(k_x, k_y) = \int \mathrm{d}x\,\mathrm{d}y\, E_o(x,y)\, e^{-ik_x x - ik_y y} = \mathcal{A}\,(4\pi\sigma^2)\, e^{-\sigma^2(k_x^2+k_y^2)}\,, \tag{33}$$

fully specifying the fourier decomposition in Eq. (29).

(e) The time averaged angular momentum per length is

$$\boldsymbol{L} = \frac{1}{c}\int dx \int dy\, \langle \boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B})\rangle\ . \tag{34}$$

The integrand of the z-component of the angular momentum involves

$$\hat{\boldsymbol{z}}\cdot(\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}) = (\hat{\boldsymbol{z}}\cdot\boldsymbol{E})(\boldsymbol{r}\cdot\boldsymbol{B}) - (\boldsymbol{r}\cdot\boldsymbol{E})(\hat{\boldsymbol{z}}\cdot\boldsymbol{B})\,. \tag{35}$$

We see that because of the $\hat{\boldsymbol{z}}\cdot\boldsymbol{E}$ and $\hat{\boldsymbol{z}}\cdot\boldsymbol{B}$ terms the angular momentum necessarily involves the first correction, $\boldsymbol{E}^{(1)}$ and $\boldsymbol{B}^{(1)}$. The time averaged angular momentum involves

$$\langle\hat{\boldsymbol{z}}\cdot(\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}))\rangle = \frac{1}{2}\mathrm{Re}[(\hat{\boldsymbol{z}}\cdot\boldsymbol{E})(\boldsymbol{r}\cdot\boldsymbol{B})^*] - \frac{1}{2}\mathrm{Re}[(\boldsymbol{r}\cdot\boldsymbol{E})(\hat{\boldsymbol{z}}\cdot\boldsymbol{B})^*]\,. \tag{36}$$

Straightforward steps yield

$$\begin{aligned}
\mathrm{Re}[(\hat{\boldsymbol{z}}\cdot\boldsymbol{E})(\boldsymbol{r}\cdot\boldsymbol{B})^*] =& \mathrm{Re}[(\hat{\boldsymbol{z}}\cdot\boldsymbol{E}^{(1)})(\boldsymbol{r}\cdot\boldsymbol{B}^{0})^*]\,, &(37)\\
=& \mathrm{Re}[\frac{1}{2ik}\left(\partial_x E_o + i\partial_y E_o\right)\left(-ixE_o + yE_o\right)^*]\,, &(38)\\
=& \frac{1}{2k}\left(xE_o\partial_x E_o + yE_o\partial_y E_o\right)\ , &(39)\\
=& \frac{1}{4k}\left(x\partial_x E_o^2 + y\partial_y E_o^2\right)\ . &(40)
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{Re}[(\boldsymbol{r}\cdot\boldsymbol{E})(\hat{\boldsymbol{z}}\cdot\boldsymbol{B})^*] =& \mathrm{Re}[(\boldsymbol{r}\cdot\boldsymbol{E}^{(0)})(\hat{\boldsymbol{z}}\cdot\boldsymbol{B}^{1})^*]\,, &(41)\\
=& \mathrm{Re}\left[(xE_o + iyE_o)\frac{1}{2k}\left(\partial_x E_o + i\partial_y E_o\right)^*\right]\ , &(42)\\
=& \frac{1}{2k}\left(xE_o\partial_x E_o + yE_o\partial_y E_o\right)\ , &(43)\\
=& \frac{1}{4k}\left(x\partial_x E_o^2 + y\partial_y E_o^2\right)\ . &(44)
\end{aligned}$$

Thus

$$\begin{aligned}
\langle L^z\rangle =& \frac{1}{c}\int dx\int dy\ \langle\hat{\boldsymbol{z}}\cdot(\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}))\rangle\ , &(45)\\
=& \int dx\int dy \frac{1}{4ck}\left(x\partial_x E_o^2 + y\partial_y E_o^2\right)\ . &(46)
\end{aligned}$$

Integrating by parts we find

$$\begin{aligned}
\langle L^z\rangle =& \frac{1}{2ck}\int dx\int dy\, E_o^2\,, &(47)\\
=& \frac{1}{ck}\mathcal{A}^2(\pi\sigma^2)\,. &(48)
\end{aligned}$$

(f) For the required ratio we find

$$\frac{\langle L^z \rangle}{\langle U \rangle} = \frac{1}{\omega} \,. \tag{49}$$

This is consistent with our quantum expectation. Each photon of definite frequency ω and wave number $\boldsymbol{k} \simeq \frac{\omega}{c}\,\hat{\boldsymbol{z}}$ carries energy $E = \hbar\omega$ and spin angular momentum $\hbar$:

$$\frac{\langle L^z \rangle}{\langle U \rangle} = \frac{\hbar}{\hbar\omega} \,. \tag{50}$$

Quantum Mechanics 1

A charged massive spinless particle in a magnetic field

Consider the spinless particle with charge e and mass m in a constant magnetic field B directed along the z-axis

1. (4 points) Write down the non-relativistic Hamiltonian describing this problem, and find the operator of particle velocity $\hat{\mathbf{v}}$.

2. (4 points) Establish the commutation relations for the spatial components of this operator $[\hat{v}_i, \hat{v}_j]$, and for $[\hat{v}_i, \hat{x}_j]$, where $\hat{x}_j$ is the coordinate operator. Explain physically what these commutators imply about measurements of the system.

3. (8 points) (i) Write down the Schrödinger equation describing the problem and find the energy spectrum. (ii) Using the Schrödinger equation from (i), write down the wave function of the lowest Landau level explicitly. (iii) Determine the degeneracy of this energy level for a system of area $A = L_x L_y$ perpendicular to the magnetic field.

4. (4 points) Evaluate the commutator of the angular momentum component $\hat{l}_z$ and velocity component $\hat{v}_z$. Provide a physical interpretation of the result.

Solution

1. We choose $\vec{A} = (-By, 0, 0)$ so that $\vec{\nabla} \times \vec{A} = B\hat{z}$. The Hamiltonian is

$$H = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} = \frac{\left(p_x + \frac{e}{c}By\right)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} \tag{51}$$

The operator of particle velocity is

$$\vec{v} = \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right) \tag{52}$$

2. The commutators are ($\omega = eB/mc$)

$$\begin{aligned} [v_i, v_j] &= \frac{i\hbar\omega}{m}\epsilon_{ijz} \\ [v_i, x_j] &= \frac{-i\hbar}{m}\delta_{ij} \end{aligned} \tag{53}$$

The first commutator indicates that the pair v_x, v_y cannot be measured simultaneously, and the second commutator indicates that all the pairs v_i, x_i cannot be measured simultaneously, i.e.

$$\begin{aligned} \Delta v_x \Delta v_y &\geq \frac{1}{2}\frac{\hbar\omega}{m} \\ \Delta v_x \Delta x &\geq \frac{\hbar}{2m} \ etc. \end{aligned} \tag{54}$$

3. The Shrodinger equation for the stationary states $\Psi(x, y, z) = e^{-iEt/\hbar}e^{ik_x x + ik_z z}\varphi(y)$ is ($\bar{y} = c\hbar k_x/eB$)

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2(y + \bar{y})^2 + \frac{\hbar^2 k_z^2}{2m}\right)\varphi(y) = E\varphi(y) \tag{55}$$

The spectrum and wavefunctions are ($a = (\hbar/m\omega)^{\frac{1}{2}}$)

$$\begin{aligned} E(n, k_z) &= \left(n + \frac{1}{2}\right)\hbar\omega + \frac{\hbar^2 k_z^2}{2m} \\ \Psi(x, y, z) &= \frac{e^{ik_x x + ik_z z}}{2\pi}\frac{e^{-\frac{1}{2}\frac{(y-\bar{y})^2}{a^2}}}{(2a\sqrt{\pi})^{\frac{1}{2}}} \end{aligned} \tag{56}$$

with degeneracy $d = BA/ch/e$ where A is the area of the plane transverse to the magnetic field.

4. The angular momentum L_z is given by

$$L_z = xp_y - yp_x = m(xv_y - yv_x) + \frac{eB}{c}y^2 \to m(xv_y - yv_x) + \frac{eB}{2c}(x^2 + y^2) \tag{57}$$

where the rightmost equation corresponds to the cylindrical gauge $\vec{A} = (-\frac{B}{2y}, \frac{B}{2x}, 0)$ for manifest rotational symmetry in the plane. Now, we have

$$[L_z, v_z] = m\epsilon_{zij}\,[x_i v_j, v_z] = ie\hbar\left(\vec{x}\cdot\vec{B} - zB\right) = 0 \tag{58}$$

which also shows that $[L_z, H] = 0$. So L_z is conserved.

Quantum Mechanics 2

Parity violation in atomic hydrogen

This problem is about the weak interactions, but its solution requires only non-relativistic quantum mechanics.

In atomic hydrogen, weak currents associated with the exchange of Z_0 bosons ($m_Z =$ 92.6 GeV/c^2) give rise to an additional interaction between the electron and the proton of the form

$$H_w = \beta_w[\vec{s}\cdot\vec{p}\,\delta^3(\vec{r}) + \delta^3(\vec{r})\;\vec{s}\cdot\vec{p}] \quad \text{with } \beta_w \approx 1.4\times 10^{-8}\ \mathrm{m}^4/\mathrm{Js} \tag{59}$$

Here, $\vec{s}$ and $\vec{p}$ are the spin and the momentum operators of the bound electron, and the proton is assumed to be fixed at $\vec{r} = 0$. The weak interaction leads to a modification of selection rules for electric-dipole transitions, and may thus e.g. be characterized via optical spectroscopy.

(a) (5 points) (i) Assuming that H_w has no effect, calculate the lifetime of *both* the $2p(m = 0)$ and the $2s$ states due to spontaneous decay with an electric-dipole approximation. (ii) Use your result to estimate the lifetimes of these states in seconds. [Hint: The decay rate, or Einstein A coefficient, is $A = \omega_0^3|D|^2/3\pi\epsilon_0\hbar c^3$, where D is the electric dipole moment and $\hbar\omega_0$ the energy difference. A list of hydrogen wave functions and useful integrals is given at the end of this problem.]

(b) (2 points) Estimate the spatial range of the weak interaction and compare it to the Bohr radius of the bound electron. In view of this result, how justified is the use of the δ functional in the expression for H_w?

(c) (9 points) (i) Using symmetry arguments, show that H_w violates parity. (ii) Give a rough estimate for the (leading-order) modification to the lifetime of an electron in $2s$ [Hint: the $2s_{1/2}$ and $2p_{1/2}$ states are separated in energy by the Lamb shift, $\Delta E \simeq 4\times 10^{-6}$ eV and mix; very crudely assume $\vec{s}\cdot\vec{p} \sim (\hbar/2)p_r$ and ignore angular dependencies]

(d) (4 points) Determine whether H_w can produce a permanent electric-dipole moment in the 2s state. If so, what would this imply for time-reversal symmetry?

QM2: Wave-functions and integrals

- The hydrogen wave functions read:

$$\psi_{1s}(r,\theta,\phi) \propto e^{-r/a_0} \tag{60a}$$

$$\psi_{2s}(r,\theta,\phi) \propto (2 - r/a_0)e^{-r/2a_0} \tag{60b}$$

$$\psi_{2p,m=\pm1}(r,\theta,\phi) \propto \frac{r}{a_0}e^{-r/2a_0}\sin\theta e^{\pm i\phi} \tag{60c}$$

$$\psi_{2p,m=0}(r,\theta,\phi) \propto \frac{r}{a_0}e^{-r/2a_0}\cos\theta \tag{60d}$$

- A relevant integral

$$\int_0^\infty \mathrm{d}x\, x^n e^{-x} = n! \tag{61}$$

Solution

(a) The decay rate is given by the Einstein-A coefficient, where $\vec{D} = \langle\psi_f|\, e\vec{r}\,|\psi_i\rangle$. Since we are in $n = 2$, decay is to the 1S state.

(1) To evaluate $\vec{D}$ between $2P(m = 0)$ and $2S$, use the (well-known) hydrogen wavefunctions

$$\psi_{210}(\vec{r}) = \Lambda_2 \frac{r}{a_0} e^{-r/2a} \cos\theta; \quad \psi_{100}(\vec{r}) = \Lambda_1 e^{-r/a_0} \tag{62}$$

with $\Lambda_1 = (\pi a_0^3)^{-1/2}$ and $\Lambda_2 = (32\pi a_0^3)^{-1/2}$. Switching to spherical coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$), it is clear that $D_{x,y} = 0$ since both wavefunctions are even in ϕ. This leaves the calculation of D_z:

$$\begin{aligned} D_z &= \int \mathrm{d}^3r\ \psi_{210}(\vec{r})\ er\cos\theta\ \psi_{100}(\vec{r}) && (63)\\ &= \frac{e}{a_0}\Lambda_1\Lambda_2\frac{4\pi}{3}\int_0^\infty \mathrm{d}r\ r^4\ e^{-kr}, \quad \text{where } \kappa = \frac{3}{2a_0} && (64)\\ &= \frac{e}{a_0}\Lambda_1\Lambda_2\frac{4\pi}{3}\frac{\partial}{\partial\kappa^4}\int_0^\infty \mathrm{d}r\ e^{-\kappa r} && (65)\\ &= \frac{e}{a_0}\Lambda_1\Lambda_2\frac{4\pi}{3}\frac{4!}{\kappa^5} = \frac{128\sqrt{2}}{243}ea_0 \approx 0.74ea_0 && (66) \end{aligned}$$

For the calculation of A, the transition frequency ω_0 between 2P and 1S is needed. Using the (well-known) Rydberg formula $\hbar\omega_0 = E_1(1/2^2 - 1/1^2)$ yields $\omega_0 = 2\pi \times 2.47 \times 10^{15}$ Hz. Plugging D and ω_0 into the expression for the A coefficient then yields the decay rate

$$\Gamma_{2P1S} = 6.26 \times 10^8 \mathrm{s}^{-1} = \frac{1}{1.6\ \mathrm{ns}} \tag{67}$$

(2) On the other hand, since ψ_{200} has no angular dependence (i.e. is even), $\vec{D}$ for decay from 2S to 1S (also even) is zero, such that there is no spontaneous optical decay from 2S,

$$\Gamma_{2S1S} = 0. \tag{68}$$

(b) First use the Heisenberg uncertainty $\Delta E\Delta t \sim \hbar$, where $\Delta E = m_Z c^2$ is the rest energy of the Z_0 boson, to determine the lifetime Δt. Assuming propagation at c, this gives the range of the Z_0

$$\lambda_z \sim \frac{c\hbar}{m_Z c^2} = \frac{\hbar}{m_Z c}, \tag{69}$$

which is just its (reduced) Compton wavelength. Comparing λ_Z to the Bohr radius of the bound electron

$$a_0 = \frac{1}{\alpha}\frac{\hbar}{m_e c} \tag{70}$$

yields

$$\frac{\lambda_z}{a_0} = \alpha\frac{m_e}{m_z} = \frac{1}{137}\frac{5.1 \times 10^5}{9.3 \times 10^{10}} = 4 \times 10^{-8}, \tag{71}$$

such that the use of the δ function in H_w for a point-like interaction at the origin is certainly very well justified.

(c) (1) Parity is violated since $H_w \propto \vec{s}\cdot\vec{p}$. Spin $\vec{s}$ is an axial vector that does not change sign under P, while $\vec{p}$ is a polar vector that changes sign. In other words, $P^\dagger H_w P = -H_w$, or $[H_w, P] = 2H_w P \neq 0$.

(2) The weak interaction H_w mixes the eigenstates of the hydrogen atom. To lowest order this leads to a perturbation of the 2S state

$$|2S\rangle' = |2S\rangle + \frac{\langle 2P|\, H_w\, |2S\rangle}{\Delta E}\, |2P\rangle \tag{72}$$

where ΔE is the differential Lamb shift. Evaluation of the matrix element yields

$$\begin{aligned}
\langle 2P|\, H_w\, |2S\rangle &= \beta_w \frac{\hbar}{2}\left[\langle 2P|\, \hat{p}_r \delta^3(\vec{r})\, |2S\rangle + \langle 2P|\, \delta^3(\vec{r})\hat{p}_r\, |2S\rangle\right] && (73)\\
&= \beta_w \frac{\hbar}{2}\left[\langle 2P|\, \frac{\hbar}{i}\frac{\partial}{\partial r}\delta^3(\vec{r})\, |2S\rangle + 0\right] && (74)\\
&= -i\frac{\beta_w \hbar^2}{2}\left[\frac{\partial \psi_{210}}{\partial r}\right]_{r=0} \psi_{200}(0) && (75)\\
&= -i\frac{\beta_w \hbar^2}{2} \times \frac{\Lambda_2}{a_0} \times 2(2a_0)^{-3/2} && (76)\\
&= -i\frac{\beta_w \hbar^2}{16\sqrt{\pi} a_0^4} && (77)
\end{aligned}$$

such that the 2P state is mixed in with the amplitude

$$\left|\frac{\langle 2P|\, H_w\, |2S\rangle}{\Delta E}\right| = \frac{7 \times 10^{-37} J}{4 \times 10^{-6}\ \mathrm{eV}} = 1 \times 10^{-12} \tag{78}$$

Perturbations of the 1S state due to H_w are negligible since the energy differences involved are many orders of magnitude higher.

The rate for electric-dipole transitions between 2P and 1S was already calculated in (a); therefore the decay rate from $|2S\rangle'$ is now [4]

$$\Gamma_{2S'1S} = 1 \times 10^{-12} \times \Gamma_{2P1S} = 6 \times 10^{-4}\ \mathrm{s}^{-1} = \frac{1}{1.7 \times 10^3\ \mathrm{s}} \tag{79}$$

(d) A permanent electric dipole in the 2s state (and similarly in any other state) would require $\langle 2S'|\, z\, |2S'\rangle \neq 0$. However, based on the results in (c), the amplitude,

$$\langle 2P|\, H_w\, |2S\rangle / \Delta E = iA \tag{80}$$

[4] This is exceedingly hard to measure. Experiments with atomic Cesium in which the effect is amplified by an order of magnitude compared to hydrogen, employed interference with a Stark-induced transition amplitude to detect the still very small transition amplitude [Wood et al, Science 275, 1759 (1997)].

is imaginary. This means that

$$\begin{aligned}
\langle 2S'| z |2S'\rangle &= [\langle 2S| - iA \langle 2P|] z [|2S\rangle + iA |2P\rangle] && (81)\\
&= \langle 2S| z |2S\rangle + iA \langle 2S| z |2P\rangle - iA \langle 2P| z |2S\rangle + A^2 \langle 2P| z |2P\rangle && (82)\\
&= 0 + \langle 2S| z |2P\rangle (iA - iA) + 0 && (83)\\
&= 0 && (84)
\end{aligned}$$

If there were a permanent electric dipole moment, it would break time-reversal symmetry, since it would be unchanged under time reversal, other than angular momentum.

Quantum Mechanics 3

Scattering from a spherical shell

Consider a particle of mass m and energy E scattering on a 3-dimensional and spherically symmetric shell potential $V(r)$ as described by the Hamiltonian

$$H = \frac{\vec{p}^{\,2}}{2m} + V(r) = \frac{\vec{p}^{\,2}}{2m} + \alpha\,\delta(r - r_0) \tag{1}$$

a. (5 points) Determine the stationary S-state wavefunction for $E > 0$ and the corresponding phase shift.

b. (5 points) In the long wavelength limit, give the form of the phase shift and the explicit form of the scattering length.

c. (5 points) How many bound states can exist for the lowest partial wave and how does their existence depend on α? A graphical proof is acceptable.

d. (5 points) What is the scattering length when a bound state appears at $E = 0$? Describe the behavior of the scattering length as a function of α, from repulsive to attractive and give a well labeled sketch of your description.

Solution: Zahed

a. The radial part of the wavefunction solution to the Shrodinger equation for $l = 0$ is of the form $\psi(r) = \chi(r)/r$ and solves

$$\chi'' - \beta\delta(r - r_0)\chi = -k^2\chi \tag{2}$$

with $E = \hbar^2k^2/2m$ and $\beta = 2m\alpha/\hbar^2$. The finite solution is of the form

$$\chi(r) = \begin{array}{ll} \sin kr & r < r_0 \\ a\sin(kr + \delta) & r > r_0 \end{array} \tag{3}$$

and solves

$$\begin{aligned} &\sin kr_0 = a\sin(kr + \delta_0) \\ &\frac{\beta}{k}\sin kr_0 = a\cos(kr_0 + \delta) - \cos kr_0 \end{aligned} \tag{4}$$

Hence the phase shift is fixed by

$$\tan(kr_0 + \delta) = \frac{\tan(kr_0)}{1 + \frac{\beta}{k}\tan(kr_0)} \tag{5}$$

and the amplitude by

$$a^2 = 1 + \frac{\beta}{k}\sin(2kr_0) + \frac{\beta^2}{k^2}\sin^2(kr_0) \tag{6}$$

b. In the long wavelength limit $k \to 0$ Eq. 5 expands as

$$\frac{kr_0 + \tan\delta}{1 - kr_0\tan\delta} \approx \frac{kr_0}{1 + \beta r_0} \tag{7}$$

and in leading order

$$\tan\delta \approx -kr_0\frac{\beta r_0}{1 + \beta r_0} \to \delta(k) \approx Ak \tag{8}$$

with the scattering length

$$A = -\frac{r_0}{1 + \frac{1}{\beta r_0}} = -\frac{r_0}{1 + \frac{\hbar^2}{2m\alpha r_0}} \tag{9}$$

c. For bound states $E < 0$ and Eq. 2 reads

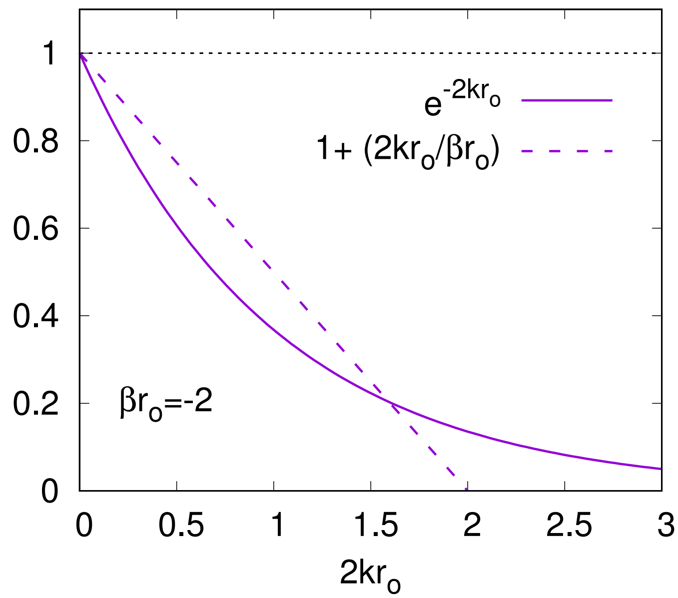


Figure 1: Graphical solution of Eq. 12

$$\chi'' - \beta\delta(r - r_0)\chi = k^2\chi \tag{10}$$

with $-E = \hbar^2 k^2/2m$ and $\beta = 2m\alpha/\hbar^2$. The finite solution is now of the form

$$\chi(r) = \begin{array}{ll} \sinh kr & r < r_0 \\ a\, e^{-kr} & r > r_0 \end{array} \tag{11}$$

and solves

$$1 + \frac{2kr_0}{\beta r_0} = e^{-2kr_0} \tag{12}$$

One bound state exists only if $\alpha < 0$ (recall $\beta = 2m\alpha/\hbar^2$) as is seen by examining Fig. 1. The condition for its existence is

$$\beta r_0 < 0 \to \alpha < -\frac{\hbar^2}{2mr_0} \tag{13}$$

d. A sketch of the scattering length as a function of α is shown in Fig. 2. From Eq. 9 it

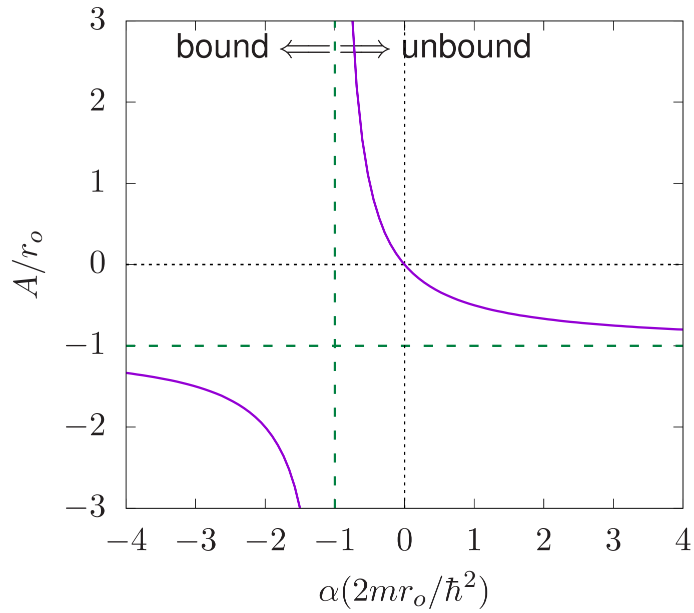


Figure 2: The scattering length given by Eq. 9 versus α.

follows that

$$\begin{aligned}
A &= 0 & \alpha &= 0 \\
A &= \pm\infty & \alpha &= \alpha_{\mp} = -\frac{\hbar^2}{2mr_0} \mp 0 \\
A &= -r_0 & \alpha &= \pm\infty
\end{aligned} \tag{14}$$

There is a bound state as $E \to +0$ for $\alpha = \alpha_{\mp}$ for which $\delta = \mp\pi/2$.

Statistical Mechanics 1

Semi-circular energy distribution

A system of identical, non-interacting, spin-½ particles has $N >> 1$ orbital single-particle eigenenergies ε, with the following semi-circular distribution function:

$$\rho(\varepsilon) = N\frac{2}{\pi}\left(\Lambda^2 - \varepsilon^2\right)^{1/2}, \qquad \text{for} - \Lambda \le \varepsilon \le +\Lambda\,, \qquad (1)$$

where $\rho(\varepsilon)d\varepsilon$ is the number of different eigenenergies within a small interval $d\varepsilon$.

A (2 points). What number N_g of particles in the system provides the lowest value of its ground state energy E_g, and what is this value?

B (8 points). Now let us temperature be different from zero, but the number of particles still have the value N_g calculated in Task A. Derive an explicit expression for the free energy F of the system. (An integral that cannot be worked out analytically in the general case is acceptable.)

C (6 points). Simplify your result for F, and calculate the entropy S of the system, in the limit of low (but still non-zero) temperature.

D (4 points). Simplify your general result for F, and calculate S, in the opposite, high-energy limit, and interpret your result.

Solution

(A) The state with the lowest energy has half filling, $N_g = N/2 \times 2 = N$.

(B) The partition function is

$$-\beta F = \log Z = \log \prod_{\epsilon_k<0} (1 + e^{-\beta\epsilon_k})^2 \tag{15}$$

$$=2 \int_{-\Lambda}^{0} \rho(E) \log(1 + e^{-\beta E}) \tag{16}$$

$$=\frac{4N}{\pi} \int_{-\pi/2}^{0} d\theta \cos^2\theta \log(1 + e^{-\beta\Lambda\sin\theta}). \tag{17}$$

(C) The leading term for $\beta \to \infty$ is

$$-\beta F = \frac{4N}{\pi} \int_{-\pi/2}^{0} d\theta \cos^2\theta(-\beta)\Lambda\sin\theta \tag{18}$$

$$=\frac{4N}{3\pi}\beta\Lambda. \tag{19}$$

However this term does not contribute to the entropy. To find the entropy we have to expand the free energy to next order in T. This is done simplest by subtracting the leading part resulting in

$$-\beta\delta F = \frac{4N}{\pi} \int_{-\pi/2}^{0} d\theta \cos^2\theta \log(1 + e^{\beta\Lambda\sin\theta}) \tag{20}$$

$$=\frac{4N}{\pi} \int_{0}^{\pi/2} d\theta \cos^2\theta \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} e^{-n\beta\Lambda\sin\theta} \tag{21}$$

$$\approx\frac{4N}{\pi} \int_{0}^{\infty} dx \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} e^{-n\beta\Lambda x} \tag{22}$$

$$=\frac{4N}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} \frac{1}{n\beta\Lambda} \tag{23}$$

$$=\frac{4N}{\pi\beta\Lambda}\frac{\pi^2}{12}. \tag{24}$$

The entropy approaches zero for $T < \Lambda/N$. The excitations above the ground state are given by particle-hole excitations, which have a level spacing of the order of single particle energies. So for $T < \Lambda/N$ pny the ground state contributes to the entropy, so that its values becomes zero.

(D)

$$-\beta F = \frac{4N}{\pi} \int_{-\pi/2}^{0} d\theta \cos^2\theta \log 2 = N \log 2 \tag{25}$$

So $F = -TN \log 2$ resulting in an entropy of $S = N \log 2$. At high temperature the bulk of the density of states determines the entropy, and is given by the logarithm of the answer of question b).

Statistical Mechanics 2

1D Boltzmann gases

Consider a monoatomic gas of identical non-relativistic particles with mass m. The particles are confined to move along a line segment of length L and they can pass through each other, so that the gas can be treated as ideal.

(a) (4 points). Calculate the chemical potential of such monoatomic gas in equilibrim at temperature T and density $n = N/L$, where $N \gg 1$ is the total number of particles.

(b) (6 points) Now assume that two atoms can form a bound state, a diatomic molecule with binding energy Δ. Calculate the density $n_{2\mathrm{at}} = N_{2\mathrm{at}}/L$ of such diatomic molecules in equilibrium with the monoatomic gas at temperature T and express it through the density $n_{1\mathrm{at}} = N_{1\mathrm{at}}/L$ of the latter. The total number of atoms $N_{1\mathrm{at}} + 2N_{2\mathrm{at}} = N$ is conserved.

(c) (5 points) Further, consider internal harmonic oscillations of the diatomic molecules from part (b) with frequency $\omega \ll \Delta/\hbar$. Calculate the chemical potential $\mu_{2\mathrm{at,osc}}$ of such diatomic gas assuming that $T \ll \Delta$.

(d) (5 points) Finally, find the density of oscillating diatomic molecules $n_{2\mathrm{at,osc}}$ in equilibrium with the monoatomic gas with density $n_{1\mathrm{at}}$ and temperature T. Compare this result to part (b) and discuss the difference qualitatively in the cases $T \ll \hbar\omega$ and $T \gg \hbar\omega$.

Solution

(a) There are at least two ways to compute the chemical potential of an ideal one-dimensional gas. One way is to use the grand canonical partition function. For one molecule, the partition function within a linear segment of length L is

$$Z_{1p} = \int \frac{dx\,dp}{2\pi\hbar} \exp\left(-\frac{p^2}{2mT}\right) = \frac{L}{2\pi\hbar} \cdot \sqrt{mT} \int dx\, e^{-x^2/2} = L\left(\frac{mT}{2\pi\hbar^2}\right)^{1/2} = \frac{L}{\lambda}\,, \tag{1}$$

where $\lambda = \sqrt{\frac{2\pi\hbar^2}{mT}}$ is the thermal wavelength of the gas. For N identical molecules, the partition function is $(Z_{1p})^N/N!$, where the N-th power of Z_{1p} comes from independent integration over the coordinates and the momenta of N particles and the factorial accounts for N-permutation of integration variables describing identical states of the system (Boltzmann's counting). Then, the grand canonical partition function

$$Z_G(\mu) = \sum_{N=0}^{\infty} e^{N\mu/T} \frac{Z_{1p}^N}{N!} = \exp\left(e^{\mu/T} Z_{1p}\right) \tag{2}$$

and the particle number can be expressed using the grand potential $\Omega(T, V, \mu) = -T\log Z_G = -e^{\mu/T}\frac{LT}{\lambda}$ as

$$\langle N\rangle = \frac{T}{Z_G}\left(\frac{\partial Z_G}{\partial\mu}\right)_{T,L} = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,L} = e^{\mu/T}\frac{L}{\lambda} \quad\Leftrightarrow\quad \mu = T\log\left(N\frac{\lambda}{L}\right) = T\log(\lambda n) \tag{3}$$

Replacement of $\langle N\rangle$ with N is justified if $N \gg 1$ and its fluctuation $\delta N = \sqrt{N} \ll N$.

Another way is to compute the Gibbs potential for N particles using the 1-dimensional equivalent of pressure $\kappa = -\left(\frac{\partial F}{\partial L}\right)_T$, where $F(T, L) = -T\log Z_N(T, L)$ is the free energy of the ideal 1-d gas:

$$F = -T\log Z_N = -T\log\frac{Z_{1p}^N}{N!} \approx TN(-\log\frac{L}{\lambda} + \log N - 1) \tag{4}$$

(Stirling's formula was used for $N!$) and the Gibbs potential $G = F + \kappa L = F - L\left(\frac{\partial F}{\partial L}\right)_T$,

$$G = F - L\left(\frac{\partial F}{\partial L}\right)_T = TN(-\log\frac{L}{\lambda} + \log N - 1) + TN = N\mu \tag{5}$$

leading to the same result for μ.

(b) Equilibrium between the monoatomic and the diatomic gases is determined by the balance of their chemical potentials,

$$\mu_{2\text{at}} = 2\mu_{1\text{at}}\,, \tag{6}$$

(It follows from the fact that the total Gibbs potential

$$G = \mu_{1\text{at}}N_{1\text{at}} + \mu_{2\text{at}}N_{2\text{at}} = \mu_{1\text{at}}(N - 2N_{2\text{at}}) + \mu_{2\text{at}}N_{2\text{at}}\,, \tag{7}$$

attains the minimum value while the diatomic molecules can dissociate or recombine, but number of atoms is conserved $N_1 + 2N_2 = N = \text{const}$, or $\Delta N_{2at} = -2\Delta N_{1at}$. The extremum condition $G = G_{min}$ leads to $dG/dN_{\rm 2at} = \mu_{\rm 2at} - 2\mu_{\rm 1at} = 0$).

The chemical potential for the diatomic gas is computed similarly to the monoatomic case except for twice the molecular mass ($2m$) and the energy counted from the ($-\Delta$) "baseline" equal to the binding energy, $\varepsilon(p) = \frac{p^2}{4m} - \Delta$. Therefore, its one-molecule partition function and chemical potential are (following the derivation in (A))

$$(Z_{1p})_{\rm 2at} = \int \frac{dx\,dp}{2\pi\hbar} \exp\left(-\frac{p^2}{4mT} + \frac{\Delta}{T}\right) = \frac{L}{\lambda_{\rm 2at}} e^{+\Delta/T}\,, \tag{8}$$

$$\mu_{\rm 2at} = T\log\frac{N_{\rm 2at}}{(Z_{1p})_{\rm 2at}} = T\log(\lambda_{\rm 2at}n_{\rm 2at}) - \Delta\,, \tag{9}$$

where the diatomic thermal wavelength is $\lambda_{\rm 2at} = \sqrt{\frac{\pi\hbar^2}{mT}}$. The balance equation for the chemical potentials $2\mu_{\rm 1at} = \mu_{\rm 2at}$ yields

$$2T\log(\lambda_{\rm 1at}n_{\rm 1at}) = T\log(\lambda_{\rm 2at}n_{\rm 2at}) - \Delta \quad \Rightarrow \quad n_{\rm 2at} = \frac{1}{\lambda_{\rm 2at}}(\lambda_{\rm 1at}n_{\rm 1at})^2 e^{\Delta/T} = n_{\rm 1at}^2 e^{\Delta/T}\sqrt{\frac{4\pi\hbar^2}{mT}} \tag{10}$$

(c) Oscillations within diatomic molecules are independent from their motion as a whole. Thus, summation over the internal degrees of freedom in the partition function can be performed independently, thus leading to the following expression for a single diatomic molecule,

$$(Z_{1p})_{\rm 2at,osc} = (Z_{1p})_{\rm 2at} \cdot Z_{\rm int}\,, \tag{11}$$

where the first factor corresponds to the translational motion of the molecule as a whole, and the second factor is a statistical sum for the internal (relative) motion of the atoms within. For harmonic oscillations with frequency ω, the energy levels are quantized as

$$E_k = -\Delta + k\hbar\omega \tag{12}$$

assuming that the ground state $E_0 = -\Delta$ corresponds to the binding energy Δ. The internal partition function is

$$Z_{\rm int} \approx e^{\Delta/T}\sum_{k=0}^{\infty} e^{-k\hbar\omega/T} = \frac{e^{\Delta/T}}{1 - e^{-\hbar\omega/T}} \tag{13}$$

where the summation over k has been extended to infinity introducing negligible exponentially suppressed correction $\propto e^{-\Delta/T} \ll 1$ because $T \ll \Delta$. The additional factor in the partition function leads to a modified expression for the chemical potential in comparison to the rigid molecule (part (b)),

$$\begin{aligned} \mu_{\rm 2at,osc} = T\log\frac{N_{\rm 2at,osc}}{(Z_{1p})_{\rm 2at,osc}} &= T\big[\log(\lambda_{\rm 2at}n_{\rm 2at}) - \frac{\Delta}{T} + \log(1 - e^{-\hbar\omega/T})\big] \\ &= \mu_{\rm 2at} + T\log(1 - e^{-\hbar\omega/T})\,. \end{aligned} \tag{14}$$

(d) Using the equilibrium condition $\mu_{2\mathrm{at,osc}} = 2\mu_{1\mathrm{at}}$, we get

$$n_{2\mathrm{at,osc}} = \frac{n_{1\mathrm{at}}^2 e^{\Delta/T}}{1 - e^{-\hbar\omega/T}} \sqrt{\frac{4\pi\hbar^2}{mT}} = \frac{n_{2\mathrm{at}}}{1 - e^{-\hbar\omega/T}} \,. \tag{15}$$

where $n_{2\mathrm{at}}$ is the equilibrium density of non-oscillating diatomic molecules computed in part (b).

If $T \ll \hbar\omega$, the internal molecular oscillations are "frozen out", and the diatomic equilibrium density $n_{2\mathrm{at,osc}} \approx n_{2\mathrm{at}}$ is the same as for the rigid molecules. In the opposite limit $T \gg \hbar\omega$, the above equation (15) can be simplified as

$$n_{2\mathrm{at,osc}} \approx \frac{n_{2\mathrm{at}}}{1 - 1 + \hbar\omega/T} = \frac{T}{\hbar\omega} n_{2\mathrm{at}} \,, \tag{16}$$

and the factor $T/(\hbar\omega)$ may be *qualitatively* interpreted as enhancement of the equilibrium density of diatomic molecules due to their internal degeneracy $\approx \frac{T}{\hbar\omega}$ because of the relative motion of the atoms.

Statistical Mechanics 3

Molecular field

The molecular-field approach in the theory of continuous phase transitions,[1] first suggested in 1908 by P.-E. Weiss, is based on taking the random variable s (say, a component of a classical "spin" variable s) in the form

$$s = \eta + \tilde{s}, \qquad \text{with } \eta \equiv \langle s \rangle, \quad \text{and } |\tilde{s}| << \langle s \rangle .$$

A (6 points). Apply the molecular-field approach to the Ising model of ferromagnetic transitions,

$$E = -J \sum_{\{j,j'\}} s_j s_{j'} - h \sum_j s_j , \qquad \text{with } J > 0 \text{ and } s_j = \pm 1 ,$$

where $\{j, j'\}$ means the pairs of nearest neighbors, on an infinite, d-dimensional cubic lattice. In particular, derive a self-consistency equation for the order parameter η.

B (3 points). Use the self-consistency equation to calculate the critical temperature T_c of the phase transition, and sketch the magnetization curves $\eta(h)$ at $T < T_c$ and $T > T_c$. How close is it to the calculated T_c to the exact values (in the same model), for d = 1, 2, and 3? Briefly explain the sign of the difference.

C (9 points). Apply the same approach to the so-called classical Heisenberg model,

$$E = -J \sum_{\{j,j'\}} \mathbf{s}_j \cdot \mathbf{s}_{j'} - \sum_j \mathbf{h} \cdot \mathbf{s}_j , \qquad \text{with } J > 0 .$$

(Here, in contrast with the Ising model, the spin of each site is modeled with a classical 3D vector **s** of a fixed length $s = 1$.) Again, calculate the critical temperature and analyze the magnetization curves at low and high temperatures.

D (2 points). Compare the molecular-field results for T_c in these two models, and interpret their difference.

[1] Sometimes it is called the mean-field theory; however, this terminology may be misleading, because it invites confusion with the Landau-type mean-field theories (such as the Ginzburg-Landau or Gross-Pitaevskii equations), which are of a higher level of phenomenology, and in particular treat T_c as a given parameter.

Solution

A (6 points). Plugging $s = \eta + \tilde{s}$ into the first term of Ising model's energy,

$$E = -J \sum_{\{j,j'\}} \left(\eta + \tilde{s}_j\right)\left(\eta + \tilde{s}_{j'}\right) - h \sum_j s_j \ .$$

multiplying the parentheses, and neglecting the term quadratic in small deviations $\tilde{s}$, we get

$$E = -J \sum_{\{j,j'\}} \left(\eta^2 + \eta\tilde{s}_j + \eta\tilde{s}_{j'}\right) - h \sum_j s_j = -J \sum_{\{j,j'\}} \left[\eta^2 + \eta\left(s_j - \eta\right) + \eta\left(s_{j'} - \eta\right)\right] - h \sum_j s_j$$
$$= NdJ\eta^2 - Jd\sum_j s_j - Jd\sum_{j'} s_{j'} - h\sum_j s_j ,$$

where N is the total number of the nodes is the lattice (which later will be taken infinite), and d is the number of the nearest neighbors per node, equal to the dimensionality of the lattice. Now changing the summation index in one of the sums from j' to j, and merging 3 similar terms, we may rewrite this expression as

$$E = NdJ\eta^2 - h_{\text{ef}} \sum_j s_j , \qquad \text{with } h_{\text{ef}} \equiv h + 2Jd\eta \ . \tag{1}$$

Besides the first, deterministic term, this is the energy of N independent "spins" in the effective field h_{ef}, which is contributed by the actual field h, and the average "molecular field" $2Jd\eta$ of the adjacent spins. The (well-known) statistics of such a system may be readily calculated from the Gibbs distribution applied to the statistical ensemble of single spins, each with just two possible states $s_\pm = \pm1$, and hence two possible energies, $E_\pm = \mp h_{\text{ef}}$:

$$W_\pm = \frac{1}{Z}\exp\left\{-\frac{E_\pm}{T}\right\} = \frac{1}{Z}\exp\left\{\mp\frac{h_{\text{ef}}}{T}\right\}, \qquad Z \equiv \sum_\pm \exp\left\{-\frac{E_\pm}{T}\right\} = \exp\left\{\frac{h_{\text{ef}}}{T}\right\} + \exp\left\{-\frac{h_{\text{ef}}}{T}\right\} = 2\cosh\frac{h_{\text{ef}}}{T},$$

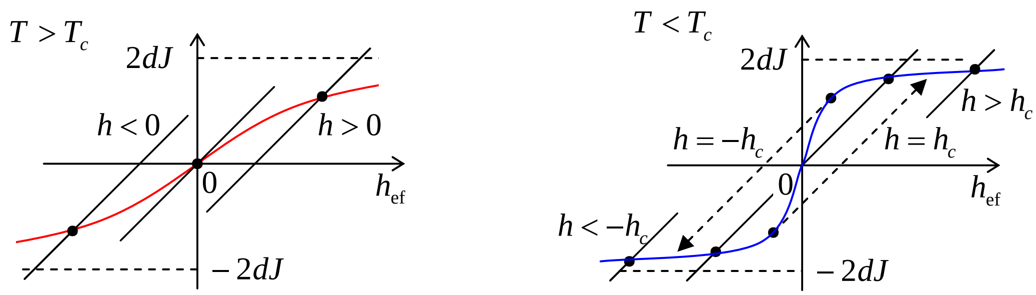
where $W_\pm$ are the probabilities of the corresponding states, and $T \equiv k_B T_K$ is temperature in energy units. In particular, for the statistical average $\eta \equiv \langle s\rangle$, which plays the role of the order parameter, we get

$$\eta = \langle s \rangle = \sum_\pm s_\pm W_\pm = \frac{(+1)\exp\{+h_{\text{ef}}/T\} + (-1)\{+h_{\text{ef}}/T\}}{2\cosh(h_{\text{ef}}/T)} \equiv \tanh\frac{h_{ef}}{T} \ . \tag{2}$$

Plugging this result into the above definition of h_{ef}, we get the following self-consistency equation:

$$h_{\text{ef}} - h = 2JD\tanh\frac{h_{\text{ef}}}{T} \ . \tag{3}$$

B (3 points). Sketching the left-hand part and right-hand part of Eq. (3) as functions of h_{ef} , for various h and T (see Fig. below),



we see that the magnetization curves $\eta(h)$ become hysteretic (and hence there is a nontrivial solution $\eta \neq 0$, describing the ordered ferromagnetic state in the absence of the external field) only if $T < T_c$, where

$$T_c = 2Jd.$$

This value of T_c is higher than the exact values of the ratio: 0 for $d = 1$, $2.27J$ for $d = 2$, and $4.51J$ for $d = 3$. The reason for this difference is that the molecular-field theory limits the thermally-induced fluctuations of spins s_j to very small values, and hence prevents a fair description of their accumulation with the growth of T, which eventually results in the destruction of the ferromagnetic state, i.e. to the phase transition, at $T = T_c$.

C (9 points). Let us align the axis z with the direction of magnetic field $\mathbf{h}$; then the state energy may be rewritten as

$$E_m = -J \sum_{\{j,j'\}} \left(s_{xj} s_{xj'} + s_{yj} s_{yj'} + s_{zj} s_{zj'} \right) - h \sum_j s_j \ .$$

In the molecular-field theory, each Cartesian components of the spin should be represented in a form similar to $s = \eta + \tilde{s}$, with a different statistical average for each component. Due to the symmetry of the problem with respect to reflections $x \to -x$ and $y \to -y$, such average, $\langle s_z \rangle \equiv \eta$, may be different from zero only for the z-component. Hence the first two terms under the double sum include only the squares of fluctuation terms:

$$E = -J \sum_{\{j,j'\}} \left(\tilde{s}_{xj} \tilde{s}_{xj'} + \tilde{s}_{zj} \tilde{s}_{zj'} \right) - J \sum_{\{j,j'\}} \left(\eta + \tilde{s}_{zj} \right)\left(\eta + \tilde{s}_{zj'} \right) - h \sum_j s_j , \qquad \text{with } |\tilde{s}| << 1 .$$

Multiplying the parentheses under the first sum, and neglecting the terms quadratic in small fluctuations, we get Eq. (1) again, but with s_j replaced with s_{zj}, which can take any real values from -1 to + 1.

As a result, the statistics is now different from the Ising model, and should be described by a continuous probability density $w(s_z)$, with the density of states uniformly distributed over all directions of the vector $\mathbf{s}$. Calculating it from the Gibbs distribution with $E = -h_{ef} s_z = -h_{\text{ef}} \cos\theta$, we get

$$w = \frac{1}{Z} \exp\left\{ \frac{-E}{T} \right\} = \frac{1}{Z} \exp\left\{ \frac{h_{ef} \cos\theta}{T} \right\}, \qquad Z = \oint_{4\pi} \exp\left\{ -\frac{E}{T} \right\} d\Omega = \oint_{4\pi} \exp\left\{ \frac{h_{ef} \cos\theta}{T} \right\} d\Omega ,$$

so that the order parameter η may be calculated as

$$\eta \equiv \langle s_z \rangle = \langle \cos\theta \rangle = \oint_{4\pi} w \cos\theta d\Omega = \frac{\oint_{4\pi} \cos\theta \exp\{h_{ef} \cos\theta / T\} d\Omega}{\oint_{4\pi} \exp\{h_{ef} \cos\theta / T\} d\Omega} = \frac{\int_{-1}^{+1} \cos\theta \exp\{h_{ef} \cos\theta / T\} d(\cos\theta)}{\int_{-1}^{+1} \exp\{h_{ef} \cos\theta / T\} d(\cos\theta)}$$

$$= \frac{T}{h_{\text{ef}}} \frac{\int_{-h_{\text{ef}}/T}^{+h_{\text{ef}}/T} e^{\xi} \xi d\xi}{\int_{-h_{\text{ef}}/T}^{+h_{\text{ef}}/T} e^{\xi} d\xi} = \frac{T}{h_{\text{ef}}} \frac{\left[e^{\xi}(\xi - 1)\right]_{-h_{\text{ef}}/T}^{+h_{\text{ef}}/T}}{\left[e^{\xi}\right]_{-h_{\text{ef}}/T}^{+h_{\text{ef}}/T}} = \coth\frac{h_{\text{ef}}}{T} - \frac{T}{h_{\text{ef}}} \to \begin{cases} 1, & \text{for } h_{\text{ef}}/T \to \infty, \\ h_{\text{ef}}/3T, & \text{for } h_{\text{ef}}/T \to 0. \end{cases}$$

This function $\eta(h_{\text{ef}})$ is qualitatively, but not quantitatively similar to that given by Eq. (2) for the Ising model; most importantly, it has a three-fold lower slope at the origin:

$$\left.\frac{\partial\eta}{\partial h_{\text{ef}}}\right|_{h_{\text{ef}}=0} = \frac{1}{3T}.$$

This difference immediately maps on the phase transition temperature T_c, giving

$$T_c = \frac{2Jd}{3}.$$

D (2 points). The lower values of T_c in the classical Heisenberg model is a natural result of the spin-to-field interaction weakening due to the availability of intermediate values, $-1 < s_z < +1$, of the field-aligned spin component s_z. In turn, this availability is an immediate result of taking into account two other Cartesian components of the vector **s**.