STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination

January 2015 (in 4 separate parts: CM, EM, QM, SM)

General Instructions (for each part):
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

Problem on Lagrangian and Hamiltonian mechanics

Consider the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} \frac{\dot{x}^2 + \dot{y}^2 + 2(x\dot{y} - y\dot{x})}{x^2 + y^2}$$

a. (6 pts.) Compute the Hamiltonian $H(x, y, p_x, p_y)$. The final form can be written as

$$\frac{1}{2} f(x, y)[(p_x - A_x(x, y))^2 + (p_y - A_y(x, y))^2]$$

for some f, A_x, A_y. Find the vector potential $\vec{A}$ and then compute the corresponding magnetic field. (Hint: recall that $\vec{B} = \vec{\nabla} \times \vec{A}$).

b. (4 pts.) Prove that the Lagrangian $L(x, y, \dot{x}, \dot{y})$ is invariant under two symmetries: rotations and scale transformations.

c. (4 pts.) Derive the conserved quantities for both symmetries.

d. (6 pts.) Rewrite the Lagrangian in polar coordinates, compute the Euler-Lagrange equations and solve them. Can you give a physical interpretation of the system?

Solution

a. The momenta are:

$$p_x = \frac{\dot{x} - y}{x^2 + y^2} \quad , \quad p_y = \frac{\dot{y} + x}{x^2 + y^2}$$

We need to find $\dot{x}(p_x, p_y, x, y), \dot{y}(p_x, p_y, x, y)$:

$$\dot{x} = (x^2 + y^2)p_x + y \quad , \quad \dot{y} = (x^2 + y^2)p_y - x$$

Then

$$\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} - L \\
&= (x^2 + y^2)(p_x^2 + p_y^2) + (p_x y - p_y x) - \frac{1}{2}\left((x^2 + y^2)(p_x^2 + p_y^2) - 1\right) \\
&= \frac{1}{2}(x^2 + y^2)(p_x^2 + p_y^2) + (p_x y - p_y x) + \frac{1}{2} \\
&= \frac{1}{2}(x^2 + y^2)\left[\left(p_x + \frac{y}{x^2 + y^2}\right)^2 + \left(p_y - \frac{x}{x^2 + y^2}\right)^2\right] \qquad (1)
\end{aligned}$$

Hence

$$\vec{A} = \frac{y\hat{x} - x\hat{y}}{x^2 + y^2}$$

and $\vec{B} = \vec{\nabla} \times \vec{A} = 0$ away from the origin.

b. Since the Lagrangian is homogeneous, it is obviously invariant under $x \to e^\lambda x$, $y \to e^\lambda y$. For rotations, the only term that is not manifestly rotationally invariant is the term linear in time derivatives. However, under

$$\delta x = -\theta y \quad , \quad \delta y = \theta x$$

we have:

$$\delta(x\dot{y} - y\dot{x}) = (-\theta y)\dot{y} + x(\theta\dot{x}) - (\theta x)\dot{x} - y(-\theta\dot{y}) = 0$$

c. The infinitesimal scale transformations are

$$\delta x = \lambda x \quad , \quad \delta y = \lambda y$$

Since the Lagrangian is invariant, there is no total derivative term, and the Noether charge is simply

$$Q_{scale} = p_x x + p_y y = \frac{x\dot{x} + y\dot{y}}{x^2 + y^2}$$

for scale transformations, and

$$Q_{rot} = xp_y - yp_x = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} + 1$$

for rotations.

d. Writing $x = r\cos\phi$, $y = r\sin\phi$, we find:

$$L = \frac{1}{2}\left(\frac{\dot{r}}{r}\right)^2 + \frac{1}{2}\dot{\phi}^2 + \dot{\phi} = \frac{1}{2}(\dot{\rho}^2 + (\dot{\phi} + 1)^2 - 1)$$

where $\rho = \ln r$. The general solution is obviously:

$$\rho - \rho_0 = v_0 t \quad , \quad \phi - \phi_0 = \omega_0 t$$

where $\rho_0, v_0, \phi_0, \omega_0$ are integration constants.

Since r goes from $0 \to \infty$, $\rho = \ln r$ goes from $-\infty \to \infty$. The system is a free particle constrained to move on a cylinder; there is no magnetic field, but there is a vector potential which would lead to an Aharanov-Bohm phase; in the ρ, ϕ coordinates, $\vec{A} = \hat{\phi}$.

Classical Mechanics 2

A particle in an attractive central potential

Consider the motion of a particle of mass m in an attractive central potential of the form

$$V(r) = \alpha\, r^k\,, \tag{1}$$

where k and α are real constants of the same sign (both positive or both negative).

a. (1 pt.) Write down the Lagrangian using polar coordinates (r, φ).

b. (3 pts.) Using conservation of the angular momentum, reduce the problem of determining the radial motion to an effective one-dimensional problem.

c. (4 pts.) Determine the radius and period of the circular orbits.

d. (2 pts.) For which values of k is the circular orbit *stable*?

e. (5 pts.) Assuming that the circular orbit is stable, consider a small perturbation around it. Find the period of the small oscillations. In the approximation of small oscillations, for which values of k will the orbit *close*?

f. (5 pts.) Go back to the full $2d$ problem for r and φ (the polar coordinates in the plane of the orbit). Eliminate the time dependence and write a differential equation for the orbit.

Solution

a. Write down the Lagrangian using polar coordinates (r, φ).

$$\mathcal{L} = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \alpha\, r^k\,. \tag{2}$$

b. Using conservation of the angular momentum, reduce the problem of determining the radial motion to an effective one-dimensional problem.

The angular momentum

$$\ell \equiv p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \tag{3}$$

is conserved by Lagrangian equations of motion since φ is a cyclic coordinate.

Effective one-dimensional Lagrangian for the radial problem

$$\mathcal{L}_{radial} = \frac{m}{2}\dot{r}^2 - V_{eff}\,, \quad V_{eff} = \frac{\ell^2}{2mr^2} + V(r) = \frac{\ell^2}{2mr^2} + \alpha r^k\,. \tag{4}$$

c. Determine the radius and period of the circular orbits.

The radius of the circular orbit is given by the extremum of V_{eff}, or equivalently, by imposing equality of the centripetal force $-V'(r)$ with the centripetal acceleration times m.

For $k \neq -2$, V_{eff} has a unique extremum at $r = r_\star$, given by solving

$$V'_{eff}(r) = \frac{-\ell^2}{mr^3} + \alpha k r^{k-1} = 0\,. \tag{5}$$

One finds

$$r_\star = \left(\frac{\ell^2}{\alpha k m}\right)^{\frac{1}{k+2}}\,. \tag{6}$$

There are no circular orbits for $k = -2$.

To find the period, we first determine the angular velocity from (3),

$$\Omega = \dot{\varphi} = \frac{\ell}{mr_\star^2}\,. \tag{7}$$

Hence

$$T = \frac{2\pi}{\Omega} = \frac{2\pi m r_\star^2}{\ell} \tag{8}$$

d. For which values of k is the circular orbit *stable*?

The orbit is stable for $V''_{eff}(r_\star) > 0$. One finds

$$V''(r_\star) = \frac{\ell^2}{mr_\star^4}(k+2)\,, \tag{9}$$

so the orbit is stable for $k > -2$.

e. Assuming the circular orbit is stable, consider a small perturbation around it. Find the period of the small oscillations. In the approximation of small oscillations, for which values of k will the orbit *close*?

Expanding around $r_\star$,

$$V(r) = V(r_\star) + \frac{V''(r_\star)}{2}(r - r_\star)^2 + \dots \tag{10}$$

Small oscillations have angular frequency

$$\omega = \sqrt{\frac{V''(r_\star)}{m}} = \frac{\ell}{mr_\star^2}\sqrt{k+2} = \sqrt{k+2}\,\Omega\,. \tag{11}$$

and period

$$\tau = \frac{2\pi}{\omega}\,. \tag{12}$$

For the orbit to close (in this approximation), the angular frequencies Ω and ω should be commensurate, so the condition is that $\sqrt{k+2}$ should be a *rational number*. For the Kepler potential, $k = -1$, so $\omega = \Omega$, the small oscillations have the same period as the circular orbit. For the harmonic potential, $k = 2$, so $\omega = 2\Omega$, the small oscillations complete two periods during a circular orbit. In fact these are the only two cases for which the *exact* orbit closes.

f. Go back to the full $2d$ problem for r and θ (the polar coordinates in the plane of the orbit). Eliminate the time dependence and write a differential equation for the orbit.

   The radial equation of motion is

$$m\frac{d^2r}{dt^2} = -V'(r) + \frac{\ell}{mr^3}\,. \tag{13}$$

   From the chain rule and (3),

$$\frac{d}{dt} = \dot{\varphi}\frac{d}{d\varphi} = \frac{\ell}{mr^2}\frac{d}{d\varphi}\,. \tag{14}$$

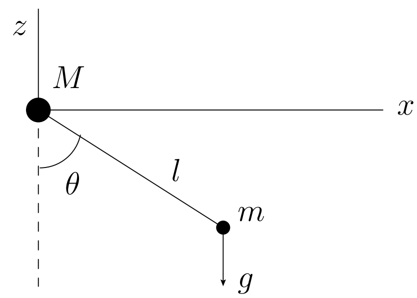
   Applying the chain rule twice in (13) one finds a differential equation for $r(\varphi)$. The equation takes a simpler form if one makes the change of variables $u = 1/r$. Then one finds

$$-\frac{\ell^2u^2}{m}\frac{d^2u}{d\varphi^2} = -V'(r = 1/u) + \frac{\ell^2u^3}{m}\,. \tag{15}$$

Classical Mechanics 3

The Sliding Pendulum

Consider a sliding pendulum, consisting of a mass M which can move without friction along a horizontal bar, and which is connected by a massless rod of length l to another mass m.

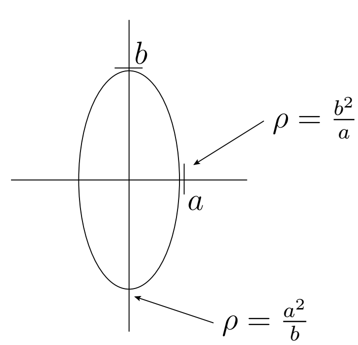


a. (4 pts.) Derive the angular frequency of small oscillations in the $x - z$ plane.

b. (4 pts.) Next consider the sliding conical (also called spherical) pendulum, for which the mass M can move without friction in the horizontal $x - y$ plane. Consider circular motion of both masses for a fixed, not necessarily small, angle θ. What is the angular frequency as a function of θ? Check your result for small θ.

c. (6 pts.) Now consider the inverted sliding pendulum. As it starts falling from rest at $\theta = \pi$, there comes a point where the tension T in the rod becomes zero. Find the value of θ when this happens, assuming for simplicity that M is infinitely large.

d. (6 pts.) What is the tension at $\theta = \frac{\pi}{2}$ and $\theta = 0$ if M and m are arbitrary?
Hint: For an ellipse given by $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, the curvature at the extremum with $x = a$ has a radius $\rho = \frac{b^2}{a}$, and at $y = -b$ one has $\rho = \frac{a^2}{b}$.



Solution

a. Choose the z-axis such that the center of mass remains on the z-axis. Then

$$x = \frac{M}{M+m} l \sin\theta\,; \quad y = -l\cos\theta\,; \quad X = -\frac{m}{M+m} l \sin\theta\,.$$

$$\begin{aligned} L &= \tfrac{1}{2} M\dot{X}^2 + \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} m\dot{y}^2 - mgy \\ &= \tfrac{1}{2}\frac{Mm}{M+m} l^2 \cos^2\theta\dot{\theta}^2 + \tfrac{1}{2} ml^2 \sin^2\theta\dot{\theta}^2 + mgl\cos\theta\,. \end{aligned}$$

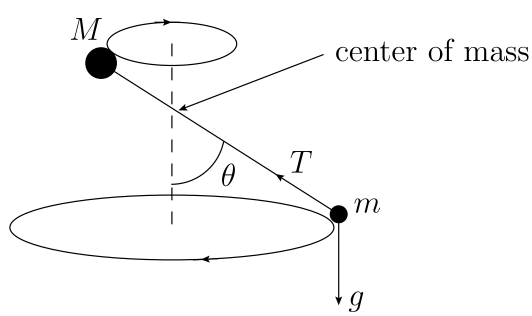
The equation of motion for small θ reads

$$\frac{Mm}{M+m} l^2 \ddot{\theta} + mgl\theta = 0\,.$$

Hence

$$\omega^2 = \frac{g}{l}\frac{M+m}{M}\,.$$

b. The mass m moves on a circle parallel to the $x - y$ plane with radius $r = \frac{M}{M+m} l \sin\theta$. If the tension in the bar is T, we get from the equations of motion for m



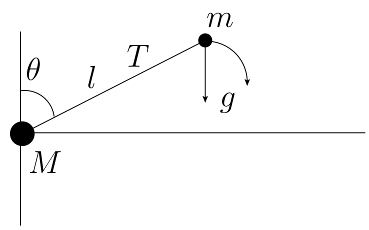
$$\frac{mv^2}{r} = T\sin\theta\,; \quad T\cos\theta = mg\,.$$

Taking the ratio we get, using $v = \omega r$

$$\tan\theta = \frac{mv^2}{r}\frac{1}{mg} = \frac{\omega^2 r}{g} = \omega^2 \frac{M}{M+m}\frac{l}{g}\sin\theta\,.$$

Hence $\omega^2 = \frac{M+m}{M}\frac{g}{l}\frac{1}{\cos\theta}$. For small θ we find the result of a) back, which should be the case, as for small θ the circular motion is a linear combination of motion in the xz plane and motion in the yz plane with phase difference $\frac{\pi}{2}$.

c. When the inverted pendulum starts falling down, we have the equation of motion for the mass m in the limit $M \to \infty$



$$m\frac{d\vec{v}}{dt} = m\vec{g} + \vec{T}$$

The reaction force (the tension) is along the radius and pointing outward, so considering motion in the radial direction we get

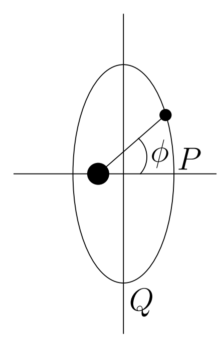
$$\frac{mv^2}{l} = mg\cos\theta + T\,.$$

Using energy conservation: $\frac{1}{2}mv^2 + mgy = mgl$, we get $v^2 = 2g(l-y)$. Then $T = \frac{mg}{l}(2l - 3y)$, hence the tension T vanishes if $y = \frac{2}{3}l$, or $\cos\theta = \frac{2}{3}$.

d. The mass m moves on an ellipse, as is clear from a), given by

$$\frac{x^2}{\left(\frac{M}{M+m}l\right)^2} + \frac{y^2}{l^2} = 1.$$

At the extremum P, the equation of motion of m yields



$$\frac{mv^2(\phi=0)}{\rho(\phi=0)} = T(\phi=0)\,.$$

At Q we find

$$\frac{mv^2\left(\phi=-\frac{\pi}{2}\right)}{\rho\left(\phi=-\frac{\pi}{2}\right)} + mg = T\left(\phi=-\tfrac{\pi}{2}\right)\,.$$

From energy conservation we get v^2:

$$\tfrac{1}{2}mv^2(\phi=0) = mgl$$

$$\tfrac{1}{2}mv^2\left(\phi=-\tfrac{\pi}{2}\right) + \tfrac{1}{2}M\left(\frac{mv}{M}\left(\phi=-\tfrac{\pi}{2}\right)\right)^2 = 2mgl\,,$$

where we used that at $\phi = -\frac{\pi}{2}$ the center of mass is at rest. Furthermore

$$\rho(\phi=0) = \frac{l^2}{\frac{M}{M+m}l} = \frac{M+m}{M}l$$

$$\rho\left(\phi=-\tfrac{\pi}{2}\right) = \left(\frac{M}{M+m}l\right)^2\frac{1}{l} = \left(\frac{M}{M+m}\right)^2 l\,.$$

So

$$T(\phi=0) = 2g\frac{mM}{M+m}$$

$$T\left(\phi=-\tfrac{\pi}{2}\right) = 4g\frac{m(M+m)}{M} + mg\,.$$

Electromagnetism 1

Fields of a non-relativistic particle

A charge particle of charge q moves non-relativistically with trajectory $\boldsymbol{R}(t)$:

(a) (6 pts.) Show that two of the four Maxwell equations are satisfied by expressing the fields $\boldsymbol{E}, \boldsymbol{B}$ in terms of the scalar and vector potentials, $A^\mu = (\varphi, \boldsymbol{A})$. Use the remaining Maxwell equations to derive the equations for the scalar and vector potentials in the Lorentz gauge.

(b) (8 pts.) Recall that the Green function of the wave equation is*

$$G(t - t_o, \boldsymbol{r} - \boldsymbol{r}_o) = \frac{\theta(t - t_o)}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|}\delta\left(t - t_o - \frac{|\boldsymbol{r} - \boldsymbol{r}_o|}{c}\right) . \tag{2}$$

Use this Green function to derive the potentials φ and $\boldsymbol{A}$ that are appropriate in the far field and the non-relativistic limit. Explicitly explain how the non-relativistic and far-field approximations are used at various points in the derivation to arrive at the final result.

(c) (4 pts.) If the particle is speeding up along the z axis

$$\boldsymbol{R}(t) = \left(v_o t + \frac{1}{2}at^2\right)\hat{\boldsymbol{z}} \, ,$$

determine the electric field in the far field as measured on the x-axis. What is the polarization of the radiated field when measured on this axis?

(d) (2 pts.) Assuming the motion as in part (c), determine the power radiated per solid angle in the $\hat{\boldsymbol{x}}$ direction.

Solution

(a) The source free Maxwell equations are satisfied because partial derivatives commute:

$$\partial_i B^i = \partial_i \epsilon^{ijk}\partial_j A_k = 0 \tag{3}$$

$$-\frac{1}{c}\partial_t B^i - (\nabla \times E)^i = -\partial_t \epsilon^{ijk}\partial_j A_k - \epsilon^{ijk}\partial_j\left(-\frac{1}{c}\partial_t A_k - \partial_k \varphi\right) \tag{4}$$

$$= 0 \tag{5}$$

* The Green function satisfies

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) G(t, \boldsymbol{r}) = \delta(t)\delta^3(\boldsymbol{r}) \tag{1}$$

The first sourced Maxwell equation

$$-\nabla\cdot\boldsymbol{E}=\rho\,, \tag{6}$$

becomes with $\boldsymbol{E}=-\frac{1}{c}\partial_t\boldsymbol{A}-\nabla\varphi$

$$-\Box\varphi-\frac{1}{c}\partial_t(\frac{1}{c}\partial_t\varphi+\nabla\cdot\boldsymbol{A})=\rho\,. \tag{7}$$

Then, writing the second sourced Maxwell equation

$$\nabla\times\boldsymbol{B}=\frac{\boldsymbol{j}}{c}+\frac{1}{c}\partial_t\boldsymbol{E} \tag{8}$$

in terms of $\boldsymbol{A}$ and ϕ, using

$$\nabla\times\nabla\times\boldsymbol{A}=-\nabla^2\boldsymbol{A}+\nabla(\nabla\cdot\boldsymbol{A})\,, \tag{9}$$

yields

$$-\Box\boldsymbol{A}+\frac{1}{c}\nabla(\frac{1}{c}\partial_t\varphi+\nabla\cdot\boldsymbol{A})=\frac{\boldsymbol{j}}{c}\,. \tag{10}$$

In the Lorentz gauge,

$$\frac{1}{c}\partial_t\varphi+\nabla\cdot\boldsymbol{A}=0\,, \tag{11}$$

we find two wave equations

$$-\Box\varphi=\rho\,, \tag{12}$$

$$-\Box\boldsymbol{A}=\frac{\boldsymbol{j}}{c}\,. \tag{13}$$

(b) Using the Green function of the wave equation

$$\varphi(t,\boldsymbol{r})=\int dt_o d^3\boldsymbol{r}_o\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\delta(t-t_o-\frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c})e\delta^3(\boldsymbol{r}_o-\boldsymbol{R}_o(t_o)) \tag{14}$$

$$\boldsymbol{A}(t,\boldsymbol{r})=\int dt_o d^3\boldsymbol{r}_o\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\delta(t-t_o-\frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c})e\boldsymbol{v}(t_o)\delta^3(\boldsymbol{r}_o-\boldsymbol{R}_o(t)) \tag{15}$$

Integrating over $\boldsymbol{r}_o$

$$\varphi(t,\boldsymbol{r})=\int dt_o\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{R}(t_o)|}\delta(t-t_o-\frac{|\boldsymbol{r}-\boldsymbol{R}(t_o)|}{c})\,e \tag{16}$$

$$\boldsymbol{A}(t,\boldsymbol{r})=\int dt_o d^3\boldsymbol{r}_o\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{R}(t_o)|}\delta(t-t_o-\frac{|\boldsymbol{r}-\boldsymbol{R}(t_o)|}{c})\,e\boldsymbol{v}(t_o) \tag{17}$$

In the far field we approximate

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{R}(t_o)|}\simeq\frac{1}{4\pi r} \tag{18}$$

Integrating over t_o involves

$$\delta(t - t_o - \frac{|\boldsymbol{r} - \boldsymbol{R}(t_o)|}{c}) = \frac{1}{1 - \boldsymbol{n}\cdot\beta(T)}\delta(t_o - T)\,. \tag{19}$$

where T (the retarded time) satisfies

$$T = t - \frac{|\boldsymbol{r} - \boldsymbol{R}(T)|}{c} \simeq t - r/c - \frac{\boldsymbol{n}\cdot\boldsymbol{R}(T)}{c} \tag{20}$$

The last approximation is a far field approximation. In a non relativistic limit

$$T \approx t - \frac{r}{c} \tag{21}$$

and

$$\delta(t - t_o - \frac{|\boldsymbol{r} - \boldsymbol{R}(t_o)|}{c}) \approx (1 + \boldsymbol{n}\cdot\beta(t - r/c))\delta(t_o - (t - r/c))\,. \tag{22}$$

So to linear order in v/c, we have

$$\boldsymbol{A}(t, \boldsymbol{r}) \simeq \frac{1}{4\pi r} e \frac{\boldsymbol{v}(t - r/c)}{c}\,. \tag{23}$$

For the scalar potential φ, we integrate over t_o and expand to first order in v/c:

$$\varphi(t, \boldsymbol{r}) = \frac{1}{4\pi r}\frac{e}{1 - \boldsymbol{n}\cdot\boldsymbol{v}(T)/c} \simeq \frac{e}{4\pi r}(1 + \boldsymbol{n}\cdot\boldsymbol{v}(t - r/c)/c)\,. \tag{24}$$

(c) Computing the electric field we have to leading order in $1/r$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{A} - \nabla\varphi\,, \tag{25}$$

$$\approx \frac{e}{4\pi r c^2}(-\boldsymbol{a} + \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{a}))\,, \tag{26}$$

where we used

$$\nabla r = \boldsymbol{n}\,, \tag{27}$$

when differentiating the potentials of part (b).

Then for a particle speeding up in the z-direction, $\boldsymbol{n} = \hat{\boldsymbol{x}}$ and we see that $\boldsymbol{E}$ is polarized in the negative z direction. The electric field on the x-axis is

$$\boldsymbol{E} = -\frac{1}{4\pi r c^2} a\,\hat{\boldsymbol{z}} \tag{28}$$

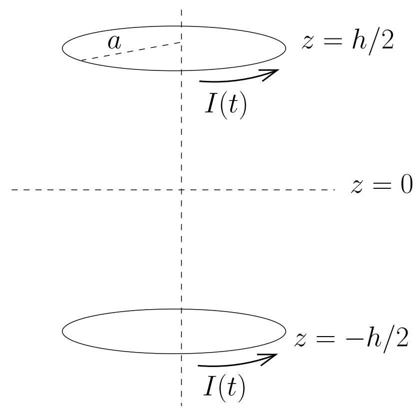
(d) The power radiated on the $x-$axis is

$$\frac{dP}{d\Omega} = c|rE|^2 = \frac{e^2}{16\pi^2 c^3}\,a^2\,. \tag{29}$$

Electromagnetism 2

Electric and magnetic fields of a solenoid

Two identical circular coils of wire of radius a are separated by a length h. The coils each carry a slowly varying sinusoidal current $I(t) = I_o \cos(\omega t)$. The axis of the coils is aligned with the $z-$axis and the geometry is centered at $z = 0$ (see below).



a. (5 pts.) At lowest order in the frequency, a magnetostatic approximation is valid. Using this approximation, show that close to the axis, and near $z = 0$, the Taylor series for the axial and radial components of the slightly *off-axis* magnetic field take the approximate form:

$$B_z \simeq \sigma_o + \frac{1}{2}\sigma_2(z^2 - \frac{\rho^2}{2}) + \dots \qquad B_\rho \simeq -\frac{1}{2}\sigma_2 z\rho + \dots\,, \tag{1}$$

where σ_0 and σ_2 are determined by the Taylor series of the *on-axis* magnetic field

$$B_z(z) \simeq \sigma_0 + \frac{1}{2}\sigma_2 z^2 + \dots\,. \tag{2}$$

Here $\rho = \sqrt{x^2 + y^2}$.

b. (5 pts.) Using the magnetostatic approximation, determine the magnetic field in the z direction close to the axis of the solenoid, and near $z = 0$, to quadratic order in z and ρ. Describe the magnetic field when $h = a$.

c. (5 pts.) Determine the electric field close to the axis of the solenoid *at* $z = 0$ to the lowest non-trivial order in the frequency and ρ.

d. (5 pts.) Briefly answer the following:

1. (3pts.) For parts b. and c., give an estimate for the size of finite-frequency corrections.

2. (2pts.) For part b., estimate at what large z the magnetostatic approximation breaks down when computing the magnetic field on the z-axis.

Solution

a. We use

$$\nabla \cdot \boldsymbol{B} = 0 \tag{3}$$

By symmetry under $z \leftrightarrow -z$ and analyticity of B as a function of x, y, a taylor series of B^z around $z = 0$ and $\rho = 0$ must take the following form

$$B^z = B_0 + \frac{a}{2} z^2 + \frac{b}{2} \rho^2 \tag{4}$$

Substituting this into the zero divergence condition

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho B^\rho + \partial_z B^z = 0 \tag{5}$$

we find upon integrating to find B^ρ

$$B^\rho \simeq -\frac{1}{2} a \rho z \tag{6}$$

Similarly demanding that $\nabla \times \boldsymbol{B} = 0$ gives that

$$\frac{\partial B^\rho}{\partial z} - \frac{\partial B^z}{\partial \rho} = 0 \tag{7}$$

So integrating with respect to ρ at $z = 0$ gives

$$B^z = B_0 - \frac{1}{4} a \rho^2 \tag{8}$$

Comparison with Eq. (4) gives the expected result $b = -(1/2)a$.

b. The magnetic field from a ring (on axis is

$$B^z = \frac{2m^z}{4\pi (z^2 + a^2)^{3/2}} \tag{9}$$

where $m^z = (I/c)\pi a^2$ is the magnetic moment. So the two rings together give

$$B^z = \frac{I a^2}{2c} \left[\frac{1}{((z + h/2)^2 + a^2)^{3/2}} + \frac{1}{((z - h/2)^2 + a^2)^{3/2}} \right] \tag{10}$$

Expanding with $R \equiv \sqrt{(h/2)^2 + a^2}$

$$((h/2 \pm z)^2 + a^2)^{-3/2} = ((h/2)^2 \pm hz + z^2 + a^2)^{-3/2} \tag{11}$$

$$\simeq \frac{1}{R^3} \left[1 \mp \frac{1}{R^2} \frac{3}{2} hz + \frac{15}{8} \frac{(hz)^2}{R^4} - \frac{3}{2} \frac{z^2}{R^2} \right] \tag{12}$$

So we find

$$B^z \simeq \frac{Ia^2}{c}\left[\frac{1}{R^3} + \frac{z^2}{R^5}\left(\frac{15}{8}h^2 - \frac{3}{2}R^2\right)\right] \tag{13}$$

$$\simeq \frac{Ia^2}{c}\left[\frac{1}{R^3} + \frac{z^2}{R^7}\frac{3}{2}\left(h^2 - a^2\right)\right] \tag{14}$$

Thus

$$B_o \equiv \frac{Ia^2}{cR^3} \qquad \sigma_2 \equiv \frac{Ia^2}{c}\frac{3(h^2-a^2)}{R^7}\,. \tag{15}$$

We can use the results of part (a) to express the fields.

When $h = a$, the field is uniform to quadratic order in coordinates since σ_2 vanishes.

c. Using the Faraday Law

$$E_\phi 2\pi\rho = -\frac{1}{c}\partial_t \int B^z(t,z)\cdot da \tag{16}$$

$$E_\phi \simeq -\frac{1}{c}\partial_t \frac{\pi\rho^2}{2\pi\rho}\left[B_o(t)\right] + O(\rho^3) \tag{17}$$

$$\simeq \left(\frac{\omega\rho}{2c}\right)\frac{I_o a^2}{cR^3}\sin(\omega t) \tag{18}$$

d. Corrections

(a) The corrections are of order $(\omega R/c)^2$ to the magnetic field, and $(\omega R/c)^3$ to the electric field

(b) When $z \sim c/\omega$ (the far field) then the quasi-static approximation breaks down.

Electromagnetism 3

Waves in metals

Consider an ohmic metal with high (but not infinite) conductivity σ and magnetic permeability[†] $\mu = 1$.

a. (6 pts.) Show that for harmonic time dependence, and high conductivity[‡] $\sigma \gg \omega$, that damped wave like solutions propagating in z-direction in the metal take the approximate form:

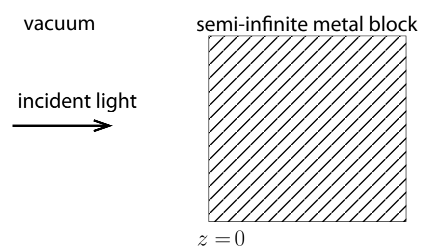
$$\boldsymbol{H}(t,z) = \boldsymbol{H}_c e^{-i\omega t + ik_c z} \tag{1}$$

where[§]

$$k_c = \frac{1+i}{\sqrt{2}} \frac{\sqrt{\sigma\omega}}{c} \tag{2}$$

b. (4 pts.) The electric field obeys a similar equation, $\boldsymbol{E}(t,z) = \boldsymbol{E}_c e^{-i\omega t + ik_c z}$. Use the Maxwell equations to express the amplitude of the electric field $\boldsymbol{E}_c$ in terms of the magnetic field $\boldsymbol{H}_c$.

c. (4 pts.) Now consider a linearly polarized plane wave in vacuum of frequency ω, which is normally incident upon a semi-infinite metal block with *infinite* conductivity as shown below.



When the metal has infinite conductivity, the amplitude of the reflected equals equals the amplitude of the incident wave, but the polarization of the reflected wave is inverted. Explain this familiar fact using the appropriate boundary conditions.

d. (6 pts.) Now consider the same reflection problem as in part 3, but this time the metal has a large (but finite) conductivity σ. Determine the electric and magnetic fields in the metal to leading order in ω/σ. The amplitude of the incident wave is E_o.

† In SI units this reads $\mu = \mu_o$

‡ In SI units this condition reads $(\sigma/\epsilon_o) \gg \omega$

§ This is written in Heaviside-Lorentz units. In SI units $k_c = (1+i)/\sqrt{2}\,\sqrt{\omega(\sigma/\epsilon_o)}\,/c$, while in Gaussian units, $k_c = (1+i)/\sqrt{2}\,\sqrt{4\pi\sigma\omega}\,/c$.

Solution

a. Writing the Maxwell Equations for harmonic fields

$$\nabla \cdot \boldsymbol{E} = \rho \tag{3}$$

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{J}}{c} - i\omega \boldsymbol{E} \tag{4}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{5}$$

$$\nabla \times \boldsymbol{E} = + i\frac{\omega}{c}\boldsymbol{B} \tag{6}$$

we then use $\boldsymbol{J} = \sigma \boldsymbol{E}$, and substitute $\boldsymbol{E} = \boldsymbol{E}_c\, e^{ik\boldsymbol{n}\cdot\boldsymbol{x}}$, with $\boldsymbol{n} = \hat{\boldsymbol{z}}$, and $\boldsymbol{H} = \boldsymbol{H}_c e^{ik\boldsymbol{n}\cdot\boldsymbol{x}}$, we have then

$$ik\boldsymbol{n} \times \boldsymbol{B}_c = \frac{\sigma}{c}\boldsymbol{E}_c - i\omega \boldsymbol{E}_c \tag{7}$$

$$ik\boldsymbol{n} \times \boldsymbol{E}_c = + \frac{i\omega}{c}\boldsymbol{B}_c \tag{8}$$

So dropping the second term on the first line (since $\sigma \gg \omega$), taking $ik\boldsymbol{n}\times$(the first equation), using the second, manipulating the cross product with the "b (ac) - c (ab)" rule, and using that $k\boldsymbol{n} \cdot \boldsymbol{B} = 0$ gives

$$k^2 \boldsymbol{B}_c = \frac{i\sigma\omega \boldsymbol{B}_c}{c^2} \tag{9}$$

Or

$$k = \sqrt{\frac{i\sigma\omega}{c^2}} = e^{i\phi}\frac{\sqrt{\sigma\omega}}{c} \tag{10}$$

where $e^{i\phi} = (1+i)/\sqrt{2}$.

b. Using

$$ik\boldsymbol{n} \times \boldsymbol{E}_c = \frac{i\omega}{c}\boldsymbol{B}_c \tag{11}$$

Fom the $\nabla \cdot \boldsymbol{E} = \rho$ equation we get $\boldsymbol{n} \cdot \boldsymbol{E}_c = 0$ after writing using current convservation, $\rho = ik\boldsymbol{n} \cdot \boldsymbol{E}/(i\omega)$. Thus we make cross both sides with $\boldsymbol{n}$, use "b(ac) - (ab) c" rule to find:

$$\boldsymbol{E}_c = \frac{\omega}{ck}\left(-\boldsymbol{n} \times \boldsymbol{B}_c\right) \tag{12}$$

This says that

$$\boldsymbol{E}_c = \sqrt{\frac{\omega}{\sigma}} e^{-i\phi}\left(-\boldsymbol{n} \times \boldsymbol{B}_c\right) \tag{13}$$

c. We write the electric field in vacuum as a sum of the incident and reflected wave

$$\boldsymbol{E}_{\text{vac}} = E_I \hat{\boldsymbol{x}} e^{ikz - i\omega t} + E_R \hat{\boldsymbol{x}} e^{-ikz - i\omega t} \tag{14}$$

$$\boldsymbol{H}_{\text{vac}} = E_I \hat{\boldsymbol{y}} e^{ikz - i\omega t} - E_R \hat{\boldsymbol{y}} e^{-ikz - i\omega t} \tag{15}$$

while inside the metal the electric fields are zero. Thus the boundary condition

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0\,, \tag{16}$$

yields

$$\boldsymbol{E}_{\rm vac}\big|_{z=0} = 0\,. \tag{17}$$

Or

$$E_I = -E_R \tag{18}$$

d. The boundary values of the vacuum fields are

$$\boldsymbol{E}_{\rm vac} =(E_I + E_R)\hat{\boldsymbol{x}} \tag{19}$$

$$\boldsymbol{H}_{\rm vac} =(E_I - E_R)\hat{\boldsymbol{y}} \tag{20}$$

Inside the conductors, the boundary values of the conductor fields

$$\boldsymbol{E}_c =H_c e^{-i\phi}\sqrt{\frac{\omega}{\sigma}}\hat{\boldsymbol{x}} \tag{21}$$

$$\boldsymbol{H}_c =H_c\hat{\boldsymbol{y}} \tag{22}$$

The boundary conditions

$$\boldsymbol{n} \times (\boldsymbol{E}_c - \boldsymbol{E}_{\rm vac}) = 0 \tag{23}$$

and

$$\boldsymbol{n} \times (\boldsymbol{H}_c - \boldsymbol{H}_{\rm vac} \tag{24}$$

So

$$E_I + E_R =H_c\sqrt{\frac{\omega}{\sigma}}e^{-i\phi} \tag{25}$$

$$E_I - E_R =H_c \tag{26}$$

And solving

$$H_c \simeq 2E_I\left(1 - \sqrt{\frac{\omega}{\sigma}}e^{-i\phi}\right) \tag{27}$$

while

$$E_c \simeq 2E_I\sqrt{\frac{\omega}{\sigma}}e^{-i\phi} \tag{28}$$

e. So the energy loss per incident flux is found by evaluating the Poynting vector just inside the metal

$$\frac{\langle \boldsymbol{S}\cdot\boldsymbol{z}\rangle}{\frac{c}{2}|E_I|^2} = \frac{{\rm Re}[E_cH_c^*]}{|E_I|^2} = 4\sqrt{\frac{\omega}{\sigma}}{\rm Re}[e^{i\phi}] = 2\sqrt{2}\sqrt{\frac{\omega}{\sigma}} \tag{29}$$

Quantum Mechanics 1

A particle in a perturbed harmonic potential

A particle of mass m in two dimensions is confined by an isotropic harmonic oscillator potential of frequency ω, while subject to a weak and anisotropic perturbation of strength $\alpha \ll 1$. The total Hamiltonian describing the motion of this particle is

$$H = H_0 + V = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + \alpha\, m\omega^2\, xy \tag{1}$$

a. (2 pts.) What are the energies and degeneracies of the three lowest-lying unperturbed states?

b. (5 pts.) Use perturbation theory to correct the energies to first order in α.

c. (5 pts.) Find the exact spectrum of H.

d. (4 pts.) Check that the perturbative results in part b. are recovered.

e. (4 pts.) Assume that 2 identical electrons are subject to the same anisotropic Hamiltonian (1). Write down the explicit wave-functions and degeneracies of the 2 lowest energy states.

Solution

a. The unperturbed states of H_0 are two dimensional harmonic oscillator states $|n_x n_y>$ with energies

$$E_{n_x n_y} = \hbar\omega(n_x + n_y + 1) \tag{2}$$

The three lowest energy states and their degeneracy d are

$$|00> \qquad \hbar\omega \qquad d = 1 \tag{3}$$
$$|01> \qquad 2\hbar\omega \qquad d = 2 \tag{4}$$
$$|10> \qquad 2\hbar\omega \qquad d = 2 \tag{5}$$

b. The state $|00>$ is non-degenerate. The energy shift is given by non-degenerate first order perturbation theory:

$$\Delta E_{00} =< 00|V|00 >= \alpha m\omega^2 < 0|x|0 >< 0|y|0 >= 0 \tag{6}$$

The states $|10>$ and $|01>$ are doubly degenerate with $d = 2$. Their energy shifts follow from degenerate perturbation theory. The interaction V in the degenerate subspace is

$$V_{2\times 2} = \begin{pmatrix} < 01|V|01 > & < 01|V|10 > \\ < 10|V|01 > & < 10|V|10 > \end{pmatrix} = \frac{1}{2}\alpha\hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

Thus, the $2\hbar\omega$ states are split

$$2\hbar\omega + \frac{1}{2}\alpha\hbar\omega \qquad \frac{1}{\sqrt{2}}(|10> + |01>) \tag{8}$$

$$2\hbar\omega - \frac{1}{2}\alpha\hbar\omega \qquad \frac{1}{\sqrt{2}}(|10> - |01>) \tag{9}$$

$$\tag{10}$$

c. By changing to the canonical variables: $X = (x+y)/\sqrt{2}$ and $Y = (x-y)/\sqrt{2}$ we can rewrite H

$$H = \frac{P_X^2}{2m} + \frac{P_Y^2}{2m} + \frac{1}{2}m\omega^2\left((1+\alpha)X^2 + (1-\alpha)Y^2\right) \tag{11}$$

and the exact spectrum is

$$E_{(n_X,n_Y)} = \hbar\omega\sqrt{1+\alpha}(n_X + \frac{1}{2}) + \hbar\omega\sqrt{1-\alpha}(n_Y + \frac{1}{2}) \tag{12}$$

d. The exact states in Taylor expansion are

$$E_{00} = \frac{1}{2}\hbar\omega\left(\sqrt{1+\alpha} + \sqrt{1-\alpha}\right) \approx \hbar\omega \tag{13}$$

$$E_{10} \approx \hbar\omega(1+\frac{\alpha}{2})(1+\frac{1}{2}) + \hbar\omega(1-\frac{\alpha}{2})(0+\frac{1}{2}) = 2\hbar\omega + \frac{1}{2}\alpha\hbar\omega \tag{14}$$

$$E_{01} \approx \hbar\omega(1+\frac{\alpha}{2})(0+\frac{1}{2}) + \hbar\omega(1-\frac{\alpha}{2})(1+\frac{1}{2}) = 2\hbar\omega - \frac{1}{2}\alpha\hbar\omega \tag{15}$$

in agreement with perturbation theory.

e. Let $\varphi_{1,2}(x)$ be the wave function associated with the deformed Hamiltonian (1) with the 2 lowest eigenvalues $\hbar\omega$ and $2\hbar\omega - \alpha\hbar\omega/2$ respectively. For 2 identical electrons the two lowest energy states are

$$d = 1 \qquad E = 2\hbar\omega : \qquad \varphi_1(1)\varphi_1(2)\,\frac{1}{\sqrt{2}}\,(|\uparrow\downarrow> -|\downarrow\uparrow>) \tag{16}$$

$$\begin{aligned}
&d = 4 \qquad E = 3\hbar\omega - \frac{1}{2}\alpha\hbar\omega : \\
&\frac{1}{\sqrt{2}}\,(\varphi_1(1)\varphi_2(2) + \varphi_2(1)\varphi_1(2))\ \frac{1}{\sqrt{2}}\,(|\uparrow\downarrow> -|\downarrow\uparrow>) \\
&\frac{1}{\sqrt{2}}\,(\varphi_1(1)\varphi_2(2) - \varphi_2(1)\varphi_1(2))\ \left(|\uparrow\uparrow>, \frac{1}{\sqrt{2}}\,(|\uparrow\downarrow> +|\downarrow\uparrow>)\,, |\downarrow\downarrow>\right)
\end{aligned} \tag{17}$$

Quantum Mechanics 2

Scattering of a particle from two static delta functions

Consider a one-dimensional non-relativistic particle of mass m and kinetic energy E scattering off the potential barrier $U(x)$ composed of two static delta-functions

$$U(x) = \beta_1 \left(\delta(x) + \delta(x-a)\right) \tag{1}$$

a. (3 pts.) Can the particle tunnel through this barrier without reflection? Explain your answer.

b. (5 pts.) If so, at what value(s) of the kinetic energy does this happen?

Now consider a three-dimensional non-relativistic particle of mass m and kinetic energy E scattering off the potential $U(\vec{r})$ composed of two static delta-functions

$$U(\vec{r}) = \beta_3 \left(\delta^{(3)}(\vec{r}-\vec{r}_1) + \delta^{(3)}(\vec{r}-\vec{r}_2)\right) \tag{2}$$

Suppose that for each of the delta-functions in (2) the S-wave scattering length $a < 0$.

c. (5 pts.) Write explicitly the S-wave bound wave-function near each of the centers. Check that each does not support a bound state for $a < 0$.

d. (7 pts.) Can the potential $U(\vec{r})$ with two centers support a bound state? If so, under what conditions? Explain your answer.

Hint: For a hard core potential of radius R, the S-wave scattering length a is defined as $d\ln\chi(R)/dR = -1/a$, with $\chi(r)$ the S-wave reduced wave-function.

Solution

a., b. Consider the particle approaching from the left ($x < 0$) with momentum $k = \sqrt{2mE}/\hbar$. Let us assume that we have found the wave function describing the state that tunnels through the barrier without reflection. At $x < 0$ this wave function is given by $\Psi_k(x) = \exp(ikx)$, i.e. it consists only of the incident plane wave, and no reflected one is present. In the region in between the delta-functions, $0 < x < a$, the wave function is the superposition $\Psi_k(x) = A \sin kx + B \cos kx$. Finally, at $x > a$, the wave function is again a plane wave $\Psi_k(x) = C \exp\left[ik(x-a)\right]$.

We have to match the wave functions at points $x = 0$ and $x = a$ to determine the constants A, B and C. By considering the Schrodinger equation with a delta-function potential $U_0\delta(x-x_0)$ at $x = x_0$ and integrating the wave function from $x = x_0 - \epsilon$ to

$x = x_0 + \epsilon$ we conclude that the derivative of the wave function changes at x_0 by the amount

$$\Delta\Psi'(x_0) \equiv \lim_{\epsilon\to 0}(\Psi'(x_0+\epsilon) - \Psi'(x_0-\epsilon)) = \frac{2m\beta}{\hbar^2}, \tag{3}$$

whereas the wave function itself is continuous, $\Psi'(x_0-\epsilon) = \Psi'(x_0+\epsilon)$ as $\epsilon \to 0$.

Performing the matching at $x = 0$ and $x = a$ we get

$$B = 1,\; kA - ik = \frac{2m\beta}{\hbar^2},\; A\sin ka + B\cos ka = C, \tag{4}$$

$$ikC - kA\cos ka + kB\sin ka = \frac{2m\beta C}{\hbar^2}. \tag{5}$$

This system has a solution only when

$$\tan ka = -\frac{k\hbar^2}{\beta m}; \tag{6}$$

the solutions of this equation determine the values of energy $E = \hbar^2 k^2/2m$ at which the particle penetrates the barrier without reflection.

c., d. The criterion for the existence of the bound state for a point-like potential can be cast in terms of the short-distance $r \to 0$ behavior of the $l = 0$ wave function $\chi(r) \equiv r\Psi(r)$:

$$\frac{1}{\chi}\frac{d\chi}{dr}\,|_{r=0} = -\frac{1}{a}, \tag{7}$$

where a is the scattering length that has to be positive for the bound state exist. Let us assume that the scattering centers are located at points $\vec{r}_1$ and $\vec{r}_2$. We are interested in finding a bound state, and will look for solutions of the form

$$\Psi(\vec{r}) \sim \frac{e^{-k|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} + \frac{e^{-k|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} \tag{8}$$

Let us now choose $\vec{r} \to \vec{r}_1$ and introduce $r_{12} = |\vec{r}_1 - \vec{r}_2|$. Expanding the exponential in the wave function (8) at $\vec{r} \to \vec{r}_1$ we get

$$\Psi(\vec{r} \to \vec{r}_1) \sim \frac{1}{|\vec{r}-\vec{r}_1|} - k + \frac{e^{-kr_{12}}}{r_{12}}. \tag{9}$$

Computing the derivative (7) of the wave function (9) multiplied by r we find

$$-\frac{1}{a} = \frac{1}{\chi}\frac{d\chi}{dr}\,|_{\vec{r}\to\vec{r}_1} = \frac{e^{-kr_{12}}}{r_{12}} - k. \tag{10}$$

We see that even if $a < 0$ (no bound state for a single potential) this equation does have solutions with $k > 0$ corresponding to the bound state, if r_{12} is sufficiently small. The critical condition for the bound state to exist is determined by $k = 0$, i.e. $r_{12} = -a$.

Quantum Mechanics 3

Harmonic oscillator subject to a transient external force

Consider a one-dimensional quantum-mechanical harmonic oscillator with mass m and resonance frequency ω. The oscillator initially (at $t \to -\infty$) is in its ground state. It is then subjected to a transient classical force $F(t)$, with $F(t \to \pm\infty) \to 0$.

a. (6 pts.) Write down the Hamiltonian $\hat{H}$ of the forced oscillator described above in terms of the usual ladder operators $\hat{a}$ and $\hat{a}^\dagger$, and solve their equations of motion in the Heisenberg picture. Show that the Hamiltonian for $t \to \pm\infty$ takes the form $\hat{H} = \hbar\omega(\hat{a}^\dagger_{\pm\infty}\hat{a}_{\pm\infty} + 1/2)$, where $\hat{a}^\dagger_{\infty} = \hat{a}^\dagger_{-\infty} - \alpha^\star$ and $\hat{a}_\infty = \hat{a}_{-\infty} + \alpha$, and determine the complex term α.

b. (6 pts.) At $t \to \pm\infty$, the ladder operators act on the states $|n_{\pm\infty}\rangle = (1/\sqrt{n!})(\hat{a}^\dagger_{\pm\infty})^n|0_{\pm\infty}\rangle$, where $|0_{\pm\infty}\rangle$ denote the vacuum with respect to $\hat{a}_{\pm\infty}$ and $\hat{a}^\dagger_{\pm\infty}$.

   Determine the probabilities $|c_n|^2$ that the oscillator has undergone a transition from the initial ground state to the n-th excited state at the end of the time evolution.

c. (3 pts.) What is the expectation value of the energy at the end of the time evolution?

d. (5 pts) Now assume that the force has a Gaussian profile, $F(t) = F_0 \exp(-t^2/(2\sigma_t^2))$, with amplitude $F_0 = \eta\hbar\omega/a_{ho}$, where η is a dimensionless parameter, and $a_{ho} = \sqrt{\hbar/m\omega}$ is the harmonic-oscillator length.

   For short pulses with $\sigma_t\omega \ll 1$, determine the maximum pulse strength η for which less than 1% of the population gets lost from the ground state. Show explicitly that in the limit $\sigma_t\omega \gg 1$, losses can be suppressed for any given value of η. Give a physical interpretation of these results.

Solution

a. The force term in the Hamiltonian is given by $F(t)\hat{x}$. Using $\hat{a} = \frac{1}{\sqrt{2}}\left[\frac{1}{a_{ho}}\hat{x} + i\frac{a_{ho}}{\hbar}\hat{p}\right]$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left[\frac{1}{a_{ho}}\hat{x} - i\frac{a_{ho}}{\hbar}\hat{p}\right]$, where $a_{ho} = \sqrt{\hbar/m\omega}$, one obtains

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + F(t)a_{ho}\frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) \tag{1}$$

   The Heisenberg equation of motion $\frac{d}{dt}\hat{a} = \frac{i}{\hbar}\left[\hat{H}, \hat{a}\right]$ yields

$$\frac{d}{dt}\hat{a} = -i\omega\hat{a}(t) + i\frac{1}{\sqrt{2}}\frac{a_{ho}}{\hbar}F(t), \tag{2}$$

and by integrating this equation (using Fourier transforms), one obtains

$$\hat{a}(t) = e^{-i\omega t}\left(\hat{a}_{-\infty} + i\frac{1}{\sqrt{2}}\frac{a_{ho}}{\hbar}\int_{-\infty}^{t} F(t')e^{i\omega t'}dt'\right) \overset{t\to\infty}{\equiv} e^{-i\omega t}\left(\hat{a}_{-\infty} + \alpha\right) \tag{3}$$

b. The transition amplitudes c_n are given by $c_n = \langle n_\infty | 0_{-\infty}\rangle$. Inserting the expressions for $\langle n_{\pm\infty}|$ and $\hat{a}_\infty$, and the fact that $\hat{a}_{-\infty}|0_{-\infty}\rangle = 0$, one obtains $c_n = (1/\sqrt{n!})\alpha^n\langle 0_\infty|0_{-\infty}\rangle$, where the bra-ket term is the coefficient c_0. From the normalization $1 = \sum |c_n|^2$ one then obtains $|c_0|^2 = e^{-|\alpha|^2}$, and thus

$$|c_n|^2 = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} \tag{4}$$

c. The probability distribution $|c_n|^2$ is Poissonian, with expectation value $\langle n\rangle = |\alpha|^2$. Therefore,

$$\langle E_\infty\rangle = \sum |c_n|^2\hbar\omega\left(n + \frac{1}{2}\right) = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right) \tag{5}$$

Alternatively, this result can also be directly obtained from the information given in part 1.

d. Up to prefactors, the expression for α is the Fourier transform of $F(t)$. It is straightforward to calculate, $\int_{-\infty}^{\infty} F(t')\exp(i\omega t)dt = F_0\sqrt{2\pi}\sigma_t\exp(-(\sigma_t\omega)^2/2)$. We obtain

$$|\alpha| = \eta\sqrt{\pi}(\omega\sigma_t)\exp\left(-\frac{(\omega\sigma_t)^2}{2}\right) \overset{\omega\sigma_t\ll 1}{\approx} \eta\sqrt{\pi}(\omega\sigma_t) \tag{6}$$

A survival probability $0.99 = |c_0|^2 = e^{-|\alpha|^2} \approx 1 - |\alpha|^2$ means $|\alpha| \approx 0.1$, and therefore for short pulses we require

$$\eta \lesssim \frac{0.1}{\sqrt{\pi}}\frac{1}{\omega\sigma_t} \tag{7}$$

In other words, the force must be so weak that the maximum energy change $\eta\hbar\omega \equiv \Delta E$, effected by F_0 over a_{ho}, remains well within the natural energy uncertainty (spectral width) $\hbar/\sigma_t$ of the pulse.

For long pulse durations $\sigma_t\omega \gg 1$, $|\alpha|$ is strongly suppressed by $\exp[-(\omega\sigma_t)^2/2]$, which can easily overcompensate the linear prefactor for any finite η. With the very coarse linear approximation $\left|\frac{\partial H}{\partial t}\right| \sim \frac{\eta\hbar\omega}{\sigma_t}$, the condition $\sigma_t\omega \gg 1$ can be re-written as

$$\frac{\left|\frac{\partial H}{\partial t}\frac{1}{\omega}\right|}{\Delta E} \ll 1 \tag{8}$$

which is the usual criterion for adiabaticity. In other words, the Hamiltonian change over one oscillation period must be small compared to the maxium energy difference.

Statistical Mechanics 1

Heat capacity and heat conductance in a ballistic system

Consider a one-dimensional system of free massless bosons with one polarization, and the dispersion relation $E_k = \hbar v|k|$, where v is the particle velocity, k – wavevector, E_k – energy. The particles are not interacting either among themselves or with external scattering potentials. If the system is in equilibrium at temperature T,

a. (6 pts.) calculate the specific heat capacity C.

b. (6 pts.) calculate the heat conductance G_{th}.

c. (8 pts.) repeat the calculations in (a) and (b) for massive fermions, for which $E_k = \hbar^2 k^2/2m$, and the chemical potential μ is far above the bottom of the energy spectrum: $\mu \gg k_B T$. (Consider one spin direction for the fermions.)

Hints: The heat conductance for the ballistic systems is defined, as usual, as the ratio of the energy flux flowing through a system over the infinitesimal temperature difference across the system that drives this flux. To evaluate the integrals needed to obtain the final numerical constants, you might find useful the following formulas:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

Solution

In equilibrium, the occupation probability of a state with energy E is given by the Bose distribution $f(E_k)$ with, as appropriate for zero-mass bosons, vanishing chemical potential. The number of particles $n(E_k)$ in the energy interval dE_k per unit length moving in one direction (left or right) then is

$$n(E_k) = \frac{1}{2\pi\hbar v} f(E_k) dE_k .$$

a. Using this relation, one can directly write down the average energy per unit length for particle moving in one direction as

$$U = \frac{1}{2\pi\hbar v} \int_0^\infty E_k f(E_k) dE_k = \frac{(k_B T)^2}{2\pi\hbar v} \int_0^\infty \frac{x dx}{e^x - 1} .$$

Evaluating the integral with the help of the given series:

$$\int_0^\infty \frac{x dx}{e^x - 1} = \sum_{k=1}^{\infty} \int_0^\infty dx x e^{-kx} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

we obtain

$$U = \frac{\pi (k_B T)^2}{12\hbar v}\,.$$

From this, we get the total (for left plus right movers) heat capacitance C per unit length:

$$C = 2dU/dT = k_B \frac{\pi k_B T}{3\hbar v}\,.$$

b. To find the heat conductance, we assume that the left-moving and right moving particles have the temperatures different by an infinitesimal amount δT. Then the total energy flux J carried by the system is

$$J = v[U(T+\delta T) - U(T)] = v(dE/dT)\delta T\,.$$

This means that the heat conductance G_{th} is:

$$G_{th} = J/\delta T = v dE/dT = Cv/2 = \frac{\pi k_B^2 T}{6\hbar}\,,$$

the result that depends only on the fundamental constants and temperature, and is independent of the properties (e.g., the velocity v) of the particle in the system.

c. For the nonlinear dispersion relation, particle velocity v is the function of energy:

$$\hbar v(E_k) = dE_k/dk\,.$$

With this modification, and taking $f(E_k)$ to be the Fermi distribution instead of the Boze distribution, relations from parts (a) and (b) remain valid. In particular, the average energy per unit length for particle moving in one direction is

$$U = \frac{1}{2\pi\hbar}\int_0^\infty E_k f(E_k)\frac{dE_k}{v(E_k)}\,.$$

At low temperatures, when $\mu \gg k_B T$, the chemical potential μ coincides with the Fermi energy E_F, and for the relevant excitations near the Fermi energy, $v = v(E_F) \equiv v_F$. In this regime, the temperature-dependent part $U(T)$ of the energy U needed for calculation of the heat capacity can be obtained as follows:

$$U(T) = \frac{1}{2\pi\hbar}\int_0^\infty E_k[f(E_k) - \Theta(E_k - E_F)]\frac{dE_k}{v(E_k)} = \frac{1}{2\pi\hbar v_F}\int_0^\infty (E_k - E_F)[f(E_k) - \Theta(-E_k)]dE_k$$

$$= \frac{1}{2\pi\hbar v_F}\int_{-\infty}^\infty (E_k - E_F)[f(E_k) - \Theta(E_k - E_F)]d(E_k - E_F) = \frac{1}{\pi\hbar v_F}\int_0^\infty (E_k - E_F) f(E_k) d(E_k - E_F)$$

$$= \frac{(k_B T)^2}{\pi\hbar v_F}\int_0^\infty \frac{x dx}{e^x + 1}\,.$$

Evaluating this integral in the same way as in part(a), we see that the energy U and the heat capacity C are given by the same expressions as for the bosons above (with

v equal to the Fermi velocity v_F). Then, the same logic as in part (b) shows that the thermal conductance is also the same. This coincidence is a manifestation of the general fact that in one dimension, thermodynamic properties are not sensitive to the exchange statistics.

Statistical Mechanics 2

Ising model in an external magnetic field

Consider the Ising model of N spins $\sigma_i = \pm 1$ in an external magnetic field h. Within the mean field approximation, its Hamiltonian can be written as

$$\mathcal{H}_{\mathcal{MF}} = \frac{1}{2} N J m^2 - (Jm + h) \sum_i \sigma_i \, , \tag{1}$$

where the co-ordination number of the lattice has been absorbed into the coupling constant J, and m is the magnetization.

The magnetization, specific heat and magnetic susceptibility are defined, respectively, as

$$m = \frac{\partial f}{\partial h}, \quad C = \frac{\partial U}{\partial T}, \quad \chi = \frac{\partial m}{\partial h} \, ,$$

where T is the temperature, U the internal energy, and f the free energy per site.

a. (6 pts.) Derive the (mean field) partition function following from eq. (1) and hence calculate the free energy of the system.

b. (8 pts.) Derive the magnetization, and graphically solve it for $h = 0$. Discuss the physical nature of the various solutions as a function of the temperature T. Identify a critical temperature T_c in terms of the system parameters, and discuss its physical meaning.

c. (6 pts.) Derive the expression for the dependence of the magnetization, the specific heat and the magnetic susceptibility on the quantity

$$t = \frac{T - T_c}{T_c} \, ,$$

where T_c is the critical temperature, and thereby determine the mean-field critical exponents $\alpha_c, \beta_c, \gamma_c$ which are defined through the relations

$$m \sim |T - T_c|^{\beta_c} \, , \quad C \sim |T_c - T|^{-\alpha_c} \, , \quad \chi = \frac{\partial m}{\partial h} \, |_{h=0} \sim |T_c - T|^{-\gamma_c} \, .$$

For the calculation of the specific heat, note that, within the mean field approximation near the critical point, the internal energy is $U \propto Jm^2$.

Solution

a. Starting with the system hamiltonian

$$\mathcal{H}_{\mathcal{MF}} = \frac{1}{2}NJm^2 - (Jm + h)\sum_i \sigma_i, \tag{2}$$

the partition function is given by:

$$\begin{aligned}
\mathcal{Z}_{\mathcal{MF}} &= \sum_{\sigma_1}\sum_{\sigma_2}\cdots\sum_{\sigma_N} e^{-\beta\mathcal{H}_{\mathcal{MF}}} \\
&= e^{-\frac{1}{2}\beta NJm^2}\left[\sum_{\sigma=\pm1} e^{\beta(Jm+h)\sigma}\right]^N \\
&= e^{-\frac{1}{2}\beta NJm^2}\left(2\cosh[\beta(Jm+h)]\right)^N
\end{aligned} \tag{3}$$

The free energy is then:

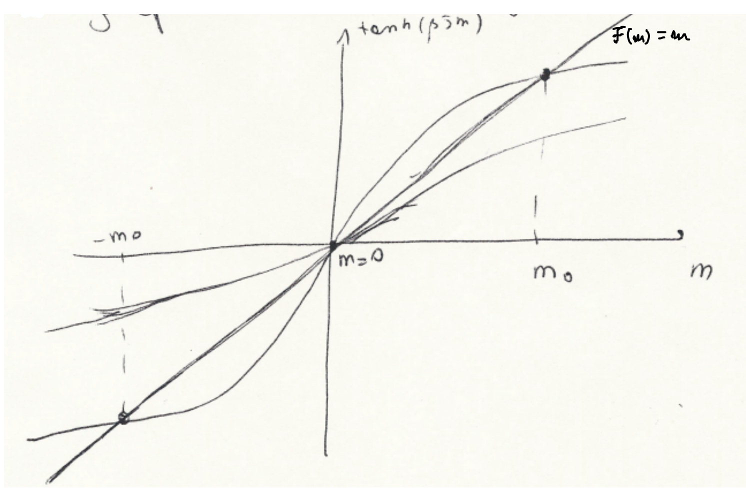
$$\begin{aligned}
\mathcal{F} &= -\frac{1}{N\beta}\ln\mathcal{Z}_{\mathcal{MF}} \\
&= -\frac{1}{N\beta}\left[-\frac{\beta}{2}NJm^2 + N\ln(2\cosh[\beta(Jm+h)])\right] \\
&= \frac{1}{2}Jm^2 - \frac{1}{\beta}\ln(2\cosh[\beta(Jm+h)])
\end{aligned} \tag{4}$$

b. The magnetization is given by:

$$\begin{aligned}
m &= \frac{\partial\mathcal{F}}{\partial h} \\
&= -\frac{1}{\beta}\frac{1}{2\cosh[\beta(Jm+h)]}(-2\sinh[\beta(Jm+h)])\beta \\
&= \tanh[\beta(Jm+h)]
\end{aligned} \tag{5}$$

such that for $h = 0$ we have $m = \tanh(\beta Jm)$.

A graphical solution gives:



Start by noticing that the solution $m = 0$ is always present (paramagnetic regime). However, two more solutions (corresponding to finite magnetization) can also be present if the slope of the hyperbolic tangent exceeds 1 at $m = 0$. The slope is:

$$\frac{d}{dm}\tanh(\beta J m)|_{m=0} = \frac{\beta J}{\cosh(\beta J m)}|_{m=0} = \beta J \tag{6}$$

The slope is $= 1$ for $\beta J = J/(k_B T) = 1$, which gives the critical temperature for the system:

$$T \equiv T_c = \frac{J}{K} \tag{7}$$

For $T > T_c$, the only state available to the system is that with $m = 0$. For $T < T_c$, two more solutions are possible, with $m = \pm m_0$; hence the system can exhibit a finite magnetization at $h = 0$.

c. The mean field relation $m = \tanh[\beta(Jm+h)]$ can be expanded for small m (equivalently small argument of the tanh) near the critical point. Since

$$\begin{aligned} \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \frac{2(x + x^3/6 + \dots)}{2(1 + x^2/2 + \dots)} \approx x(1+\frac{x^2}{6})(1-\frac{x^2}{2}) \approx x(1-\frac{x^2}{3}) \end{aligned} \tag{8}$$

we have

$$m = \beta(Jm+h)\left[1 - \frac{[\beta(Jm+h)]^2}{3} + \dots\right]. \tag{9}$$

Setting $h = 0$ for a moment, we have (for $T < T_c$)

$$1 = \beta J - \frac{(\beta J)^3 m^2}{3} \tag{10}$$

$$\sqrt{\frac{-3(1-\beta J)}{(\beta J)^3}} = m \approx \left(\frac{T_c - T}{T}\right)^{1/2} \times \text{(nonsingular nonzero pieces)} \tag{11}$$

so ($T < T_c$):

$$\beta_c = \frac{1}{2} \tag{12}$$

Using the relation

$$\frac{U}{N} = \frac{-J}{2}m^2 \tag{13}$$

so

$$\frac{C}{Nk_B} = \frac{-J}{2}\frac{\partial(m^2)}{\partial T} \tag{14}$$

but we just found $m \sim (T_c - T)^{1/2}$, so $m^2 \sim (T_c - T)$, so

$$\frac{C}{Nk_B} = \frac{-J}{2} \times \text{constant(T)} \tag{15}$$

so

$$\alpha_c = 0 \tag{16}$$

Finally, differentiating eq.(9) with respect to h yields

$$\frac{\partial m}{\partial h} = \beta(1 + J\frac{\partial m}{\partial h})\left[1 - (\beta(H + Jm))^2\right] \tag{17}$$

We then set $h = 0$ and for $T \approx T_c$, m is small so we can neglect the $(\beta Jm)^2$ term compared to the 1 in the brackets. Thus,

$$\frac{\partial m}{\partial h}(1 - \beta J) = \beta \tag{18}$$

or

$$\frac{\partial m}{\partial h}|_{h=0} = \frac{\beta}{1 - \beta J} = \frac{1}{T - T_c} \tag{19}$$

so

$$\gamma_c = 1 \tag{20}$$

Statistical Mechanics 3

Vapor pressure

a. (2 pts.) Write down the condition for thermodynamic equilibrium between the liquid and gas phases along a liquid-gas coexistence curve.

b. (14 pts.) Using your solution to part (a) and taking into account the entropy change at a liquid-gas phase transition, derive the relation for the vapor pressure along a liquid-gas coexistence curve.

c. (4 pts.) Calculate the vapor pressure of water at 27° C. Some relevant physical constants include the gas constant, $R = 8.315$ J/mol/K and the latent heat of vaporization of water, $L = 2.26 \times 10^3$ J/g or equivalently, $L = 4.06 \times 10^4$ J/mol.)

Solution

a. The condition for thermal equilibrium is that the Gibbs free energy $G = G(T, p)$ must be the same on either side of the phase coexistence curve, where T and p denote the temperature and pressure . Let us label the two phases as 1 and 2. Now $dG = SdT + Vdp$, where S and V denote the entropy and volume, respectively. Hence, $-S_1dT + V_1dp = -S_2dT + V_2dp$, i.e., $(S_2 - S_1)dT = (V_2 - V_1)dp$. So the condition is that

$$\frac{dp}{dT} = \frac{\Delta S}{\Delta V} \tag{1}$$

b. The entropy change across the phase boundary is $\Delta S = L/T$, where L is the latent heat of the phase transition. Furthemore, for the liquid-gas phase transition, $V_g >> V_\ell$ (where g and ℓ denote gas and liquid), so we have

$$\frac{dp}{dT} = \frac{S_g - S_\ell}{V_g - V_\ell} = \frac{L}{T(V_g - V_\ell)} \simeq \frac{L}{TV_g} \tag{2}$$

From the ideal gas law $pV = Nk_BT = \nu RT$, where $\nu = N/N_{Avog.}$ is the number of moles and $R = N_{Avog.}k_B$ is the gas constant, we have $V_g = \nu RT/p$. Combining this with Eq. (2), we obtain

$$\frac{dp}{p} = \frac{LdT}{\nu RT^2} = \frac{L_{mol.}}{RT^2} \tag{3}$$

where $L_{mol.}$ is the molar latent heat. Integrating both sides, we get, for the vapor pressure of liquid water on the phase coexistence curve with water vapor,

$$p = p_{atm} \exp\left[-\frac{L_{mol.}}{R}\left(\frac{1}{T} - \frac{1}{T_b}\right)\right] \tag{4}$$

where we have used the fact that the vapor pressure is equal to 1 atm. $= 1.01 \times 10^5$ Pascal at $T = T_{boil} = 373$ K.

c. Using the values $T = 27^\circ$ C $= 300$ K, $R = 8.315$ J/mol., and $L_{mol.} = 4.06 \times 10^4$ J/mol in Eq. (4), we get

$$p = 4.14 \times 10^{-2} \text{ atm} = 4.19 \times 10^3 \text{ Pa} \tag{5}$$